Karlsruhe Institute of Technology

SS 2022

Mathematical Methods of Theoretical Physics

Lecture: Prof. Dr. K. Melnikov
Exercises: Dr. C. Brønnum-Hansen

Exercise Sheet 0

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This sheet is a "warm-up" sheet. It does not give any points and will not count towards the total points and total number of sheets in the end.

Exercise 1: Integration using the residue theorem

Calculate the following integrals using the residue theorem.

$$(a) \quad \int_0^{2\pi} \frac{\sin^2\varphi}{5+4\cos\varphi}\,\mathrm{d}\varphi\,, \tag{1.1}$$

$$(b) \quad \int_{-\infty}^{+\infty} \frac{x^6}{(1+x^4)^2}\,\mathrm{d}x\,, \tag{1.2}$$

$$(c) \quad \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+2x+2}\,\mathrm{d}x\,. \tag{1.3}$$

Exercise 2: Power series solutions to differential equations

Find a power series solution to the differential equation

$$y'(x) + 2xy(x) = 0 \tag{2.1}$$

around $x = 0$.

(a) Proceed by making the ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^n\,, \tag{2.2}$$

inserting it into the differential equation and then deriving a recurrence relation for the coefficients a_n by comparing coefficients in x.

(b) Try to derive a solution to the recurrence and sum up the series. Verify that the solution satisfies the differential equation.

Exercise 3: Multiplication of power series

Given two (formal) power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \,, \qquad\qquad B(x) = \sum_{n=0}^{\infty} b_n x^n \,, \tag{3.1}$$

find an expression for c_n so that the product can be written as

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n \,. \tag{3.2}$$

Exercise 4: Equating coefficients

Consider the function

$$f(x) = \frac{\mathrm{e}^{-cx}}{a + bx} \,. \tag{4.1}$$

Calculate the power series expansion of $f(x)$ around $x = 0$ and fix the coefficients a, b and c such that the expansion starts as

$$f(x) = 4 + 2x + x^2 + \frac{x^3}{6} + \mathcal{O}(x^4) \,. \tag{4.2}$$

What is the next term in the expansion?

Solution of exercise 1

In this exercise we will use the residue theorem to calculate integrals,

$$\oint_{\mathcal{C}} f(z)\mathrm{d}z = (2\pi i)\sum_k \mathrm{Res} f(z_k)\,, \tag{1.4}$$

where $f(z)$ is some function of complex variable z and z_k are represent poles of the function $f(z)$ that lie inside the closed contour $\mathcal{C}$.

The residue $\mathrm{Res}f(z_k)$ can be calculated using the formula

$$\mathrm{Res}f(z_k) = \frac{1}{(m-1)!}\lim_{z\to z_k}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\Big[(z-z_k)^m f(z)\Big]\,, \tag{1.5}$$

where m is such that the limit is well-defined.

We now apply the residue theorem to a few examples.

(a) We want to calculate the integral

$$I_a = \int_0^{2\pi}\frac{\sin^2\varphi}{5+4\cos\varphi}\,\mathrm{d}\varphi\,. \tag{1.6}$$

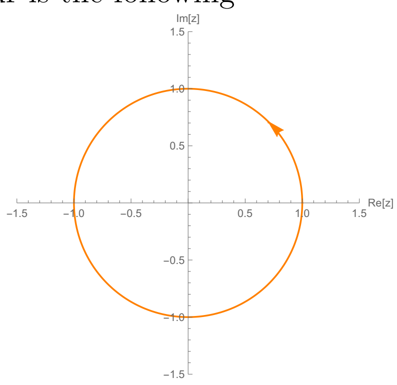
To this end we change the integration variable $\varphi \to z = e^{i\varphi}$. This implies

$$\mathrm{d}\varphi = -\,i\,\frac{\mathrm{d}z}{z}\,, \tag{1.7}$$

$$\sin\varphi = \frac{e^{+i\varphi}-e^{-i\varphi}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)\,, \tag{1.8}$$

$$\cos\varphi = \frac{e^{+i\varphi}+e^{-i\varphi}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)\,. \tag{1.9}$$

