Problem 1 (4 Points)

K^+ mesons are produced in a height of 10 km above ground. The half-life time of K^+ mesons is $\tau_{1/2} = 8.6 \cdot 10^{-9}$ s.

a) Calculate the minimum velocity of the K^+ mesons, for which at least half of the particles are expected to reach the ground.
b) What is the time of flight t_E of the K^+ mesons in the reference frame of the earth?
c) How is the time of flight t_E related to the half-life time $\tau_{1/2}$ of the K^+ mesons?

a) The velocity of the K^+ mesons should be without relativity at least

$$v = \frac{10 \cdot 10^3\,\mathrm{m}}{8.6 \cdot 10^{-9}\,\mathrm{s}} = 1.16 \cdot 10^{12}\,\mathrm{m/s} >> c.$$

Relativity is necessary to understand why K^+ mesons reach ground. The condition for the time of flight is

$$t = \frac{\ell(v)}{v} < \tau_{1/2},$$

Therefore due to length contraction

$$t = \frac{\ell_0\sqrt{1-(v/c)^2}}{v} < \tau_{1/2},$$

and

$$1-(v/c)^2 < \left(\frac{v\tau_{1/2}}{\ell_0}\right)^2 = (\mathrm{v/c^2})\left(\frac{c\tau_{1/2}}{\ell_0}\right)^2$$

and

$$\left(\frac{v}{c}\right) > \frac{1}{\sqrt{1+\left(\frac{c\tau_{1/2}}{\ell_0}\right)^2}},$$

$$\mathrm{v} > \mathrm{c}\left(1-\frac{1}{2}\left(\frac{3 \cdot 10^8\,\mathrm{m/s} \cdot 8.6 \cdot 10^{-9}\,\mathrm{s}}{10^4\,\mathrm{m}}\right)^2\right) = \mathrm{c}(1-3.125 \cdot 10^{-8}).$$

Remark: It would be better to ask for the minimum energy of the K^+ *mesons. The problem avoids this additional complication:*

$$E = \frac{m_0c^2}{\sqrt{1-(v/c)^2}} > m_0c^2/(c\tau_{1/2}/\ell_0) = 3{,}88 \cdot 10^3\, m_0c^2.$$

b) The time of flight in the frame of the earth is

$$t_E = \frac{10\,\mathrm{km}}{c} = \frac{10^4\,\mathrm{m}}{3 \cdot 10^8\,\mathrm{m/s}} = 33.3\,\mu\mathrm{s}.$$

c) In the reference frame of the earth is the time of flight smaller than the half-life time of the K^+ mesons, which is prolonged due to the effect of time dilation

$$t_E \le \tau_{1/2}(v) = \tau_{1/2}\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

Problem 2 (4 Points)

The kinetic energy of cold neutrons is 0.1 meV.

a) Calculate the speed of these neutrons.
b) Calculate the de Broglie wavelength of these neutrons.
c) Spherical viruses form a crystal with the lattice constant $d = 1.0 \cdot 10^{-8}$ m. Calculate for these neutrons the angle under which the 1st order diffraction maximum can be observed with respect to the incident neutron beam.
d) Calculate the length of the corresponding vector of the reciprocal lattice.

a) The speed of the neutrons is

$$E_{\text{kin}} = \frac{1}{2} m_{\text{n}} c^2 \left(\frac{v}{c}\right)^2 \to v = c\sqrt{\frac{2E_{\text{kin}}}{m_{\text{n}}c^2}} = 3 \cdot 10^8\,\text{m/s}\sqrt{\frac{0.2 \cdot 10^{-3}\,\text{eV}}{939 \cdot 10^6\,\text{eV}}} = 138.4\,\text{m/s}$$

b) The de Broglie wave length is

$$\lambda = \frac{h}{p} = \frac{h}{mc^2\left(\frac{v}{c^2}\right)} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,(3 \cdot 10^8\,\text{m/s})^2}{939 \cdot 10^6\,\text{eV}\,138.4\,\text{m/s}} = 2.87 \cdot 10^{-9}\,\text{m}.$$

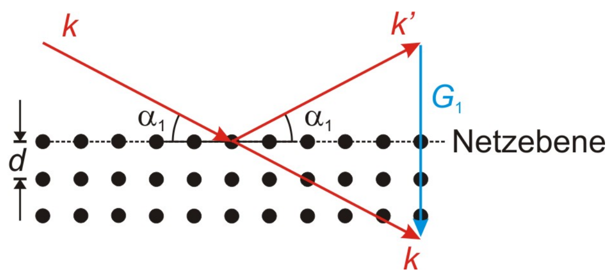
c) With Bragg's law

$$\lambda = 2d \sin\alpha_1 \to \sin\alpha_1 = \frac{\lambda}{2\text{d}} = \frac{2.87 \cdot 10^{-9}\,\text{m}}{10^{-8}\,\text{m}} = 0.287 \to \alpha_1 = 16.7°$$

and the angle with respect to the incident beam is $2\alpha_1 = 33.4°$.

