Written Examination in Physics **Spring 2023**
Modern Physics (KSOP)

Dear Students,

if you would like to see your exam, please send me an email by April 06 at the latest (Bernd.Pilawa@kit.edu) together with a copy of your KIT identity card. I will then send you a pdf scan of your exam. If you wish, I will check the correction of your exam and then determine the final grade.

With best regards,
Bernd Pilawa

Problem 1 (4 Points)

a) What energy is required to accelerated an electron to 90 % of the speed of light?
b) What is the momentum of the electron after it has been accelerated to 90 % of the speed of light?
c) An observer moves at 50 % of the speed of light perpendicular to the motion of the electron.
   i) What is the momentum of the electron perpendicular to the velocity of the observer?
   ii) What is the energy of the electron in the observer's frame of reference?

a)

$$E = mc^2 = \frac{m_e c^2}{\sqrt{1-0.9^2}} = 2.29 m_e c^2$$

The energy to accelerate is

$$E_{\text{beschl.}} = E - m_e c^2 = 1.29 m_e c^2 = 1.29 \cdot 500\,\text{keV} = 645\,\text{keV}$$

b) momentum of the electron

$$\vec{p} = m\vec{v} = \frac{m_e \vec{v}}{\sqrt{1-0.9^2}} \quad \to \quad p = 2.29 \cdot 9.1 \cdot 10^{-31}\,\text{kg}\, 0.9 \cdot 3 \cdot 10^8\,\text{ms}^{-1}$$

$$= 56.3 \cdot 10^{-23}\,\text{kgms}^{-1} = 1053\,\text{keV}/c$$

c) Lorentz transformation for the coordinates (Invariant: $(ct)^2 + \vec{r}^2 = s^2$)

$$x' = \gamma(\text{x} - \frac{\text{v}_{\text{obs.}}}{\text{c}} ct), \quad y' = \text{y}, \quad z' = \text{z}, \quad ct' = \gamma(ct - \frac{\text{v}_{\text{obs.}}}{\text{c}} \text{x})$$

Lorentz transformation for the components of the momentum
(Invariant: $(\frac{E}{c})^2 + \vec{p}^{\,2} = m_e^2$)

$$p_x' = \gamma(p_x - \frac{\text{v}_{\text{obs.}}}{\text{c}^2} E), \quad p_y' = p_y, \quad p_z' = p_z, \quad \frac{E'}{c} = \gamma(\frac{E}{c} - \frac{\text{v}_{\text{obs.}}}{\text{c}} p_x)$$

and $\gamma = \frac{1}{\sqrt{1-(\frac{v_{\text{obs.}}}{c})^2}}$.

i) The momentum perpendicular to the velocity of the observer does not change

$$p'_{\text{y}} = p_{\text{y}} \quad \text{and} \quad p'_{\text{z}} = p_{\text{z}} \quad \rightarrow \quad p_{\perp} = 56{,}3 \cdot 10^{-23}\,\text{kgms}^{-1}$$

ii) The energy of the electron in the frame of reference of the observer is with $p_{\text{x}} = 0$

$$E' = \gamma E = \frac{E}{\sqrt{1 - v_{\text{obs.}}^2/c^2}} = \frac{E}{\sqrt{1-0.5^2}} = 1.15 \cdot E = 1.15 \cdot 2.29 m_{\text{e}} c^2$$

$$E' = 2.63 \cdot m_{\text{e}} c^2 = 2.63 \cdot 500\,\text{keV} = 1.3\,\text{MeV}$$

