Written Examination in Physics
Modern Physics (KSOP)

Problem 1 (4 Points)

The velocity of an electron in the reference frame of the Earth is in an experiment 60% of the velocity of light.

a) Calculate for the electron the time which is in the reference frame of the Earth necessary to pass a distance of 40 km.
b) How long is the distance of 40 km in a frame of reference moving with the electron?
c) Which time is necessary in that frame to pass this distance?
d) Describe an experiment to test the effect experimentally.

Problem 2 (4 Points)

Electrons at rest are accelerated by a voltage of 1,0 kV.

a) What is the velocity of the electrons after the acceleration? Justify that relativistic effects can be neglected.
b) Calculate the de Broglie wavelength of the electron beam.
c) The electrons pass a crystal lattice with a lattice constant $d = 3{,}0 \cdot 10^{-10}$ m. Calculate for the 1^{st} diffraction maximum the angles of observation with respect to the direction of the electron beam.
d) What is the effect of the spatial spread of the atomic electron clouds on the diffraction pattern?

Problem 3 (4 Points)

Photons with the energy 1,0 MeV hit electrons at rest.

a) Calculate the wave length of an electromagnetic wave formed by these photons.
b) Give the energy of a photon deflected by 30° from the direction of the incident photon beam.
c) Give the energy of an electron after the collision with this photon.
d) Give the momentum of this electron.

Problem 4 (4 Points)

a) Give the temperature on the surface of the Sun, if the maximum of the electromagnetic spectrum is measured at $\lambda = 500$ nm.
b) Give the total intensity of the electromagnetic radiation due to the Sun on Earth.
c) Calculate the temperature of the Earth. Assume that the Earth is a sphere formed by a perfect thermal conductor.

Problem 5 (4 Points)

Spherical coordinates r, ϑ, φ are used to solve the Schrödinger equation of an electron in the electric field of a nucleus.

a) Sketch and write up the equation of the effective potential energy $\phi_{\text{eff}}(r)$ of the electron.
b) Explain the quantum numbers characterizing the radial part of the wave function.
c) Sketch the radial wave functions of the first excited state of the electron.

d) Which additional quantum numbers are necessary to characterize the quantum state of the electron?

Problem 6 (4 Points)

One basic assumption of the Sommerfeld model for electrons in a metallic solid is that electron waves are plane waves propagating freely and isotropically in all directions.

a) Explain the meaning of Pauli's principle for the Sommerfeld model.
b) Calculate the energy of an electron described by a plane wave.
c) Calculate the Fermi energy of Cu at $T = 0$. Hint: There is one valence electron per Cu atom. The volume of a wavenumber state is $(2\pi)^3/V$ and V denotes the volume of the solid.
d) Sketch for the probability that a plane wave is occupied by an electron as a function of energy for $T = 0$.

Problem 7 (not discussed in the current lecture) (4 Points)

For the description of quantum particles, particle annihilation $\boldsymbol{a}$ and creation $\boldsymbol{a}^\dagger$ operators are used, which act on quantum states $|n\rangle$ simply characterized by the number n of quantum particles.

a) Give the result of $\boldsymbol{a}^\dagger |n\rangle$ and $\boldsymbol{a}|n\rangle$ for bosons and fermions.
b) Give the particle number operator $\boldsymbol{N}|n\rangle = n|n\rangle$ in terms of the particle annihilation and creation operators.
c) Give the Hamilton operator for a system of photons with wave vector $\vec{k}$.
d) Give the electric field of an electromagnetic wave formed by photons with wave vector $\vec{k}$ in terms of the annihilation and creation operators. Ignore the prefactor!

Problem 8 (4 Points)

A crystal is formed by N unit cells each containing n atoms.

a) Give the number of acoustic and optic phonon branches, respectively.
b) Give the number of existing phonon $\vec{q}$-modes.
c) Explain the Debye approximation for the calculation of the specific heat.
d) Calculate the Debye temperature of Cu. Use the sound velocity $c_{\mathrm{Cu}} = 3000\,\mathrm{m/s}$.