d) The length of the reciprocal lattice vector is (compare the following sketch, $k = k' = 2\pi/\lambda$)

$$G_1 = \frac{2\pi}{d} = \frac{2\pi}{10^{-8}\,\text{m}} = 2\pi \cdot 10^8\,\text{m}^{-1}.$$



Problem 3 (4 Points)

During the decay $^{22}Na \rightarrow {}^{22}Ne$ one photon with the energy 1.28 MeV is emitted.

a) Calculate the wavelength of the corresponding electromagnetic wave.
b) Assume that the photons are scattered at electrons at rest. Calculate the energies of the scattered photons.
c) Calculate i) the smallest photon energy and ii) the highest kinetic energy of the electron after scattering.

a) The wavelength for 1.28 MeV is

$$E_\lambda = \frac{hc}{\lambda} \rightarrow \lambda = \frac{hc}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\mathrm{eVs}\, 3 \cdot 10^8\,\mathrm{m/s}}{1.28 \cdot 10^6\,\mathrm{eV}} = 9,7 \cdot 10^{-13}\,\mathrm{m}$$

b) The formula for the Compton scattering is

$$\lambda' - \lambda = \lambda_\mathrm{C}(1 - \cos\theta) \rightarrow \lambda' = \lambda + \lambda_\mathrm{C}(1 - \cos\theta).$$

λ denotes the wavelength of the incident wave, λ' the wavelength of the scattered wave and θ the scattering angle with respect to the incident beam. The energy of the scattered photon is then

$$E_{\lambda'} = \frac{hc}{\lambda'} = \frac{hc}{\lambda + \lambda_\mathrm{C}(1 - \cos\theta)} \rightarrow E_{\lambda'}^{-1} = E_\lambda^{-1} + \mathrm{E}_\mathrm{C}^{-1}(1 - \cos\theta).$$

Thereby $\mathrm{E_C}$ denotes

$$E_\mathrm{C} = hc/\lambda_\mathrm{C} = 4.14 \cdot 10^{-15}\,\mathrm{eVs}\, 3 \cdot 10^8\,\mathrm{ms^{-1}}/2.426 \cdot 10^{-12}\mathrm{m} = 5.1 \cdot 10^5\,\mathrm{eV}.$$

(This is the rest energy of the electron)

c) i) The smallest photon energy results from back-scattering i.e. $\theta = 180°$.
Therefore

$$E_\mathrm{min}^{-1} = E_\lambda^{-1} + 2\mathrm{E}_\mathrm{C}^{-1}$$

$$E_\mathrm{min} = \frac{1}{E_\lambda^{-1} + 2\mathrm{E}_\mathrm{C}^{-1}} = \frac{\mathrm{MeV}}{1.28^{-1} + 2 \cdot 0.51^{-1}} = 0.21\,\mathrm{MeV}.$$

ii) The highest kinetic energy of the scattered electron is then

$$E_\mathrm{max} = 1.28\,\mathrm{MeV} - 0.21\,\mathrm{MeV} = 1.07\,\mathrm{MeV}.$$

Problem 4 (4 Points)

On a hot day (30° C) a black car is exposed to the plain sun. The intensity of the sun is 700 W/m^2.

a) What is the temperature of the car in thermal equilibrium when only half of its surface is exposed to the sun? Assume that the car behaves like a black body and neglect the effect of thermal conduction.
b) Sketch the radiation spectrum of a black body.
c) How large is the wavelength of the maximum of the spectrum for the car in thermal equilibrium?
d) Repeat the calculation for a car which reflects 30 % of the incident radiation. How large is the equilibrium temperature of this car?