Alternatively, the velocity of the electron in the observer's reference system can be calculated:
With the Lorentz transformation

$$x' = \gamma(x - v_{\text{obs.}} \cdot t), \quad t' = \gamma(t - v_{\text{obs.}} \cdot x/c^2), \quad y' = y, \quad z' = z$$

with $\gamma = 1/\sqrt{1 - v_{\text{obs.}}^2/c^2}$ and the speed of the observer $v_{\text{obs.}} = 0{,}5\,c$.
With the velocity of the electron in the rest frame $v_{\text{x}} = 0$ $v_{\text{y}} = 0.9c$ is the velocity in the frame of the observer

$$v'_{\text{x}} = \frac{x'}{t'} = \frac{x - v_{\text{obs.}} \cdot t}{t - v_{\text{obs.}} \cdot x/c^2} = \frac{v_{\text{x}} - v_{\text{obs.}}}{1 - v_{\text{obs.}} \cdot v_{\text{x}}/c^2} = -v_{\text{obs.}}$$

and

$$v'_{\text{y}} = \frac{y'}{t'} = \frac{y}{\gamma(t - v_{\text{obs.}} \cdot x/\text{c}^2)} = \frac{v_{\text{y}}}{\gamma(1 - v_{\text{obs.}} \cdot v_{\text{x}}/c^2)} = \frac{v_{\text{y}}}{\gamma}$$

The energy of the electron is

$$E = \frac{m_{\text{e}} c^2}{\sqrt{1 - v_{\text{e}}^2}} \quad \text{und} \quad v_{\text{e}}^2 = (v'_{\text{x}})^2 + (v'_{\text{y}})^2 = v_{\text{obs.}}^2 + (v_{\text{y}})^2(1 - v_{\text{obs.}}^2/\text{c}^2)$$

$$E = m_{\text{e}} c^2 \frac{1}{\sqrt{1 - v_{\text{obs.}}^2 - (v_{\text{y}})^2(1 - v_{\text{obs.}}^2/c^2)}} = m_{\text{e}} c^2 \frac{1}{\sqrt{(1 - v_{\text{obs.}}^2)(1 - (v_{\text{y}})^2)}}$$

and

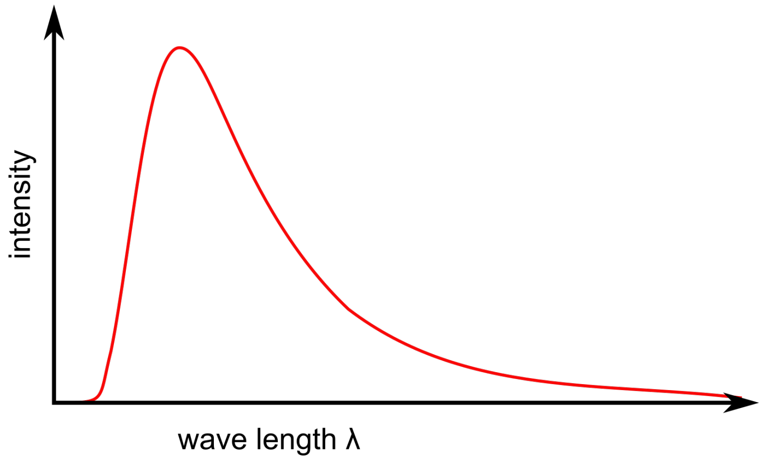
$$E = m_{\text{e}} c^2\, 1.15 \cdot 2.29$$

Problem 2 (4 Points)

a) Ideal thermal radiation is often referred to as blackbody radiation. What characterizes an ideal thermal radiation source?
b) Sketch the radiation spectrum of a blackbody as a function of wavelength.
c) How does the temperature of the radiation source affect the radiation spectrum?
d) ^{210}Po decays into lead by an α-decay and represents a heat source with the thermal power per kg of $P_{\text{decay}}/m = 141\,\text{W}/\text{kg}$. What is the equilibrium temperature on the surface of a polonium sphere (radius 1 cm) in an environment at room temperature $T_0 = 300$ K if the sphere is only cooled by thermal radiation? The density of polonium is $\rho = 9{,}2 \cdot 10^3\,\text{kgm}^{-3}$.