Required physical constants:

Velocity of light:	$c = 3 \cdot 10^8\,\mathrm{m/s}$
Mass of the electron:	$m_e = 500\,\mathrm{keV}/c^2$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\mathrm{eVs}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\mathrm{As}$
Compton wavelength:	$\lambda_C = 2.426 \cdot 10^{-12}\,\mathrm{m}$
Stefan-Boltzmann's constant:	$\sigma = 5.67 \cdot 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$
Wien's constant:	$b = 2.9\,\mathrm{mm \cdot K}$
Boltzmann constant:	$k_B = 8.6 \cdot 10^{-5}\mathrm{eV/K}$
Avogadro's number:	$N_A = 6 \cdot 10^{23}\,\mathrm{mol^{-1}}$
Radius of the Earth:	$r_E = 6.37 \cdot 10^6\,\mathrm{m}$

Radius of the Sun:	$r_S = 6.96 \cdot 10^8$ m
Distance Sun-Earth:	$r_{SE} = 1.5 \cdot 10^{11}$ m
Density of Copper:	$\rho_{Cu} = 9\,g/cm^3$
Molar mass of Cu:	$m_{Cu} = 63.5\,g/mol$

Problem 1 (4 Points)

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b) How long is the distance of 40 km in a frame of reference moving with the electron?
c) Which time is necessary in that frame to pass this distance?
d) Describe an experiment to test the effect experimentally.

a) Time for 40 km:

$$t_0 = 40 \cdot 10^3\,\mathrm{m}/(0.6 \cdot\ 3 \cdot 10^8\,\mathrm{m/s}) = 22.2 \cdot 10^{-5}\,\mathrm{s}$$

b) The distance of 40 km moves by with the velocity $-0.6 \cdot c \rightarrow$ length contraction

$$\ell(v) = 40\,\mathrm{km}\sqrt{1 - 0.6^2} = 40\,\mathrm{km} \cdot 0.8 = 32\,\mathrm{km}.$$

c) The time in the frame of the electron is

$$t_\mathrm{e} = \frac{\ell(v)}{0.6 \cdot\ 3 \cdot 10^8\,\mathrm{m/s}} = t_0\sqrt{1 - 0.6^2} = 22.2 \cdot 10^{-5}\,\mathrm{s}\,0.8 = 17.8 \cdot 10^{-5}\,\mathrm{s}$$

d) The effect of time dilation, i.e. $t(v) = t_0 = \frac{t_\mathrm{e}}{\sqrt{1-0.6^2}}$ can be tested by replacing the electron by an particle with a short life time (e.g. the μ^-) and counting the starting and arriving particles.

Problem 2 (4 Points)

Electrons at rest are accelerated by a voltage of 1,0 kV.

a) What is the velocity of the electrons after the acceleration? Justify that relativistic effects can be neglected.
b) Calculate the de Broglie wavelength of the electron beam.
c) The electrons pass a crystal lattice with a lattice constant $d = 3{,}0 \cdot 10^{-10}$ m. Calculate for the 1st diffraction maximum the angles of observation with respect to the direction of the electron beam.
d) What is the effect of the spatial spread of the atomic electron clouds on the diffraction pattern?

a) The kinetic energy is $E_{\mathrm{kin}} = \mathrm{e}U_0$.

$$\mathrm{v} = c\sqrt{\frac{2\mathrm{e}U_0}{m_\mathrm{e}c^2}} = c\sqrt{\frac{2 \cdot 10^3\,\mathrm{eV}}{500 \cdot 10^3\,\mathrm{eV}}} = 0.063 \cdot c = 1.9 \cdot 10^7\,\mathrm{m/s}.$$

v is very much small than the velocity of light. Therefore relativistic effects can be ignored.

b) The de Broglie wavelenth is

$$\lambda = \frac{h}{p_\mathrm{e}} = \frac{h}{m_\mathrm{e}\,\mathrm{v}} = c\frac{4.14 \cdot 10^{-15}\,\mathrm{eVs}}{500 \cdot 10^3\,\mathrm{eV}\,0.063} = 3 \cdot 10^8\,\mathrm{m}\,13.14 \cdot 10^{-20} = 4 \cdot 10^{-11}\,\mathrm{m}.$$

c) Bragg's law $n\lambda = 2d\sin\theta_n$

$$\sin\theta_1 = \frac{\lambda}{2d} = \frac{4 \cdot 10^{-11}}{2 \cdot 3 \cdot 10^{-10}\,\mathrm{m}} = 0.067 \approx \theta_1$$

The angles with respect to the electron beam are $\pm 2\theta_1 \widehat{=} \pm 7.46°$.

d) Similar to the diffraction on a grating the diffraction pattern has to be multiplied by the Fourier-transform of the electron cloud (slit-function). The intensity of higher order diffraction peaks will be reduced.