a) In equilibrium one can write with the Stefan-Boltzmann law

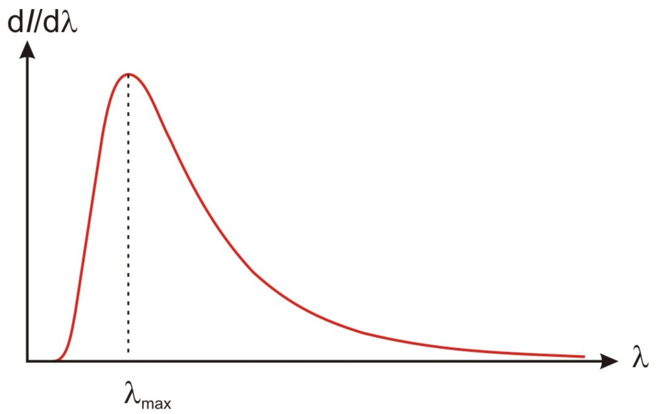
$$700\,\mathrm{Wm}^{-2}\frac{A}{2} = \mathrm{A}\sigma(T_\mathrm{C}^4 - T_0^4).$$

Thereby T_0 denotes the temperature of the environment and T_C the equilibrium temperature of the car

$$T_\mathrm{C} = \left(\frac{350\,\mathrm{Wm}^{-2}}{\sigma} + T_0^4\right)^{1/4} = \left(\frac{350\,\mathrm{Wm}^{-2}}{5.67\cdot 10^{-8}\mathrm{Wm}^{-2}\mathrm{K}^{-4}} + (273\,\mathrm{K} + 30\,\mathrm{K})^4\right)^{1/4}$$
$$= 347.6\,\mathrm{K}$$

This is a temperature of 74.6° C.

b) The radiation spectrum of a black body is



$I = P/A$ denotes the intensity of the radiation.

c) With Wien's law λ_max becomes

$$\lambda_\mathrm{max} = \frac{b}{T} = \frac{2.9\,\mathrm{mmK}}{347.6\,\mathrm{K}} = 8.3\cdot 10^{-6}\,\mathrm{m}.$$

d) With problem a)

$$T_\mathrm{C} = \left(\frac{0.7\cdot 350\,\mathrm{Wm}^{-2}}{5.67\cdot 10^{-8}\mathrm{Wm}^{-2}\mathrm{K}^{-4}} + (273\,\mathrm{K} + 30\,\mathrm{K})^4\right)^{1/4} = 336\,\mathrm{K}$$

the equilibrium temperature is 63° C.

Problem 5 (4 Points)

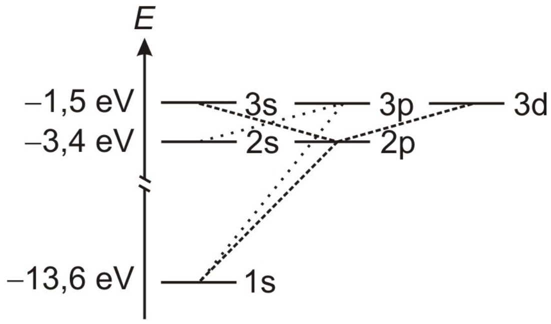
a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.
d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

a) The energy is with the Rydberg unit of energy

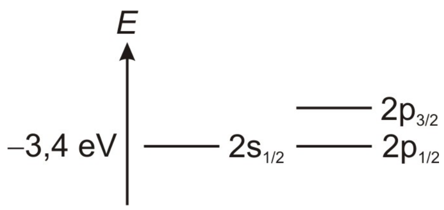
$$E_\mathrm{n} = -\frac{R}{\mathrm{n}^2},$$

i.e. $E_1 = -13.6\,\mathrm{eV}$, $E_2 = -3.4\,\mathrm{eV}$ and $E_3 = -1.5\,\mathrm{eV}$.

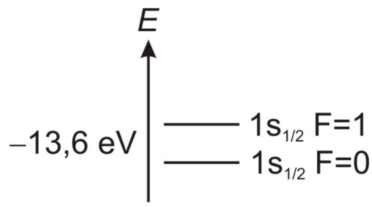
b) The energy level scheme and the allowed electric dipole transitions (dashed lines). The selection rule is $\Delta\ell = \pm 1$.



c) The energy level scheme of the first excited state. The p-orbital is splitted due to the spin-orbit-coupling into a state with the total angular momentum $j = 1/2$ and a state with $j = 3/2$.



d) The ground state of the hydrogen atom is splitted due to the hyperfine interaction with the proton. There are two states with the total angular momentum $F = 0$ and $F = 1$.