a) blackbody radiation
   i) The electromagnetic radiation is in thermal equilibrium with the rest of matter.
   ii) The surface does not affect the radiation, i.e. electromagnetic radiation can leave or enter the thermal radiation source unhindered.

b) Spectrum of ideal thermal radiation



c) The wavelength of the maximum λ_{max} is inversely proportional to the temperature of the radiation source, i.e. $\lambda_{\text{max}} \propto 1/T$ (Wien's displacement law).
d) In thermal equilibrium,

$$P_{\text{heating}} + A\sigma T_0^4 = A\sigma T^4 \quad \text{with} \quad P_{\text{heating}} = (P_{\text{decay}}/m)\,\rho\,\frac{4\pi r^3}{3} \quad \text{and} \quad A = 4\pi r^2$$

surface temperature

$$T = (\frac{P_{\text{heating}}}{\sigma A} + {T_0}^4)^{(1/4)} = (\frac{141\,\text{Wkg}^{-1}\,9.2 \cdot 10^3\,\text{kgm}^{-3}\,4\pi\,10^{-2}\,\text{m}}{5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}\,3 \cdot 4\pi} + 300^4\,\text{K}^4)^{1/4}$$

$$T = (763 \cdot 10^8\,\text{K}^4 + 300^4\,\text{K}^4)^{1/4} = (763 + 81)^{1/4} \cdot 100\,\text{K} = 539\,\text{K}$$

Problem 3 (4 Points)

a) How is the orbital angular momentum defined in Newtonian mechanics and how does the orbital angular momentum of classical physics differ from the orbital angular momentum in quantum physics?
b) Write down the orbital angular momentum operator of quantum physics and give the eigenvalue equations for the orbital angular momentum.
c) Which values can the eigenvalues of the orbital angular momentum have?
d) A mass *m* can rotate at a distance *r* around a fixed point in space. Write down the Schrödinger equation of the mass *m* and give the energy eigenvalues of this mass.

a) Newton's definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p}.$$

Here, $\vec{r}$ denotes the position vector and $\vec{p}$ the momentum of a particle. With

Newton's momentum, all three components of the vector can be assigned numerical values. In the case of angular momentum in quantum physics, only one component, commonly referred to as the z-component, can be assigned a fixed value. In addition, in quantum physics, the length of the angular momentum vector can also be assigned a fixed value.

b) With the momentum operator $\hat{\vec{p}} = -i\hbar\nabla$ one can write for the angular momentum operator

$$\hat{\vec{L}} = -i\,\hbar\vec{r} \times \nabla$$

The eigenvalue equations are

$$\begin{aligned}\hat{L}_z Y_{\ell,m} &= m\hbar Y_{\ell,m} \\ \hat{\vec{L}}^2 Y_{\ell,m} &= \ell(\ell+1)\hbar^2 Y_{\ell,m}\end{aligned}$$

The eigenfunctions $Y_{\ell,m} = Y_{\ell,m}(\theta,\varphi)$ are spherical harmonics.

c) The angular momentum quantum number ℓ can assume the values 0, 1, 2 and so on. The angular momentum quantum number m varies in steps of 1 in the interval $-\ell \leq m \leq +\ell$.

d) The Schrödinger equation of a rotating mass is

$$E_\ell Y_{\ell,m}(\theta,\varphi) = \frac{\hat{\vec{L}}^2}{2mr^2} Y_{\ell,m}(\theta,\varphi)$$

The energy eigenvalues are

$$E_\ell = \frac{\ell(\ell+1)\hbar^2}{2mr^2}$$

Problem 4 (4 Points)

The electron has an intrinsic angular momentum: the spin.

a) Give the spin operator of the electron.
b) Since the eigenfunctions of the spin operator are not wave functions, Dirac introduced a generalized notation of the quantum states. Explain the Dirac notation and use it to write down the eigenvalue equations of the spin.
c) Which values can the eigenvalues of the electron spin have?
d) The spin of the electron is linked to a magnetic moment. Calculate the change in energy of the electron when a magnetic field of strength $B_0 = 1\,\mathrm{T}$ is applied.

