Prof. Bernd Pilawa, Sören Bieling - soeren.bieling@kit.edu

Mock exam
(Problem Sheet 10)

Time for mock exam: 90 min. (strict!)
Write your name on every sheet of paper.
Whenever a final result is numerical, please underline it.
All online participants have 10 min. after the exam to photo-scan their exam answers and email them as pdf to me at soeren.bieling@kit.edu.

Physical constants:

Speed of light: $c = 3 \times 10^8\,\mathrm{m/s}$
Planck's constant: $h = 6.626 \times 10^{-34}\,\mathrm{Js}$
Elementary charge: $e = 1.602 \times 10^{-19}\,\mathrm{C}$
Mass of the electron: $m_e = 511\,\mathrm{keV/c^2} = 9.11 \times 10^{-31}\,\mathrm{kg}$
Compton wavelength of the electron: $\lambda_{C,e} = \frac{h}{m_e c} = 2.426\,\mathrm{pm}$
Boltzmann constant: $k_B = 1.381 \times 10^{-23}\,\mathrm{J/K}$
Stefan-Boltzmann constant: $\sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2} = 5.67 \times 10^{-8}\,\mathrm{W/m^2K^4}$
Atomic mass unit: $u = 1.66 \times 10^{-27}\,\mathrm{kg}$
Rydberg unit of energy: $R = 13.6\,\mathrm{eV}$

Problem 1: **(Points 4)**

a) Albert Einstein explained Planck's law by three fundamental processes. Explain these processes.

b) Make a sketch of Planck's law, i.e. plot the spectral radiance as a function of the wavelength λ.

c) Write down and explain Stefan-Boltzmann's law. Explain what a black body is.

d) A sphere with a radius of $r = 10\,\mathrm{cm}$ approximating a black body is exposed to the sun ($150\,\mathrm{W/m^2}$) and absorbs the whole arriving power of the sun. Calculate the temperature of the sphere in thermal equilibrium when the temperature of the surrounding is $T = 300\,\mathrm{K}$.

Problem 2: **(Points 4)**

The two stars of the binary star system β-Aurigae have almost the same mass and orbit the common centre of gravity. During the time of observation one star moves at a speed of $100\,\mathrm{km/s}$ directly towards the Earth whereas the other star recedes from Earth at this speed.

a) Calculate the Doppler shift of the red Balmer line ($\lambda = 657\,\mathrm{nm}$) due to the rotation of the two stars around the centre of gravity.

b) The red Balmer line splits in two components due to the rotation. Calculate the energy difference between the two components in units of eV.

The microwave beam of a radar trap (frequency $\nu_0 = 30\,\mathrm{GHz}$) is oriented under 45° towards the approaching traffic.

c) Calculate the frequency shift of the microwave measured in a car approaching the radar trap with a velocity of $70\,\mathrm{km/h}$.

d) A fraction of the radiation is reflected back to the radar trap. Calculate the frequency shift which is measured by the receiver.

Problem 3: **(Points 4)**

Photons with the energy $1.0\,\mathrm{MeV}$ hit electrons at rest.

a) Calculate the wavelength of these photons.

b) Calculate the energy of a photon deflected by 30° from the direction of the incident photon beam.

c) Calculate the kinetic and total energy of an electron after the collision with this photon.

d) Calculate the momentum of this electron.

Problem 4: **(Points 4)**

a) Give the wave function for a particle with momentum $\vec{p}$ and energy E.

b) Write down the quantum mechanical operators of energy and momentum.

c) Write down their eigenvalue equations.

d) Write down the Schrödinger equation for a particle of charge q and mass m in the electric potential $\Phi(\vec{r})$.