The integration contour is the following



and the integral is written as

$$\begin{aligned} I_a &= -\,i\oint\frac{\mathrm{d}z}{z}\frac{-\frac{1}{4}(z^2-2+1/z^2)}{5+2z+2/z} \\ &= +\,\frac{i}{4}\oint\frac{(z^4-2z^2+1)\mathrm{d}z}{z^2(2+z)(1+2z)}\,. \end{aligned} \tag{1.10}$$

This integral has three residues

$$z_1 = 0\,, \qquad z_2 = -1/2\,, \qquad z_3 = -2\,. \tag{1.11}$$

Two of them, z_1 and z_2, lie inside the integration contour.
Using the prescription given in Eq. (1.5) we have

$$\mathrm{Res} f(z_1) = \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[z^2 f(z)\right] = \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^4-2z^2+1}{(2+z)(1+2z)}\right] = -\frac{5}{4}\,, \tag{1.12}$$

$$\mathrm{Res} f(z_2) = \lim_{z\to -\frac{1}{2}} \left[\left(z-\tfrac{1}{2}\right) f(z)\right] = \lim_{z\to -\frac{1}{2}} \left[\frac{z^4-2z^2+1}{2z^2(2+z)}\right] = +\frac{3}{4}\,, \tag{1.13}$$

which by application of the residue theorem of Eq. (1.4) leads us to

$$I_a = +\frac{i}{4}(2\pi i)\left[\mathrm{Res} f(z_1) + \mathrm{Res} f(z_2)\right]\,, \tag{1.14}$$

and to the final result

$$I_a = +\frac{\pi}{4}\,. \tag{1.15}$$

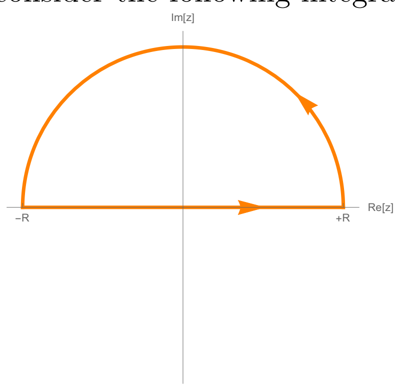
You can check the result, for example, in `Mathematica` by executing:

```
Integrate[Sin[x]^2/(5+4*Cos[x]),{x,0,2Pi}]
```

(b) Our second integral to calculate is

$$I_b = \int_{-\infty}^{+\infty} \frac{x^6\,\mathrm{d}x}{(1+x^4)^2}\,. \tag{1.16}$$

In order to use the residue theorem to calculate this integral we perform simply use $x = z$ and consider the following integration contour



where we distinguish two parts of this contour $\mathcal{C}_1$, which is a line along real axis from $-R$ to $+R$, and $\mathcal{C}_2$ which is a semicircle with radius R and origin at $z = 0$.

We now need to find poles of the function $f(z) = z^6/(1+z^4)^2$. These are

$$z_1 = e^{+i(\pi/4)}\,, \quad z_2 = e^{+i(3\pi/4)}\,, \quad z_3 = e^{+i(5\pi/4)}\,, \quad z_4 = e^{+i(7\pi/4)}\,. \tag{1.17}$$

Only z_1 and z_2 lie within of the integration contour. Therefore, using residue theorem of Eq. (1.4) we have

$$\int_{\mathcal{C}_1} \frac{z^6\,\mathrm{d}z}{(1+z^4)^2} + \int_{\mathcal{C}_2} \frac{z^6\,\mathrm{d}z}{(1+z^4)^2} = (2\pi i)\Big[\mathrm{Res}f(z_1) + \mathrm{Res}f(z_2)\Big]\,. \tag{1.18}$$