a) Spin operator of the electron

$$\hat{\vec{s}} = \frac{\hbar}{2}\begin{pmatrix}\sigma_\mathrm{x}\\ \sigma_\mathrm{y}\\ \sigma_\mathrm{z}\end{pmatrix}$$

Here $\sigma_{\mathrm{x,\,y,\,z}}$ denote the Pauli matrices.

b) Dirac introduced a special bracket symbol $|\,...\,\rangle$ to designate quantum states, in which the quantum numbers are written that determine a quantum state, e.g.

$$|\text{quantum number}_1,\ \text{quantum number}_2,\ \text{quantum number}_3,\ ...\rangle$$

Eigenvalue equations of the spin

$$\begin{aligned}\hat{s}_\mathrm{z}\,|s,\,m_\mathrm{s}\rangle &= m_\mathrm{s}\hbar\,|s,\,m_\mathrm{s}\rangle\\ \hat{\vec{s}}^{\,2}\,|s,\,m_\mathrm{s}\rangle &= s(s+1)\hbar^2\,|s,\,m_\mathrm{s}\rangle\end{aligned}$$

c) The eigenvalues of the electron spin are $s = 1/2$ and $m_\mathrm{s} = \pm 1/2$.

d) Zeeman splitting

$$\Delta E_{m_\mathrm{s}} = \pm g\mu_\mathrm{B}B_0 m_\mathrm{s}$$

with $g = 2$ (in a very good approximation) results

$$\begin{aligned}\Delta E_{m_\mathrm{s}=\pm 1/2} = \pm\mu_\mathrm{B}B_0 &= \pm 9.274\cdot 10^{-24}\,\mathrm{Am^2}\cdot 1\,\mathrm{Vsm^{-2}}\\ &= \pm 9.274\cdot 10^{-24}\,\mathrm{Ws}\\ &= \pm 5.8\cdot 10^{-5}\,\mathrm{eV}.\end{aligned}$$

Problem 5 (4 Points)

The valence electron of sodium is a 3s electron.

a) The yellow sodium line corresponds to the 3s $\leftrightarrow$ 3p transition. Explain why the yellow sodium line splits into two components and write down the transitions that correspond to the two components.
b) Which component has the smaller energy?
c) Explain why in a magnetic field one of the two components splits into four and the other into six spectral lines. Give the transitions corresponding to these spectral lines.

a) Spin-orbit coupling: in the 3p state, the magnetic moment of the electron aligns in the magnetic field caused by the orbital motion of the electron. Two settings are possible in which the orbital angular momentum with the quantum number $\ell = 1$ couple with the spin of the electron to become the total angular momentum with the quantum numbers $j = 1/2$ or $j = 3/2$.
The transitions are

$$3s_{1/2} \leftrightarrow 3p_{1/2}$$
$$3s_{1/2} \leftrightarrow 3p_{3/2}$$

b) The transition $3s_{1/2} \leftrightarrow 3p_{1/2}$ has the smaller energy, since the magnetic moments of orbit and spin are oriented antiparallel rather than parallel to each other (However, note that strictly parallel or antiparallel alignment is not possible for quantized angular momentum.).
c) The ground state splits into the two quantum states
$|3s_{1/2}, m_j = \pm 1/2\rangle$ in the magnetic field.
Likewise, the first excited state $3p_{1/2}$ splits into the two quantum states
$|3p_{1/2}, m_j = \pm 1/2\rangle$,
while the second excited state $3p_{3/2}$ splits into the four quantum states
$|3s_{3/2}, m_j = \pm 1/2, \pm 3/2\rangle$.
With the selection rule $\Delta m_j = 0, \pm 1$ and the fact that the splitting energy of the quantum states $3s_{1/2}$, $3p_{1/2}$ and $3s_{3/2}$ in the magnetic field is different, result the four transitions