Problem 3 (4 Points)

Photons with the energy 1,0 MeV hit electrons at rest.

a) Calculate the wave length of an electromagnetic wave formed by these photons.
b) Give the energy of a photon deflected by 30° from the direction of the incident photon beam.
c) Give the energy of an electron after the collision with this photon.
d) Give the momentum of this electron.

a) The wavelenth of the electromagnetic wave is

$$E_\lambda = \frac{\mathrm{hc}}{\lambda} \rightarrow \lambda = \frac{\mathrm{hc}}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\mathrm{eVs}\, 3 \cdot 10^{8}\,\mathrm{m/s}}{10^{6}\,\mathrm{eV}} = 12.42 \cdot 10^{-13}\,\mathrm{m}.$$

b) The increase of the wavelenth (i.e. the reduction of energy) due to the scattering on a free electron at rest is

$$\lambda' - \lambda = \lambda_\mathrm{C}(1 - \cos\theta) \rightarrow \lambda' = \lambda + \lambda_\mathrm{C}(1 - 0.866)$$
$$= (1.242 \cdot 10^{-12} + 2.426 \cdot 10^{-12}\, 0.134\,\mathrm{m}) = 1.567 \cdot 10^{-12}\,\mathrm{m}.$$

The energy of the scattered photon is

$$E_{\lambda'} = \frac{\mathrm{hc}}{\lambda'} = \frac{4.14 \cdot 10^{-15}\mathrm{eVs}\, 3 \cdot 10^{8}\,\mathrm{m/s}}{1.567 \cdot 10^{-12}\,\mathrm{m}} = 7.93 \cdot 10^{5}\,\mathrm{eV}$$

c) The kinetic energy of the electron is $E_\mathrm{kin} = E_\lambda - E_{\lambda'} = 207\,\mathrm{keV}$ and the total energy $E_\mathrm{e} = E_\mathrm{kin} + m_\mathrm{e}c^2 = 707\,\mathrm{keV}$.

d) With

$$E_\mathrm{e}^2 = c^2\vec{p}^2 + m_e^2 c^4$$

is the momentum

$$p = \frac{1}{c}\sqrt{E_\mathrm{e}^2 - m_\mathrm{e}^2 c^4} = \frac{1}{c}\sqrt{707^2 - 500^2}\,\mathrm{keV} = \frac{1}{c} 500\,\mathrm{keV} = m_\mathrm{e} \cdot c$$
$$= \frac{500 \cdot 10^{3} \cdot 1.6 \cdot 10^{-19}}{3 \cdot 10^{8}} \frac{\mathrm{kgm}}{\mathrm{s}} = 2.67 \cdot 10^{-22}\, \frac{\mathrm{kgm}}{\mathrm{s}}$$

Problem 4 (4 Points)

a) Give the temperature on the surface of the Sun, if the maximum of the electromagnetic spectrum is measured at $\lambda = 500\,\text{nm}$.
b) Give the total intensity of the electromagnetic radiation due to the Sun on Earth.
c) Calculate the temperature of the Earth. Assume that the Earth is a sphere formed by a perfect thermal conductor.

a) With Wien's displacement law

$$\lambda_{\text{max}} = \frac{b}{T}$$

one gets

$$T = \frac{2.9 \cdot 10^{-3}\,\text{m} \cdot \text{K}}{500 \cdot 10^{-9}\,\text{m}} = 5800\,\text{K}.$$

b) With the Stefan-Boltzmann law is the power of the Sun

$$P = 4\pi r_S^2 \sigma T^4$$

and the intensity on Earth

$$I = \left(\frac{r_S}{r_{SE}}\right)^2 \sigma T^4 = \left(\frac{6.96 \cdot 10^8\,\text{m}}{1.5 \cdot 10^{11}\,\text{m}}\right)^2 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4} 5800^4\,\text{K}^4$$
$$= (4.64 \cdot 10^{-3})^2 5.67 \cdot 10^{-8}\,\text{Wm}^{-2} 1.1 \cdot 10^{15} = 1340\,\text{W/m}^2.$$

c) The total power on Earth received from the Sun is

$$P_{ES} = \pi r_E^2 \cdot I.$$

When the Earth behaves like a black body is the total power emitted by the Earth

$$P_E = 4\pi r_E^2 \sigma T_E^4.$$

In equilibrium one has $P_{ES} = P_E$ and the temperature is

$$\pi r_E^2 \cdot I = 4\pi r_E^2 \sigma T_E^4 \rightarrow I = 4\sigma T_E^4$$

and

$$T_E = \left(\frac{I}{4\sigma}\right)^{1/4} = \left(\frac{1340\,\text{Wm}^{-2}}{4 \cdot 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}}\right)^{1/4} = 277\,\text{K}.$$