Problem 5: **(Points 5)**

Spherical coordinates (r, ϑ, φ) are used to solve the Schrödinger equation of a single electron in the radially symmetric electric field of a nucleus.

a) Sketch and write down the equation for the effective potential energy $\Phi_{eff}(r)$ of the electron.

b) Which quantum numbers determine the radial $R(r)$ and angular part $\mathcal{Y}(\vartheta, \varphi)$ of the wave function? Write down their range of values they can take as well as the eigenvalue equation of their corresponding operators.

c) Which additional quantum numbers are still necessary to fully characterize the quantum state of the electron? Write down their range as well as the eigenvalue equation of their corresponding operators.

d) Sketch the radial wave functions of the first excited state of the electron.

Problem 6: **(Points 4)**

The total angular momentum $\vec{j}$ of an electron results from the addition of its orbital angular momentum $\vec{l}$ and its spin $\vec{s}$.

a) Give the quantum numbers characterizing the total angular momentum. Write down their range as well as the eigenvalue equation of their corresponding operators.

b) Write down the magnetic moment due to its orbital angular momentum and due to its spin.

c) Write down the Hamilton operator for the spin-orbit coupling (coupling constant ξ) and calculate its energy eigenvalues.

d) In its ground state Eu^{3+} has the electronic configuration [Xe] $4f^6$. Derive its orbital angular momentum and spin quantum number L and S. Write down the atomic notation of the quantum states of its spin-orbit multiplet sorted in order of increasing energy.

Happy new year, welcome to 2022 and good luck!

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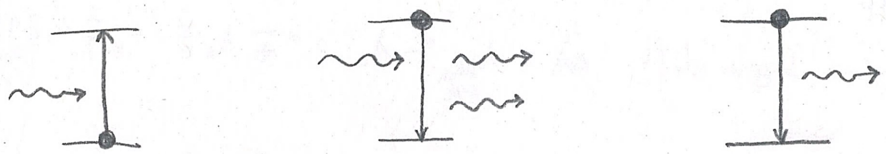
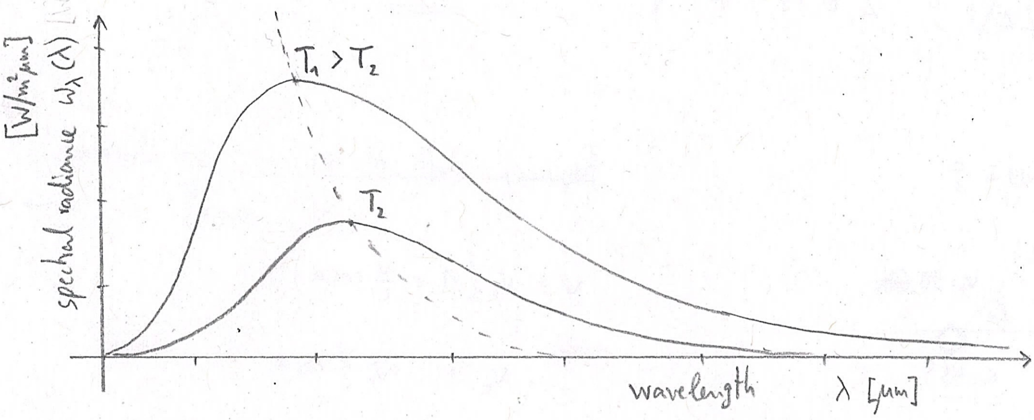
Ex. 10.1: a) The 3 processes are: absorbtion, stimulated emission & spontaneous emission

1P



b) sketch of Planck's law of radiation

1P



c) Stefan-Boltzmann law: calculates total power emitted by em radiation by a body (ε) with temperature T

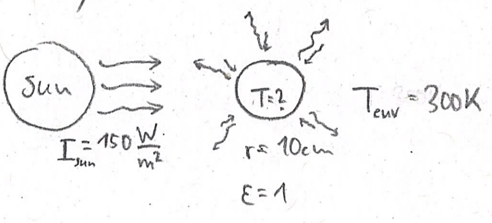
$$P = \varepsilon \cdot \sigma \cdot A \cdot T^4$$

1P

What is a black body? A black body absorbs all radiation of all wavelengths.