$$|3s_{1/2}, \pm 1/2\rangle \leftrightarrow |3p_{1/2}, \pm 1/2\rangle$$
$$|3s_{1/2}, \pm 1/2\rangle \leftrightarrow |3p_{1/2}, \mp 1/2\rangle$$

and the six transitions

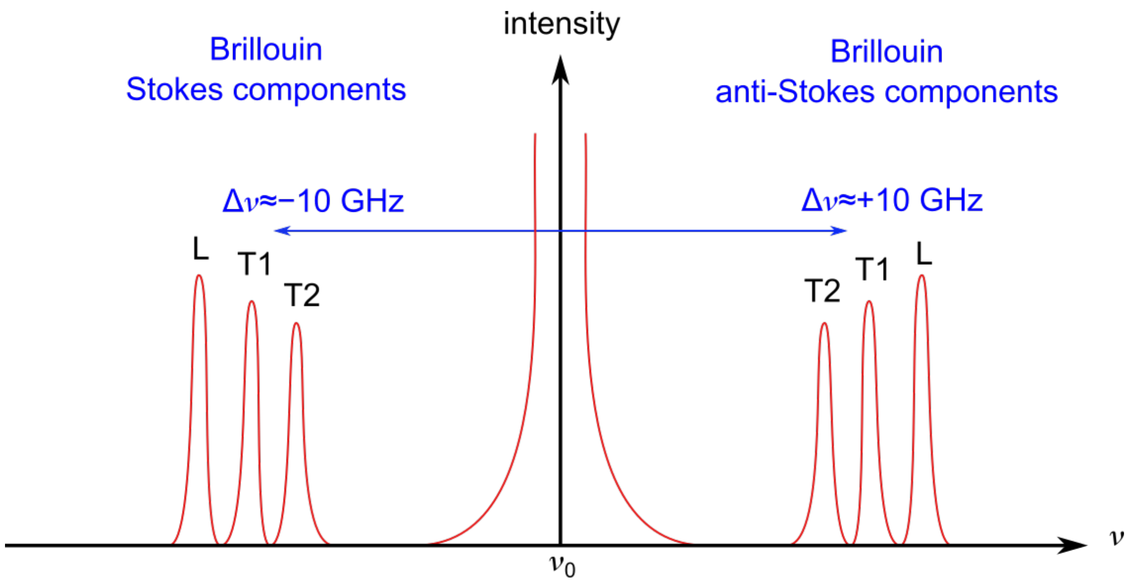
$$|3s_{1/2}, +1/2\rangle \leftrightarrow |3p_{3/2}, +3/2\rangle \quad \text{and} \quad |3s_{1/2}, -1/2\rangle \leftrightarrow |3p_{3/2}, -3/2\rangle$$
$$|3s_{1/2}, +1/2\rangle \leftrightarrow |3p_{3/2}, \pm 1/2\rangle \quad \text{and} \quad |3s_{1/2}, -1/2\rangle \leftrightarrow |3p_{3/2}, \pm 1/2\rangle\,.$$

Problem 6 (4 Points)

a) What is the difference between Rayleigh and Brillouin scattering?
b) Sketch a Brillouin spectrum, give the typical frequency range and label the Stokes and anti-Stokes components.
c) State the energy and momentum conservation laws relevant to phonon-phonon scattering.
d) What is umklapp scattering and what is the experimental evidence that umklapp scattering occurs?

a) Rayleigh scattering describes the elastic scattering of light due to density fluctuations of a material.
Brillouin scattering describes the absorption or emission of an acoustic phonon by a photon in a material.

b) Depending on the emission or absorption of a longitudinally or transversely polarized phonon, there are up to three lines. A phonon is emitted in the Stoke components, while a phonon is absorbed by a photon in the anti-Stokes components. The frequency shift of the Brillouin lines compared to the elastic scattering of the Rayleigh line is in the range of several GHz.



