Mathematical Methods of Theoretical Physics

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Exercise Sheet 9

Issue: 29.6.2022 – Submission: 6.7.2022 – Discussion: 13.7.2022

Exercise 1: Continued fractions and square roots (4 points)

In this exercise we will consider continued fractions and find how they can be used in the context of square roots and quadratic equations.

(a) Consider the following continued fraction

$$\Delta = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}\,, \tag{1.1}$$

and show that $\Delta = (1+\sqrt{5})/2$, i.e. the Golden ratio.

(b) Consider a quadratic equation $x^2 - bx - c = 0$. Derive a solution for x in terms of a continued fraction (algebraic manipulations only). How to find a second solution? When does this method fail?

(c) We want to use continued fractions to calculate a square root of a positive integer $a \in \mathbb{N}_+$. To this end, consider the quadratic equation $x^2 = a$, and derive a solution for x in terms of a continued fraction (algebraic manipulations only). *Hint:* Subtract p^2 from both sides with $p \in \mathbb{N}_+$ and factor terms involving x.

(d) To compute square roots calculators often use Newton's method that allows to find an approximate solution to the equation $0 = f(x)$ in an iterative manner, i.e.

$$x_{n+1} = x_n - f(x_n)/f'(x_n)\,, \tag{1.2}$$

where the x_n approximate the exact solution for $n \to \infty$. Derive the recurrence relation for finding the square root of $a \in \mathbb{N}_+$.

(e) Compare the convergence of the two approximations derived in the previous questions: use $a = 17$ and compare the first ten steps of the respective methods. Use $x_0 = 1$ for the starting point of Newton's method.

Exercise 2: Two-point Padé approximant (6 points)

In this exercise we will construct various Padé approximants to the function

$$f(z) = \frac{e^{-z}}{2\sqrt{z}} \int_0^z \frac{e^t\,\mathrm{d}t}{\sqrt{t}}\,. \tag{2.1}$$

We will use the knowledge of the asymptotic expansions of $f(z)$ around $z = 0$ and $z = \infty$, which are

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{+n} \quad \text{as } z \to 0\,, \qquad \text{with} \qquad a_n = \frac{(-4)^n n!}{(2n+1)!}\,, \tag{2.2}$$

$$f(z) \sim \sum_{n=0}^{\infty} b_n z^{-n} \quad \text{as } z \to \infty\,, \qquad \text{with} \qquad b_0 = 0 \text{ and } b_n = \frac{2(2n-2)!}{4^n (n-1)!}\,. \tag{2.3}$$

(a) Compute values of the function $f(z)$ for $z = 1, 16$ and 256 using numerical integration. *Hint:* You can use, for example, the function `NIntegrate` in Mathematica.

(b) Compute a diagonal Padé approximant ($N = M$) of the function $f(z)$ using the expansion around $z = 0$ and $N = 8$; let us denote it by $F_N^{(0)}(z)$. Give values of $F_N^{(0)}(z)$ for $z = 1, 16$ and 256. *Hint:* You can use expressions from the lecture notes. Solutions of linear systems of equations can be obtained using, for example, the function `LinearSolve` in Mathematica.

(c) Compute a diagonal Padé approximant ($N = M$) of the function $f(z)$ using the expansion around $z = \infty$ and $N = 8$; let us denote it by $F_N^{(\infty)}(z)$. Give values of $F_N^{(\infty)}(z)$ for $z = 1, 16$ and 256. *Hint:* You can use $z = 1/\xi$.

It is possible to construct a Padé approximant which uses expansions of $f(z)$ around two points, for example $z = 0$ and $z = \infty$. We call it a two-point Padé approximant. Here, for simplicity, we consider only a diagonal version ($N = M$), i.e.

$$F_N^{(2\text{pt})}(z) = \frac{\sum_{n=0}^{N} A_n z^n}{\sum_{m=0}^{N} B_m z^m}\,, \tag{2.4}$$

with $B_0 = 1$.

(d) Show that the coefficients $A_0, \ldots, A_N$ and $B_1, \ldots, B_N$ are determined by the following relations ($2N + 1$ equations)

$$\text{(I)} \quad A_n = \sum_{k=0}^{n} a_{n-k} B_k \quad \text{for} \quad 0 \le n \le N\,, \tag{2.5}$$

$$\text{(II)} \quad A_n = \sum_{k=n}^{N} b_{k-n} B_k \quad \text{for} \quad 1 \le n \le N\,. \tag{2.6}$$

Hint: Match Eq. (2.4) to the first $N + 1$ terms of Eq. (2.2) and to the first N terms of Eq. (2.3).

(e) Compute the Padé approximant $F_N^{(2\text{pt})}(z)$ using the relations in Eqs. (2.5) and (2.6); use $N = 8$. Give values of $F_N^{(2\text{pt})}(z)$ for $z = 1, 16$ and 256 and compare with the other Padé approximations as well as the numerical values obtained in question (a). *Hint:* You can start by computing A_0 and then rewrite the systems in Eqs. (2.5) and (2.6) in terms of $N \times N$ matrices.

Exercise 3: Padé approximants for a Borel sum (4 points)

Consider the series

$$S = \sum_{n=0}^{\infty} (-x)^n \frac{(2n)!}{2^n n!} \,. \tag{3.1}$$

(a) Find the Borel sum of this series.
This expansion may be useful, $\frac{1}{\sqrt{1-u}} = \sum_{n=0}^{\infty} \frac{u^n}{n!} \frac{(2n-1)!!}{2^n}$.

(b) Compute the Padé approximants P_N^N and P_{N+1}^N for $N = 1, 2, 3$. Compare the results for $x = \frac{1}{3}$ to the Borel sum.