$\varepsilon_{(\lambda)} \to 1$

d)



1P

In equilibrium: $P_{Absorbtion} \overset{!}{=} P_{emission}$

$$I_{sun} \frac{W}{m^2} \cdot \cancel{\pi} r^2 + \varepsilon \cdot \sigma \cdot 4\cancel{\pi r^2} \cdot T_{env}^4 = \varepsilon \cdot \sigma \cdot 4\cancel{\pi r^2}\, T^4$$

$$T_{sun}^4 = T_{env}^4 + \frac{I_{sun}}{4\varepsilon \cdot \sigma}$$

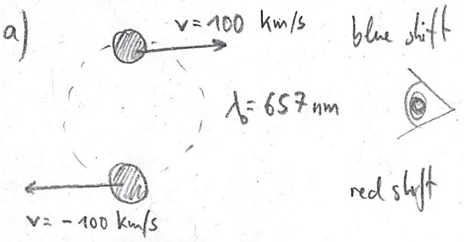
$$\sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2} = 5{,}67 \cdot 10^{-8} \frac{W}{m^2 K^4}$$

$$T = \sqrt[4]{T_{env}^4 + \frac{I_{sun}}{4\varepsilon \cdot \sigma}} \approx \underline{\underline{306\ K}} \quad (\approx 33°C)$$

Σ 4P

Ex. 10.2 Doppler shift

a)



blue shift: $\lambda_{blue} = \lambda_0 \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \xrightarrow{v \ll c} \lambda_0 \left(1-\frac{v}{c}\right)$

red shift: $(v \to -v)$ $\lambda_{red} = \lambda_0 \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \xrightarrow{v \ll c} \lambda_0 \left(1+\frac{v}{c}\right)$

$\sqrt{\underbrace{\frac{1+\beta}{1-\beta} \cdot \frac{1+\beta}{1+\beta}}_{1^2 - \beta^2 \to 0}}$

1P

Doppler shift $\Delta\lambda = \lambda_{\substack{blue \\ red}} - \lambda_0 = \mp \lambda_0 \frac{v}{c} = \mp 0{,}219 \text{ nm}$

$c = 3 \cdot 10^8 \frac{m}{s}$

b)

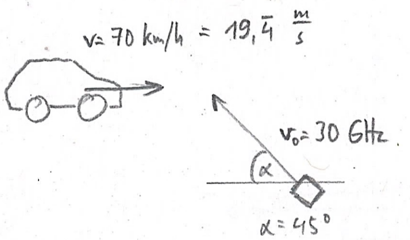
$\Delta E = h \Delta\nu \qquad \nu = \frac{c}{\lambda} \qquad \Delta\nu = -\frac{c}{\lambda^2}\Delta\lambda$

$h = 6{,}626 \cdot 10^{-34}$ J·s

$e = 1{,}602 \cdot 10^{-19}$ C

1P $\Delta E = \frac{hc}{\lambda_0^2} \cdot (2\Delta\lambda) = 2{,}017 \cdot 10^{-22}$ J $= 1{,}26$ meV

c)



frequency shift at car: is increased!

$\alpha = 45°$: $v \to v\cos\alpha$

$\nu' = \nu_0\left(1 + \frac{v}{c}\cos\alpha\right)$

1P $= \nu_0 + \nu_0 \frac{v}{c}\cos\alpha$

$= 30 \text{ GHz} + 1375 \text{ Hz}$

d) reflected wave at receiver: also blue shifted!