c) energy and momentum conservation laws

$$\hbar\omega(\vec{q}_1) + \hbar\omega(\vec{q}_2) = \hbar\omega(\vec{q}_3) \quad \text{and} \quad \vec{q}_1 + \vec{q}_2 + \vec{K} = \vec{q}_3$$

d) The momentum of a phonon is a crystal or quasi-momentum, i.e. a vector of the reciprocal lattice can be added. Only the shortest wave vector of a phonon determines the direction of propagation of the lattice wave. If the sum $\vec{q}_1 + \vec{q}_2$ can be reduced to a shorter q-vector by adding a reciprocal lattice vector, the resulting wave can propagate into the opposite direction of the original two waves. The corresponding scattering is referred to as umklapp scattering, since the energy of the two original waves can be transported in the opposite direction.

Umklapp scattering reduces the thermal conductivity of crystals at high temperatures.

Problem 7 (4 Points)

a) Explain the Sommerfeld theory of metals.
b) Calculate the Fermi wave number k_F in the Sommerfeld model.
c) Give the definition of the Fermi energy E_F.
d) Calculate the density of states $D(E)$ of the electrons in the Sommerfeld model.

a) In the Sommerfeld theory of metals, the electrons are described by plane waves that can propagate in all spatial directions. The wave number vectors are quantized according to the formula

$$\vec{k}_{n_1,n_2,n_3} = \frac{2\pi}{L}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

Here L denotes the edge length of a cube-shaped metal sample.
According to the Pauli principle, each quantum state that is determined by a wave number vector $\vec{k}_{n_1,n_2,n_3}$ can be occupied by two electrons.

b) The Fermi wave number is determined by the condition

$$N = 2 \cdot \frac{4\pi k_F^3/3}{(2\pi)^3/L^3}$$

Since all spatial directions are equal, all k-states within a sphere can be occupied by electrons up to the Fermi wave number. $4\pi k_F^3/3$ is the volume of a sphere, $(2\pi)^3/L^3$ is the volume of a k-state. The factor 2 takes into account that each k-state can be occupied by two electrons and N denotes the number of electrons. The Fermi wave number is only determined by the electron density

$$k_F = (3\pi^2 \frac{N}{L^3})^{1/3} = (3\pi^2 \frac{N}{V})^{1/3}.$$

c) The Fermi energy is given by the kinetic energy of the electrons with the Fermi wave number

$$E_F = \frac{\hbar^2 k_F^2}{2m_e}$$

d) The density of states of the electrons is

$$D(E) = \frac{1}{V}\frac{dN}{dE}.$$

Since all spatial directions are equal, the number of states dN at a certain energy is given by the number of k-states in a spherical shell with radius k and thickness dk

$$dN = 2 \cdot \frac{4\pi k^2 dk}{(2\pi)^3/V}.$$

With $E = \hbar^2 k^2/2m_\mathrm{e} \rightarrow k^2 = 2m_\mathrm{e}E/\hbar^2$ and $\mathrm{d}E = \hbar^2 k\mathrm{d}k/m_\mathrm{e} \rightarrow \mathrm{d}k = m_\mathrm{e}\mathrm{d}E/(\hbar^2 k)$

$$\mathrm{d}N = \frac{V}{\pi^2} \cdot \frac{2m_\mathrm{e}E}{\hbar^2} \cdot \frac{m_\mathrm{e}\mathrm{d}E}{\hbar^2 k} = \frac{V}{\pi^2} \cdot \frac{2m_\mathrm{e}E}{\hbar^2} \cdot \frac{m_\mathrm{e}\mathrm{d}E}{\hbar\sqrt{2m_\mathrm{e}E}}$$

and

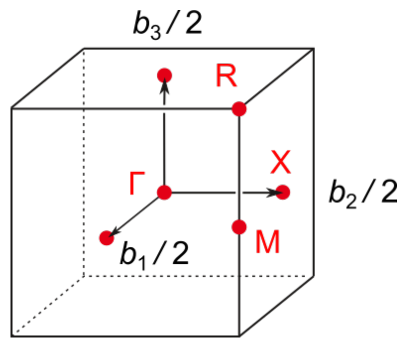
$$D(E) = \frac{1}{V}\frac{\mathrm{d}N}{\mathrm{d}E} = \frac{1}{\pi^2}\frac{\sqrt{2m_\mathrm{e}^3}}{\hbar^3}\sqrt{E}.$$