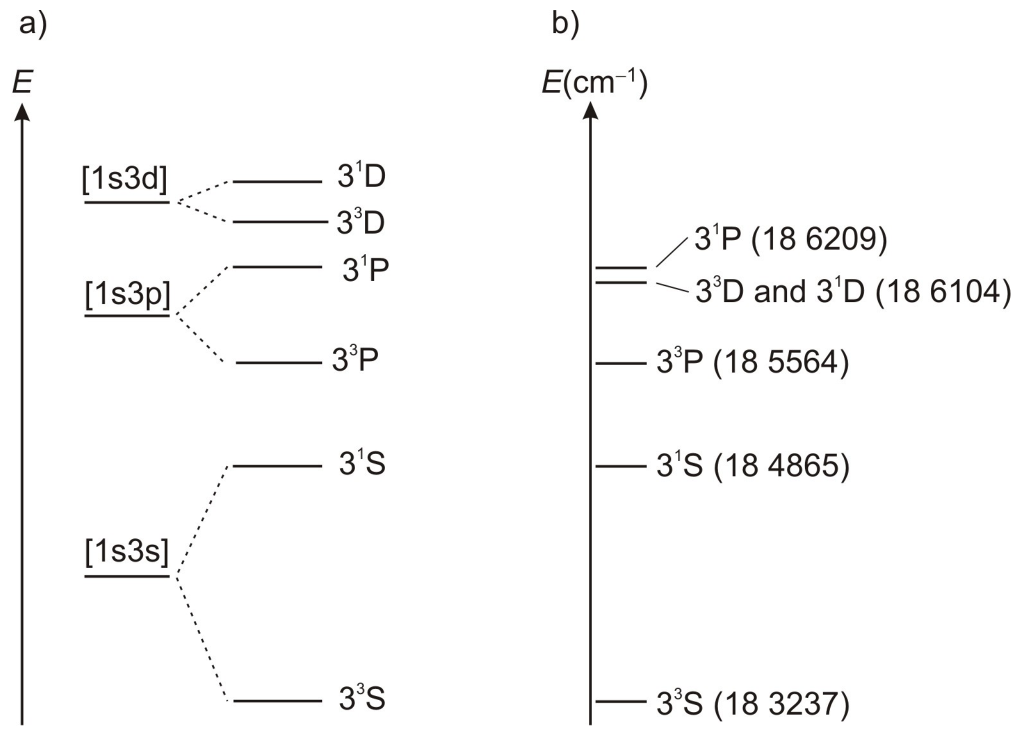


Problem 6 (4 Points)

There are two electrons in the neutral Helium atom. The excited states with lowest energy result from the excitation of one electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the expected energy level scheme and denote the energy levels, when the excited electron is lifted into a state with the quantum number $n = 3$. Ignore spin-orbit coupling.
b) Give reasons for the energy difference between states of different angular momenta L.
c) Explain the energy difference between spin singlet and triplet states.
d) Give the states of problem a), which can be excited from the groundstate by electric dipole transitions.

a) The following figure a) shows the sketch of the expected energy level scheme[1].



[1] Figure b) shows the experimental results. The exchange interaction is more effective than the screening of the nuclear charge. Without exchange is the energy of the [1s3d] configuration only slightly higher than the [1s3p] configuration. The exchange interaction of the [1s3d] configuration is very small.

b) The mean distance between the electron cloud and the nucleus increases with increasing orbital angular momentum. Therefore the screening of the nuclear charge by the 1s electron is more effective for higher quantum numbers of the angular momentum of the excited electron and the binding energy is reduced.

c) In the singlet state the electron-electron repulsion is stronger than in the triplet state, since the two electrons occupy the same orbital and the mean electron-electron distance is smaller than in the triplet state. Therefore the binding energy of the singlet state is reduced. The effect is known as exchange interaction.

d) The selection rules for electric dipole transitions are $\Delta S = 0$ and $\Delta \ell = \pm 1$. Therefore only the transition 1^1S $\leftrightarrow$ 3^1P is possible by electric dipole radiation.

Problem 7 (4 Points)

a) The vectors of a primitive unit cell of the bcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x} - \vec{y} + \vec{z})$, $\vec{a}_2 = \frac{a}{2}(\vec{y} - \vec{z} + \vec{x})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z} - \vec{x} + \vec{y})$. $\vec{x}$, $\vec{y}$ and $\vec{z}$ denote orthogonal unit vectors. What is the volume of the unit cell?

b) Give the primitive vectors of the corresponding reciprocal lattice.

c) The vectors of a primitive unit cell of the fcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x} + \vec{y})$, $\vec{a}_2 = \frac{a}{2}(\vec{y} + \vec{z})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z} + \vec{x})$. What is the volume of the unit cell?

d) Give the primitive vectors of the corresponding reciprocal lattice.

a) There are two lattice points in the simple cubic unit cell. Therefore the volume of the primitive unit cell of the bcc lattice is

$$V_{\text{Cell}} = \frac{a^3}{2}.$$

Alternatively one can calculate $\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \frac{a^3}{2}$.