We first look at the contour integrals. We immediately write

$$\int_{\mathcal{C}_1} \frac{z^6\,\mathrm{d}z}{(1+z^4)^2} = \int_{-R}^{+R} \frac{x^6\,\mathrm{d}x}{(1+x^4)^2}\,, \tag{1.19}$$

$$\int_{\mathcal{C}_2} \frac{z^6\,\mathrm{d}z}{(1+z^4)^2} = \int_{0}^{\pi} \frac{R^6 e^{6i\alpha}}{(1+R^4e^{4i\alpha})^2} Re^{i\alpha} i\mathrm{d}\alpha\,, \tag{1.20}$$

where we have parametrised the second integral using $z = Re^{i\alpha}$. If we take the limit $R \to \infty$ we obtain

$$\lim_{R\to\infty} \int_{\mathcal{C}_1} \frac{z^6\,\mathrm{d}z}{(1+z^4)^2} = I_b\,, \tag{1.21}$$

$$\lim_{R\to\infty} \int_{\mathcal{C}_2} \frac{z^6\,\mathrm{d}z}{(1+z^4)^2} = \lim_{R\to\infty} \frac{i}{R} \int_0^{\pi} e^{-i\alpha}\mathrm{d}\alpha = 0\,. \tag{1.22}$$

We calculate the residues

$$\mathrm{Res}f(z_1) = \lim_{z\to z_1} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-z_1)^2 \frac{z^6}{(1+z^4)^2}\right] = +\frac{3}{16\sqrt{2}}(1-i)\,, \tag{1.23}$$

$$\mathrm{Res}f(z_2) = \lim_{z\to z_2} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-z_2)^2 \frac{z^6}{(1+z^4)^2}\right] = -\frac{3}{16\sqrt{2}}(1+i)\,. \tag{1.24}$$

Finally, we obtain

$$I_b + 0 = (2\pi i)\left[\frac{3}{16\sqrt{2}}(1-i) - \frac{3}{16\sqrt{2}}(1+i)\right]\,, \tag{1.25}$$

which gives

$$I_b = \frac{3\pi}{4\sqrt{2}}\,. \tag{1.26}$$

You can check the result, for example, in `Mathematica` by executing:

```
Integrate[x^6/(1+x^4)^2,{x,-Infinity,+Infinity}]
```

(c) The last example is to calculate the following integral

$$I_c = \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+2x+2}\mathrm{d}x\,. \tag{1.27}$$

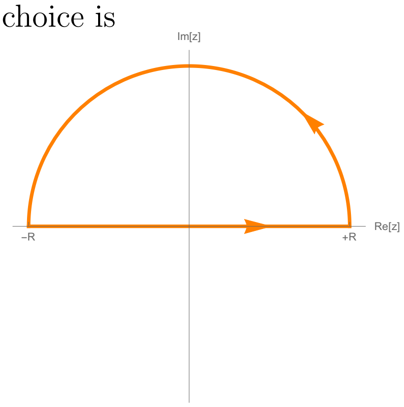
We again set $x = z$ and use $\sin z = (e^{+iz} - e^{-iz})/(2i)$. This leads us to the following integral

$$I_c = \frac{1}{2i} \int_{-\infty}^{+\infty} \frac{e^{+iz} - e^{-iz}}{(z+1)^2 + 1} \mathrm{d}z \,. \tag{1.28}$$

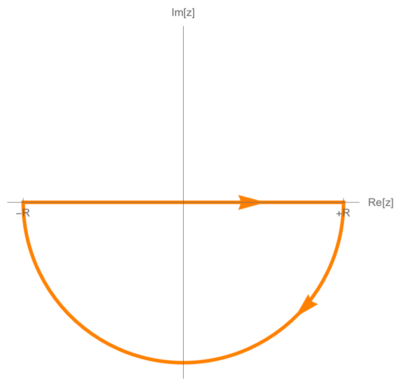
We can immediately identify that the poles of the integrand are

$$z_1 = -1 + i \,, \qquad z_2 = -1 - i \,. \tag{1.29}$$

In order to apply the residue theorem to calculate the integral I_c, we need to discuss what contour of integration is suitable for our task. We would like part of the contour to lie along the real axis, which will be directly related to integral I_c, and closed in such a way that the integral over the additional part either vanishes or is easy to calculate. In this case, a somewhat natural choice is



or



Before deciding which of the two is best for us, let us analyse the situation in a more general setup.