$\nu'' = \nu'\left(1 + \frac{v}{c}\cos\alpha\right) = \nu_0\left(1 + \frac{v}{c}\cos\alpha\right)^2 = \nu_0\left(1 + 2\frac{v}{c}\cos\alpha + \underbrace{\left(\frac{v}{c}\cos\alpha\right)^2}_{\to 0 \; v \ll c}\right)$

1P

$= 30 \text{ GHz} + 2750 \text{ Hz}$

Σ 4P

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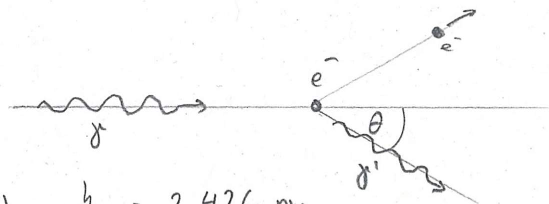
Ex. 10.3: $E_\gamma = 1\ \text{MeV}$

h, c

1P a) $E_\gamma = \frac{hc}{\lambda}$ $\quad \lambda = \frac{hc}{E_\gamma} = \underline{\underline{1,24\ \text{pm}}}$

b) Compton scattering under $\theta = 30°$



1P $\lambda_C = \frac{h}{m_e c} = 2,426\ \text{pm}$

$\Delta\lambda = \lambda' - \lambda = \lambda_c(1 - \cos\theta)$

$\lambda' = \lambda + \lambda_c(1 - \cos\theta) = \underline{\underline{1,565\ \text{pm}}}$

$E_{\gamma'} = \frac{hc}{\lambda'} = \underline{\underline{793\ \text{keV}}}$

c) kinetic energy of e^-: $E_{kin,e^-} = E_\gamma - E_{\gamma'} = 1\ \text{MeV} - 793\ \text{keV} = \underline{\underline{207\ \text{keV}}}$

$(3,32 \cdot 10^{-14}\ \text{J})$

$v_e \approx 0,9 \cdot c$

1P total energy of e^-: $E_{total,e^-} = E_{kin,e^-} + m_e c^2$

$= 207\ \text{keV} + 511\ \text{keV} = \underline{\underline{718\ \text{keV}}}$

$(1,15 \cdot 10^{-13}\ \text{J})$

d) relativistic momentum of e^-:

$E_e^2 = c^2 \vec{p}^2 + m_e^2 c^2 \quad \Rightarrow \quad p = \frac{1}{c}\sqrt{E_e^2 - m_e^2 c^4} = \frac{1}{c}\sqrt{718^2 - 511^2}\ \text{keV}$

1P $= \frac{1}{c} \cdot 504,4\ \text{keV}$

$= \underline{\underline{2,69 \cdot 10^{-22} \frac{\text{kg m}}{\text{s}}}}$

$\Sigma\ 4P$

Ex. 10.4:

a) given $\vec{p}, E$ $\Rightarrow$ $\Psi_{(\vec{r},t)} = \Psi_0 \, e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar}$ 1P

b) → operators $\hat{E} = i\hbar \frac{\partial}{\partial t}$ $\hat{\vec{p}} = -i\hbar \vec{\nabla}$ 1P

c) → eig. value equation: $\hat{E}\,\Psi_{(\vec{r},t)} = i\hbar \frac{\partial}{\partial t} \Psi_{(\vec{r},t)} = E\Psi$

$\hat{\vec{p}}\,\Psi_{(\vec{r},t)} = -i\hbar \vec{\nabla} \cdot \Psi_{(\vec{r},t)} = \vec{p}\,\Psi$ 1P

d) particle of charge q
mass m
in electric potential $\phi(\vec{r})$

$$i\hbar \frac{\partial}{\partial t} \Psi_{(\vec{r},t)} = -\frac{\hbar^2 \nabla^2}{2m} \Psi_{(\vec{r},t)} + q\,\phi(\vec{r})\,\Psi_{(\vec{r},t)}$$

