Karlsruhe Institute of Technology

SS 2022

Mathematical Methods of Theoretical Physics

Lecture: Prof. Dr. K. Melnikov
Exercises: Dr. C. Brønnum-Hansen

Exercise Sheet 8

Issue: 22.6.2022 – Submission: 29.6.2022 – Discussion: 6.7.2022

Exercise 1: π computation (4 points)

In this exercise we want to calculate π using infinite series. A simple formula to achieve this task is

$$\pi = 4 \arctan(1) \,, \tag{1.1}$$

where the $\arctan(x)$ function can be expanded in Taylor series around $x = 0$.

(a) Write down the series S_N that allows one to compute π. Calculate π for various values of N in range $0 < N \leq 1000$.

From this point onwards you can use a CAS, e.g. Mathematica.

(b) Apply the Shanks transformation to the series S_N once, twice and three times. What happens with the convergence of the transformed series? Check numerical values for $0 < N \leq 100$.

To improve convergence one can also use, for example,

$$\pi = 24 \arctan\left(\tfrac{1}{8}\right) + 8 \arctan\left(\tfrac{1}{57}\right) + 4 \arctan\left(\tfrac{1}{239}\right) . \tag{1.2}$$

(c) Use this series both with and without Shanks transformation to determine π and compare with the results obtained above.

Exercise 2: Breaking Shanks and Richardson (5 points)

Here we would like to investigate what happens when we apply the Shanks transformation or Richardson extrapolation to series that do not fulfil the assumptions on the transients that went into the respective constructions.
First, consider the geometric sum

$$G_N(x) = \sum_{k=0}^{N} (-x)^k \,, \tag{2.1}$$

which we used as the motivating example for the Shanks transformation in the lectures.

(a) What happens if you apply the Shanks transformation twice?

(b) Apply a first order Richardson extrapolation to the geometric sum. Compare the convergence towards the exact result for $N \to \infty$ between the original (unimproved) partial sums and the result of the Richardson extrapolation for $x = 0.99$. After how many terms do you get at least 5 correct digits in both cases?

Next, we deal with the sum

$$Z_N(x) = \sum_{k=1}^{N} \frac{1}{n^2} \tag{2.2}$$

that converges to $\zeta(2) = \frac{\pi^2}{6}$ for $N \to \infty$. As we saw in the lecture, this series converges slowly, but the rate of convergence can be significantly improved by applying a Richardson extrapolation.

(c) Instead, apply a Shanks transformation and plot the partial sums before and after the transformation. What is the behaviour of the transformed partial sums?

(d) Show that the remainder term after the Shanks transformation still has a $1/n$ behaviour.

Exercise 3: Borel and Euler (4 points)

Sum the series

$$\sum_{n=0}^{\infty} (-1)^n n^2 \,, \tag{3.1}$$

using Euler and Borel summation and show that the results agree. Generalise the result to sum the series

$$\sum_{n=0}^{\infty} (-1)^n n^p \,, \tag{3.2}$$

with p a positive integer.