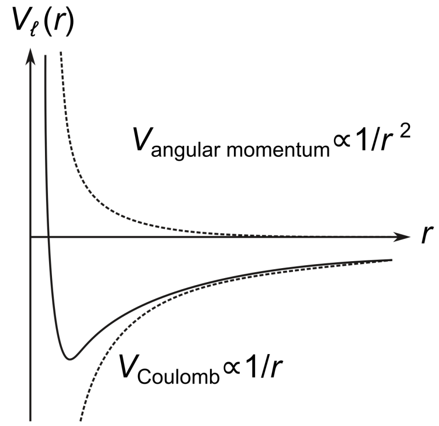
Problem 5 (4 Points)

Spherical coordinates r, ϑ, φ are used to solve the Schrödinger equation of an electron in the electric field of a nucleus.

a) Sketch and write up the equation of the effective potential energy $\phi_{\text{eff}}(r)$ of the electron.
b) Explain the quantum numbers characterizing the radial part of the wave function.
c) Sketch the radial wave functions of the first excited state of the electron.
d) Which additional quantum numbers are necessary to characterize the quantum state of the electron?

a) The potential energy of the electron is due to the Coulomb potential in the electric field of the nucleus and there is an effective potential energy due to the motion of the electron around the nucleus. This contribution is proportional to the square of the angular momentum L.

$$\phi_{\text{eff}} = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r} + \frac{L^2}{2m_e r^2}.$$



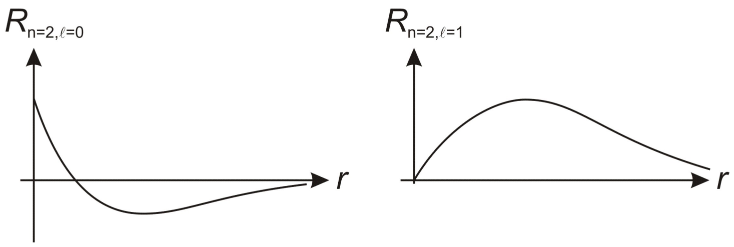
b) The radial part of the wave function is characterized by the principal quantum number n

$$E_n = -13.6\,\text{eV}\left(\frac{Z}{n}\right)^2$$

and the angular momentum quantum number ℓ

$$L^2 y_{\ell,m} = \hbar^2 \ell(\ell+1) y_{\ell,m}.$$

c) For $n = 2$ there are two radial function $R_{n=2,\ell=0}$ and $R_{n=2,\ell=1}$.
$R_{n=2,\ell=0}$ starts at$r = 0$ with a finite value, changes the sign and approaches zero for $r \to \infty$.
Due to the centri-fugal potential starts $R_{n=2,\ell=1}$ for $r = 0$ at zero and approaches after a maximum zero again for $r \to \infty$.



d) For the orbital part one needs in addition to n and ℓ

the magnetic quantum number $|m| \leq \ell$ of the angular momentum.

To characterize the spin of the electron one needs the quantum numbers $s = 1/2$ and $m_s = \pm 1/2$.

s and ℓ add up to the total angular momentum which is characterized by the quantum numbers j and m_j.

Problem 6 (4 Points)

One basic assumption of the Sommerfeld model for electrons in a metallic solid is that electron waves are plane waves propagating freely and isotropically in all directions.

a) Explain the meaning of Pauli's principle for the Sommerfeld model.
b) Calculate the energy of an electron described by a plane wave.
c) Calculate the Fermi energy of Cu at $T = 0$. Hint: There is one valence electron per Cu atom. The volume of a wavenumber state is $(2\pi)^3/V$ and V denotes the volume of the solid.
d) Sketch for the probability that a plane wave is occupied by an electron as a function of energy for $T = 0$.