1P

Σ 4P

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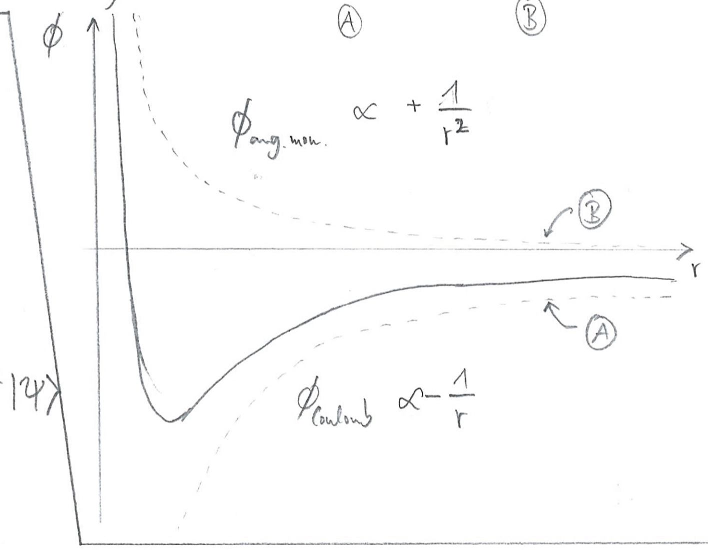
Ex. 10.5: Recall from Ex. 10.4 d): $i\hbar \frac{\partial}{\partial t} \Psi_{(\vec{r},t)} = \left[-\frac{\hbar^2}{2m}\nabla^2 + q\phi(\vec{r}) \right] \Psi_{(\vec{r},t)}$

a) (A) nucleus of charge Ze → Coulomb potential

1P (B) angular momentum L of electron's motion around nucleus

$$\phi_{eff}(r) = \underbrace{-\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}}_{A} + \underbrace{\frac{L^2}{2m_e r^2}}_{B}$$



b) $\boxed{\Psi_{(r,\vartheta,\varphi)} = R_{n\ell(r)} \cdot Y_\ell^m(\vartheta,\varphi)} \cdot \chi_{spin}$

radial part $R_{n\ell}(r)$ determined by quantum numbers:

- main quantum numbers $n = 1, \dots, \infty$ (range)

2P eig. value eq: $\hat{H}|\Psi\rangle = E_n|\Psi\rangle = -13{,}6\,eV\left(\frac{Z}{n}\right)^2|\Psi\rangle$

- angular momentum quantum number $\ell = 0, \dots, n-1$

$$\hat{L}^2 Y_\ell^m = \hbar^2 \ell(\ell+1) Y_\ell^m$$

angular part $Y_\ell^m(\vartheta,\varphi)$ determined by:

- ang. mom. quantum number ℓ (as above)
- magnetic quantum numbers $m_\ell = -\ell, \dots, 0, \dots, +\ell$ $\quad \hat{L}_z Y_\ell^m = m_\ell \hbar Y_\ell^m$