Jordan's lemma: Let us consider an integral

$$J = \int_{\mathcal{C}_+} f(z) e^{ikz} \mathrm{d}z \,, \tag{1.30}$$

where $\mathcal{C}_+$ is the upper semicircle with radius R and origin at $z = 0$, i.e. the first choice from above, parameter k is positive and $f(z)$ is some function. We would like to estimate the value of integral J. For this purpose, we parametrise

$$z = Re^{i\alpha} = R\cos\alpha + iR\sin\alpha \,, \tag{1.31}$$

$$\mathrm{d}z = iRe^{i\alpha}\, \mathrm{d}\alpha \,. \tag{1.32}$$

We can then estimate the modulus of the integral J as

$$|J| = \left| \int_{\mathcal{C}_+} f(z) e^{ikz} \mathrm{d}z \right| \leqslant \int_{\mathcal{C}_+} |f(z)| \cdot e^{-kR\sin\alpha} \cdot |\mathrm{d}z| \,. \tag{1.33}$$

Further, we can put a bound by taking the maximal value of the function $|f(z)|$ on the semicircle $\mathcal{C}_+$,

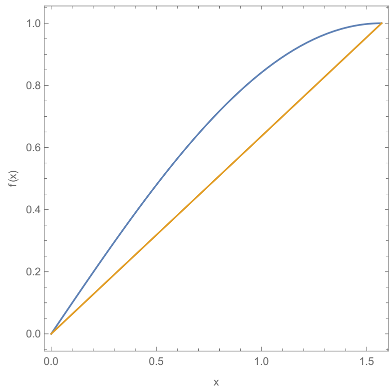
$$f_{\max} = \max_{z\in\mathcal{C}_+} |f(z)|\,. \tag{1.34}$$

This leads us to a bound

$$\begin{aligned} |J| &\leqslant f_{\max}\cdot R\int_0^{\pi} e^{-kR\sin\alpha}\mathrm{d}\alpha \\ &= 2f_{\max}\cdot R\int_0^{\pi/2} e^{-kR\sin\alpha}\mathrm{d}\alpha\,, \end{aligned} \tag{1.35}$$

where in the second step we have used the symmetry $\alpha \to \pi - \alpha$. Moreover, on the interval $\alpha \in [0, \frac{\pi}{2}]$ we have

$$\sin\alpha \geqslant \tfrac{2}{\pi}\alpha\,. \tag{1.36}$$



and this inequality allows us to estimate

$$\int_0^{\pi/2} e^{-kR\sin\alpha}\mathrm{d}\alpha \leqslant \int_0^{\pi/2} e^{-2kR\alpha/\pi}\mathrm{d}\alpha = \frac{\pi}{2kR}\Big[1 - e^{-kR}\Big]\,. \tag{1.37}$$

We finally obtain

$$|J| \leqslant \frac{\pi}{k} f_{\max}(1 - e^{-kR}) \xrightarrow{R\to\infty} \frac{\pi}{k} f_{\max}\,, \tag{1.38}$$

which means that integral

$$J = \int_{\mathcal{C}_+} f(z)e^{ikz}\mathrm{d}z \tag{1.39}$$

vanishes as soon as the maximum of function $f(z)$ vanishes on the semicircle $\mathcal{C}_+$.
Note that our analysis featured $k > 0$. In case we need to deal with negative values of k, we can perform the same analysis but closing the contour with

lower semicircle $\mathcal{C}_-$, i.e. like the choice of the contour on the right in picture above.

Being equipped with the Jordan's lemma, we can now get back to our integral I_c,

$$\begin{aligned} I_c &= \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{e^{+iz}-e^{-iz}}{(z+1)^2+1}\,\mathrm{d}z \\ &= \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{e^{+iz}}{(z+1)^2+1}\,\mathrm{d}z - \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{e^{-iz}}{(z+1)^2+1}\,\mathrm{d}z\,, \end{aligned} \tag{1.40}$$

where we have split our integral into two parts and, based on the Jordan's lemma, we will close the integration contour of the first one using upper semicircle and the contour of the second integral using lower semicircle. The residue theorem then leads us to

$$\int_{-\infty}^{+\infty} \frac{e^{+iz}}{(z+1)^2+1}\,\mathrm{d}z = +(2\pi i)\mathrm{Res}f(z_1) = +\pi e^{-1-i}\,, \tag{1.41}$$