a) The meaning of the Pauli Principle in the context of the Sommerfeld model is that each wave describes one electron with spin up ($\mathrm{m_s} = +1/2$) and one with spin down ($\mathrm{m_s} = -1/2$).
b) The potential energy of the electrons is a constant off-set since the electrons can propagate freely. The kinetic energy is with

$$\psi = \psi_0 \mathrm{e}^{\mathrm{i}(\vec{k}\vec{r}-\omega t)}$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\nabla^2}{2m_\mathrm{e}}\psi \rightarrow \hbar\omega(\vec{k}) = \mathrm{E} = \frac{\hbar^2\vec{k}^2}{2m_\mathrm{e}}$$

c) Since the electrons propagate isotropically in all directions the $\vec{k}$ states occupied by electrons are within sphere with the radius k_F

$$\frac{N}{2} = \frac{\frac{4\pi k_\mathrm{F}^3}{3}}{(2\pi)^3/\mathrm{V}} \rightarrow k_\mathrm{F} = (3\pi^2 N/V)^{\frac{1}{3}}$$

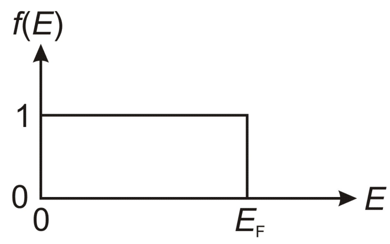
and the highest energy is the Fermi energy $E_\mathrm{F} = \hbar^2 k_\mathrm{F}^2/2m_\mathrm{e}$.
With $\rho = \frac{m}{V} = \frac{N}{V}\frac{m_\mathrm{mol}}{N_\mathrm{A}}$ one gets

$$k_\mathrm{F} = \left(\frac{3\pi^2\rho N_\mathrm{A}}{m_\mathrm{mol}}\right)^{1/3} = \left(\frac{3\pi^2 9\,\mathrm{g/cm^3} 6\cdot 10^{23}}{63.5\mathrm{g}}\right)^{1/3} = 1.36\cdot 10^8\ \mathrm{cm^{-1}}$$

and

$$E_\mathrm{F} = \frac{(4.14\cdot 10^{-15}\,\mathrm{eVs})^2(1.36\cdot 10^8\,\mathrm{cm^{-1}} 3\cdot 10^{10}\,\mathrm{cm/s})^2}{2\cdot 4\pi^2 500\cdot 10^3\,\mathrm{eV}} = 7,2\,\mathrm{eV}.$$

d) The probability is given by the Fermi function $f(E)$. At $T = 0$ all k-states with $E \leq E_F$ are occupied with the probability 1. The probability that k-states with $E > E_F$ are occupied is zero for $T = 0$.



Problem 7 (4 Points)

For the description of quantum particles, particle annihilation $\boldsymbol{a}$ and creation $\boldsymbol{a}^\dagger$ operators are used, which act on quantum states $|n\rangle$ simply characterized by the number n of quantum particles.

a) Give the result of $\boldsymbol{a}^\dagger |n\rangle$ and $\boldsymbol{a} |n\rangle$ for bosons and fermions.
b) Give the particle number operator $\boldsymbol{N} |n\rangle = n |n\rangle$ in terms of the particle annihilation and creation operators.
c) Give the Hamilton operator for a system of photons with wave vector $\vec{k}$.
d) Give the electric field of an electromagnetic wave formed by photons with wave vector $\vec{k}$ in terms of the annihilation and creation operators. Ignore the prefactor!

a) For Bosons

$$a |n\rangle = \sqrt{n} |n-1\rangle$$
$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle,$$

and for Fermions one has the additional condition

$$a^\dagger |1\rangle = 0,$$

since a quantum state can be occupied only by one Fermion.

b) The particle number operator is

$$N = a^\dagger a.$$

c) The Hamilton operator gives the energy a system, i.e.

$$E |n_{\vec{k}}\rangle = H |n_{\vec{k}}\rangle = \hbar\omega(\vec{k}) N_{\vec{k}} |n_{\vec{k}}\rangle = \hbar\omega(\vec{k}) n_{\vec{k}} |n_{\vec{k}}\rangle \, .$$

The Hamilton operator is $H = \hbar\omega(\vec{k}) N_{\vec{k}}$ and the energy of $n_{\vec{k}}$ photons simply $\hbar\omega(\vec{k}) n_{\vec{k}}$.

d) When a system is perturbed by an wave, the phase factor $\mathrm{e}^{\mathrm{i}(\vec{k}\vec{r}-\omega t)}$ is related to the absorption of a photon. Energy conservation demands

$$\omega_\mathrm{f} - \omega - \omega_\mathrm{i} = 0$$

and a photon is absorbed (annihilation) when the energy of the final state is larger than the energy of the initial state, i.e. $\omega_\mathrm{f} > \omega_\mathrm{i}$.