c) additional quantum numbers still necessary: spin quantum numbers s, m_s

- $s = \frac{1}{2}$ $\quad \hat{S}^2 \chi_{spin} = \hbar^2 s(s+1) \chi_{spin}$

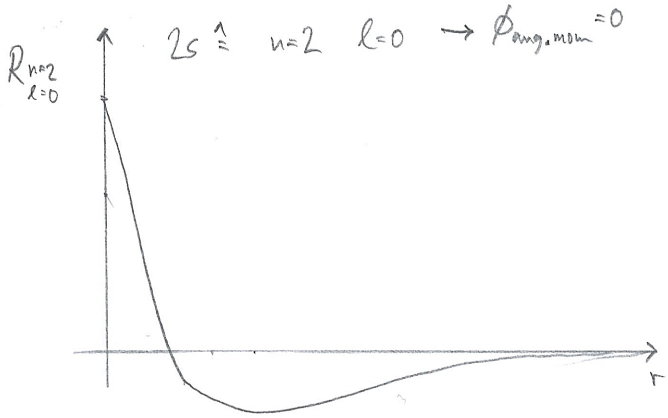
1P

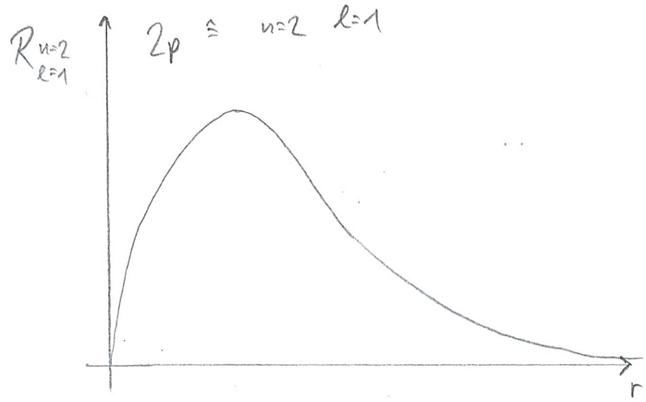
- $m_s = -s, \dots, +s = -\frac{1}{2}, \frac{1}{2}$ $\quad \hat{S}_z \chi_{spin} = m_s \hbar \chi_{spin}$

d) first excited state = 2s, 2p

$2s \mathrel{\hat=} n=2\ \ \ell=0 \rightarrow \phi_{ang.mom} = 0$ $\qquad$ $2p \mathrel{\hat=} n=2\ \ \ell=1$

1P





keep in mind: $\rho = \int \Psi^*\Psi\, dV \propto r^2 R_{n,\ell}^2(r)$ $\qquad$ # roots = $n-\ell$

$\Sigma\, 5P$ $\qquad R_{20} = 2\left(\frac{Z}{2a_0}\right)^{3/2}\left(1 - \frac{Zr}{2a_0}\right)e^{-\frac{Zr}{2a_0}}$ $\qquad R_{21} = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)e^{-\frac{Zr}{2a_0}}$

Ex. 10.6:

a) total angular momentum quantum numbers:

1P

j $\quad |l-s|, \dots, |l+s|$ $\quad \hat{j}^2 |\Psi\rangle = j(j+1)\hbar^2 |\Psi\rangle$

m_j $\quad -j, \dots, +j$ $\quad \hat{j}_z |\Psi\rangle = m_j \hbar |\Psi\rangle$

b) magnetic moments due to orb. ang. momentum $\vec{L}$ and spin $\vec{S}$

1P

$\vec{\mu}_L = -\mu_B \frac{\vec{L}}{\hbar}$ $\quad \vec{\mu}_S = -\mu_B \cdot g_s \frac{\vec{S}}{\hbar}$ $\quad$ with $g_s \hat{=} 2$, $\mu_B = \frac{e\hbar}{2m}$ (Bohr's magneton)

c) $\hat{H}_{LS} = \xi \frac{\hat{\vec{L}} \cdot \hat{\vec{S}}}{\hbar^2}$ $\quad \vec{J}^2 = (\vec{L} + \vec{S})^2 = \vec{L}^2 + 2\vec{L}\cdot\vec{S} + \vec{S}^2$

$\vec{L}\cdot\vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$

1P

$\hat{H}_{LS} |\Psi\rangle = E_{JLS} |\Psi\rangle$

$E_{JLS} = \frac{1}{2}\xi\left(J(J+1) - L(L+1) - S(S+1)\right)$

d) Eu^{3+} in $|g\rangle$: $[Xe]\,4f^6$ $\quad$ atomic notation: ${}^{2S+1}L_J$ $\quad$ $L = 0\ 1\ 2\ 3$: S P D F

$\Rightarrow n=4 \quad L=3$ $\quad S=3$ $\quad J = 0, 1, 2, 3, 4, 5, 6$

$m_L = -3, \dots, 0, \dots, 3$ $\quad$ 2nd Hund's rule: maximize S

1P

(c.f. E_{JLS}) $\quad {}^7F_0,\ {}^7F_1,\ {}^7F_2,\ {}^7F_3,\ {}^7F_4,\ {}^7F_5,\ {}^7F_6$

$\xrightarrow{\text{increasing energy}}$

Σ 4P