$$\int_{-\infty}^{+\infty} \frac{e^{-iz}}{(z+1)^2+1}\,\mathrm{d}z = -(2\pi i)\mathrm{Res}f(z_2) = +\pi e^{-1+i}\,, \tag{1.42}$$

as $z_1 = (-1+i)$ lies in the upper plane and $z_2 = (-1-i)$ in the lower plane. **Note:** When using residue theorem for the second integral we inserted a minus sign since the contour was not positively oriented, i.e. when travelling along a positively oriented contour one always has the contour interior to the left, otherwise one needs to change the orientation of the contour which in our case amounts to a minus sign.
Putting all ingredients together we obtain

$$\begin{aligned} I_c &= \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{e^{+iz}}{(z+1)^2+1}\,\mathrm{d}z - \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{e^{-iz}}{(z+1)^2+1}\,\mathrm{d}z \\ &= \frac{1}{2i}\frac{\pi}{e}\left[e^{-i}-e^{+i}\right] \\ &= -\frac{\pi}{e}\sin(1) \end{aligned} \tag{1.43}$$

You can check the result, for example, in `Mathematica` by executing:

```
Integrate[Sin[x]/(x^2+2x+2),{x,-Infinity,+Infinity}]
```

Solution of exercise 2

(a) Inserting the ansatz into the differential equation yields

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x\sum_{n=0}^{\infty} a_n x^n = 0\,. \tag{2.3}$$

Expanding out the first terms yields

$$0 = x^0 a_1 + x^1(2a_2 + 2a_0) + x^2(3a_3 + 2a_1) + \dots\,, \tag{2.4}$$

which immediately tells us that $a_1 = 0$ (and of course $a_2 = -a_0$, $a_3 = -\frac{2}{3}a_1 = 0$). Next, we return to the general n expression. We want to rewrite Eq. (2.3) such that we can immediately read off the recurrence. Therefore, we shift the first series by $i = n - 1$ and the second series by $j = n + 1$ and find

$$0 = \sum_{i=0}^{\infty}(i+1)a_{i+1}x^i + \sum_{j=1}^{\infty} 2a_{j-1}x^j \quad = a_1 + \sum_{k=1}^{\infty} x^k((k+1)a_{k+1} + 2a_{k-1})\,. \tag{2.5}$$

In the last step we simply renamed the summation indices in both series to k and merged the sums. Since each power of x^k is linearly independent, we get the recurrence we are looking for (for $k \geq 1$)

$$(k+1)a_{k+1} + 2a_{k-1} = 0\,, \tag{2.6}$$

or after shifting once more $l = k + 1$ (with $l \geq 2$)

$$la_l + 2a_{l-2} = 0\,. \tag{2.7}$$

(b) Combining the recurrence Eq. (2.7) with our previous finding that $a_1 = 0$, we immediately see that all odd coefficients vanish. To make working with the recurrence a bit easier, we rewrite $l = 2m$ and $\tilde{a}_m = a_{2m}$ to get

$$2m\tilde{a}_m + 2\tilde{a}_{m-1} = 0\,. \tag{2.8}$$

It is easy to check that

$$\tilde{a}_m = a_0 \frac{(-1)^m}{m!} \tag{2.9}$$

fulfils this recurrence. Therefore, we get

$$y(x) = \sum_{l=0}^{\infty} a_l x^l = \sum_{m=0}^{\infty} \tilde{a}_m x^{2m} = a_0 \sum_{m=0}^{\infty} \frac{(-x^2)^m}{m!} = a_0 \exp(-x^2)\,, \tag{2.10}$$

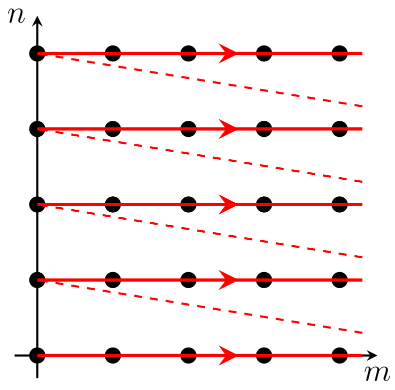
which we can easily verify fulfils the differential equation.