b) The primitive vectors of the reciprocal lattice are

$$\vec{b}_1 = \frac{2\pi}{V_{\text{Cell}}} \vec{a}_2 \times \vec{a}_3,$$

i.e.

$$\vec{b}_1 = \frac{4\pi}{4a} \left[(\vec{y} - \vec{z} + \vec{x}) \times (\vec{z} - \vec{x} + \vec{y}) \right] = \frac{\pi}{a} (\vec{x} + \vec{z} + \vec{y} + \vec{x} - \vec{y} + \vec{z}) = \frac{2\pi}{a} (\vec{x} + \vec{z})$$

By cyclic permutation one gets

$$\vec{b}_2 = \frac{2\pi}{a} (\vec{y} + \vec{x})$$

and

$$\vec{b}_3 = \frac{2\pi}{a} (\vec{z} + \vec{y})$$

c) There are four lattice points in the simple cubic unit cell. Therefore the volume of the primitive cell of the fcc lattice is

$$V_{\text{Cell}} = \frac{a^3}{4}.$$

Alternatively one can calculate $\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \frac{a^3}{4}$.

d) The primitive vectors of the reciprocal lattice are

$$\begin{aligned}\vec{b}_1 &= \frac{2\pi}{V_{\text{Cell}}}\vec{a}_2 \times \vec{a}_3 \\ &= \frac{2\pi}{a}\left[(\vec{y}+\vec{z}) \times (\vec{z}+\vec{x})\right] \\ &= \frac{2\pi}{a}\left(\vec{x}-\vec{z}+\vec{y}\right)\end{aligned}$$

and by cyclic permutation

$$\begin{aligned}\vec{b}_2 &= \frac{2\pi}{a}\left(\vec{y}-\vec{x}+\vec{z}\right) \\ \vec{b}_3 &= \frac{2\pi}{a}\left(\vec{z}-\vec{y}+\vec{x}\right).\end{aligned}$$

(Conclusion: The reciprocal lattice of the bcc lattice is a fcc lattice and vice versa.)

Problem 8 (4 Points)

a) Calculate the energy of a free electron described by a simple plane wave in a region of constant potential energy V.
b) Calculate the density of conduction electrons of potassium. Hint: The mass of the potassium atom is $39u$ and there is one conduction electron per atom.
c) Calculate the Fermi wavenumber k_F of potassium.
d) Calculate the Fermi energy of potassium.

a) With the Schrödinger equation is the energy of a plane wave

$$\psi = \psi_0 \mathrm{e}^{\mathrm{i}(\vec{k}\vec{r}-\omega t)}$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\nabla^2}{2m_\mathrm{e}}\psi + V\psi \rightarrow E = \hbar\omega(\vec{k}) = \frac{\hbar^2\vec{k}^2}{2m_\mathrm{e}} + V$$

b) With the number of atoms N and the density of potassium one gets

$$\rho_\mathrm{K} = \frac{m}{V} = \frac{N \cdot 39u}{V}$$

and the electron density is

$$\frac{N}{V} = \frac{\rho_\mathrm{K}}{39u} = \frac{0.86 \cdot 10^{-3}\,\mathrm{kg/cm^3}}{39 \cdot 1.66 \cdot 10^{-27}\,\mathrm{kg}} = 1.33 \cdot 10^{22}\,1/\mathrm{cm}^3 = 1.33 \cdot 10^{28}\,1/\mathrm{m}^3.$$

c) The Fermi wavenumber results from

$$\frac{N}{2} = \frac{4\pi k_\mathrm{F}^3 \mathrm{V}}{3(2\pi)^3} \rightarrow k_\mathrm{F} = \left(3\pi^2\frac{N}{V}\right)^{1/3} = \left(3\pi^2 1.33 \cdot 10^{28}\,1/\mathrm{m}^3\right)^{1/3} = 7.3 \cdot 10^9\,\mathrm{m}^{-1}.$$

d) The Fermi energy is

$$E_\mathrm{F} = \frac{\hbar^2 k_\mathrm{F}^2}{2m_\mathrm{e}} = \frac{h^2 k_\mathrm{F}^2}{8\pi^2 m_\mathrm{e}} = \frac{(4.14 \cdot 10^{-15}\,\mathrm{eVs})^2(7.3 \cdot 10^9\,\mathrm{m}^{-1})^2(3 \cdot 10^8)^2\,\mathrm{m^2s^{-2}}}{8\pi^2 \cdot 500 \cdot 10^3\,\mathrm{eV}}$$
$$= 2.1\,\mathrm{eV}.$$