Problem 8 (4 Points)

a) Sketch the 1st Brillouin zone of a simple cubic lattice with the lattice constant a and mark the symmetry points.
b) Calculate the Fermi wave number for the case that there are 3 conduction electrons in the primitive cell of the simple cubic lattice.
c) Calculate the Fermi energy in units $E_0 = \frac{\hbar^2}{2m_e}(\frac{\pi}{a})^2$ and sketch the band structure of the quasi-free electron gas up to the Fermi energy along the ΓX and the ΓM direction in the reduced zone scheme.
d) Sketch the intersection of the Fermi surface of the 2nd energy band with the ΓXM plane in the reduced zone scheme.

a) With the basis vectors of the simple cubic lattice $\vec{a}_1 = a\vec{e}_x$, $\vec{a}_2 = a\vec{e}_y$ and $\vec{a}_3 = a\vec{e}_z$ are the basis vectors of the reciprocal lattice $\vec{b}_1 = (2\pi/a)\vec{e}_x$, $\vec{b}_2 = (2\pi/a)\vec{e}_y$ and $\vec{b}_3 = (2\pi/a)\vec{e}_z$.

Sketch of the 1st Brillouin zone



b) Fermi wave number

$$k_F = (3\pi^2 \frac{N}{V})^{1/3} = (3\pi^2 \cdot 3/a^3)^{1/3} = \frac{4.46}{a}$$

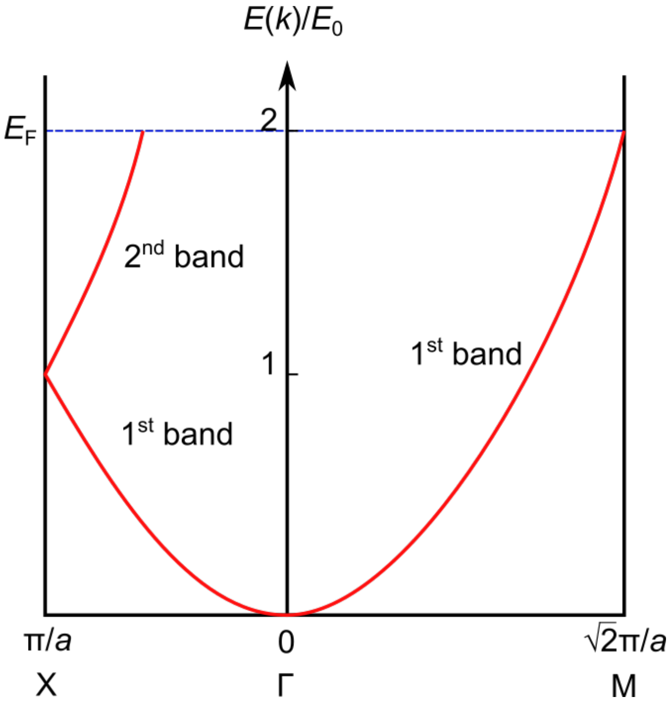
c) Fermi energy

$$E_F = \frac{\hbar^2}{2m_e} k_F^2$$

and

$$\frac{E_F}{E_0} = \left(\frac{k_F\, a}{\pi}\right)^2 = \left(\frac{4.46}{\pi}\right)^2 = 2.02$$

Band structure



d) Intersection of the Fermi surface of the 2nd energy band with the ΓXM plane in the reduced zone scheme