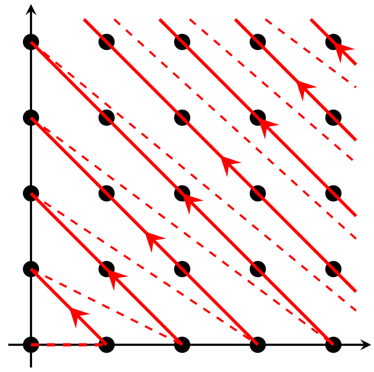
Solution of exercise 3

We start by realising that simply multiplying the two series term-wise, i.e.,

$$A(x)B(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n x^n b_m x^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m x^{n+m} \tag{3.3}$$

corresponds to summing first over columns (m running from 0 to ∞ indexes the columns) and then summing over rows (n running from 0 to ∞ indexes the rows) in the diagram shown on the left below. However we can also sum diagonally: Then n running from 0 to ∞ indexes the diagonals while m running from 0 to n indexes the element of the diagonal – as shown in the diagram on the right.





Then we need to sum

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_m b_{n-m} x^n \tag{3.4}$$

and therefore, the expression we are looking for is

$$c_n = \sum_{m=0}^{n} a_m b_{n-m} \,. \tag{3.5}$$

Solution of exercise 4

We can easily calculate the expansion of

$$f(x) = \frac{\mathrm{e}^{-cx}}{a+bx} = \frac{1}{a}\mathrm{e}^{-cx}\frac{1}{1-\frac{-bx}{a}} \tag{4.3}$$

without calculating lots of derivatives by combining the expansions of the exponential function and the geometric series

$$\mathrm{e}^{-cx} = \sum_{n=0}^{\infty}\frac{(-c)^n}{n!}x^n \,, \qquad \frac{1}{1-\frac{-bx}{a}} = \sum_{n=0}^{\infty}\left(-\frac{b}{a}\right)^n x^n \,. \tag{4.4}$$

Combining this with the Cauchy product derived in the previous exercise yields

$$f(x) = \frac{1}{a}\sum_{n=0}^{\infty} x^n \sum_{m=0}^{n}\frac{(-c)^m}{m!}\frac{(-b)^{n-m}}{a^{n-m}} = \sum_{n=0}^{\infty}(-x)^n\sum_{m=0}^{n}\frac{c^m b^{n-m}}{m! a^{n-m+1}} \tag{4.5}$$

Thus, expanding $f(x)$ through $\mathcal{O}(x^3)$ yields

$$\begin{aligned} f(x) = \frac{1}{a} - x\frac{ac+b}{a^2} + x^2\frac{a^2c^2+2abc+2b^2}{2a^3} \\ - x^3\frac{a^3c^3+3a^2bc^2+6ab^2c+6b^3}{6a^4} + \mathcal{O}(x^4)\,. \end{aligned} \tag{4.6}$$

Equating coefficients with the given expansion

$$f(x) = 4 + 2x + x^2 + \frac{x^3}{6} + \mathcal{O}(x^4) \tag{4.7}$$

yields

$$4 = \frac{1}{a} \qquad \Rightarrow a = \frac{1}{4} \tag{4.8}$$

$$2 = -16b - 4c \qquad \Rightarrow b = -\frac{1}{8} - \frac{c}{4} \tag{4.9}$$

$$1 = 1 + 2c + 2c^2 \qquad \Rightarrow c = \begin{cases} -1 \\ 0 \end{cases} \tag{4.10}$$

Inserting both solutions for c into the coefficient of x^3 yields

$$c = 0 \qquad \Rightarrow [x^3]f(x) = \frac{1}{2}\,, \tag{4.11}$$

$$c = -1 \qquad \Rightarrow [x^3]f(x) = \frac{1}{6}\,. \tag{4.12}$$

Therefore, the solution is

$$a = \frac{1}{4} \qquad b = \frac{1}{8} \qquad c = -1\,, \tag{4.13}$$

which yields

$$f(x) = 4 + 2x + x^2 + \frac{x^3}{6} + \frac{x^4}{12} + \mathcal{O}(x^5) \tag{4.14}$$