On the other hand the factor $\mathrm{e}^{-\mathrm{i}(\vec{k}\vec{r}-\omega t)}$ is related to the emission (creation) of a photon. Therefore the electric/magnetic field of an electromagnetic wave in terms of the particle operators is

$$\vec{E}_{\vec{k}} \propto a_{\vec{k}} \mathrm{e}^{\mathrm{i}(\vec{k}\vec{r}-\omega t)} + a^\dagger_{\vec{k}} e^{-\mathrm{i}(\vec{k}\vec{r}-\omega t)}.$$

Problem 8 (4 Points)

A crystal is formed by N unit cells each containing n atoms.

a) Give the number of acoustic and optic phonon branches, respectively.
b) Give the number of existing phonon $\vec{q}$-modes.
c) Explain the Debye approximation for the calculation of the specific heat.
d) Calculate the Debye temperature of Cu. Use the sound velocity $c_{\text{Cu}} = 3000\,\text{m/s}$.

a) There are 3 acoustic and $3(n-1)$ optical phonon branches.
b) There a N $\vec{q}$-modes
c) In the Debye approximation the dispersion relation of the three acoustic branches is replaced by

$$\omega(q) = c_{\text{S}} \cdot q.$$

c_{S} is the average sound velocity within the crystal. Since the approximation applies for low temperatures the shape of the 1st-Brillouin zone does not matter and the Brillouin zone is replaced by a sphere.

d) The Debye temperature is defined as

$$k_{\text{B}}\Theta_{\text{D}} = \hbar\omega(q_{\text{max}}).$$

Thereby denotes q_{max} the radius of the sphere which replaces the 1st-Brillouin zone in the Debye approximation, i.e.

$$N = \frac{\frac{4\pi q_{\text{max}}^3}{3}}{(2\pi)^3/\text{V}} \rightarrow q_{\text{max}} = (6\pi^2 N/V)^{\frac{1}{3}}$$

With $\rho = \frac{m}{V} = \frac{N}{V}\frac{m_{\text{mol}}}{N_{\text{A}}}$ and $c_{\text{S}} = c_{\text{Cu}}$

$$\begin{aligned} q_{\text{max}} &= (6\pi^2 N/V)^{\frac{1}{3}} = (6\pi^2 \rho N_{\text{A}}/m_{\text{mol}})^{\frac{1}{3}} \\ &= (6\pi^2 9\,\text{gcm}^{-3} 6 \cdot 10^{23}/63.5\,\text{g})^{\frac{1}{3}} = 1.7 \cdot 10^8\,\text{cm}^{-1} \end{aligned}$$

one gets

$$\Theta_{\text{D}} = \frac{\hbar c_{\text{S}} q_{\text{max}}}{k_{\text{B}}} = \frac{4.14 \cdot 10^{-15}\text{eVs}\, 3 \cdot 10^5\text{cm/s}\, 1.7 \cdot 10^8\,\text{cm}^{-1}}{2\pi \cdot 8.6 \cdot 10^{-5}\text{eVK}^{-1}} = 390\,\text{K}$$

Problem 1 (4 Points)

An antenna on board of a satellite emits one short pulse per second. The satellite is moving relative to a receiver with the velocity of 90 % of the velocity of light.

a) Calculate the frequency of pulses arriving at the receiver.
b) A second antenna is placed 2 m behind the first antenna in the direction of the motion of the satellite. Calculate the time interval between two pulses recorded by the receiver, when the pulses are emitted simultaneously by the two antennae on board of the satellite.
c) The mass of the satellite at rest is 1 kg. Calculate the energy which is a least necessary to accelerate the satellite from zero to 90 % of the velocity of light.
d) Calculate the time which is necessary for a power station with an output power of 1 GW to provide this amount of energy.

Problem 2 (4 Points)

Two frames of reference S and S' move with the relative velocity $\vec{v}$ with respect to each other.

a) Write up the equations for the transformation of the coordinates $\vec{r}$ and t between frame S and S'.
b) Give the equations for the transformation of the velocity $\vec{v}$ of a particle between frame S and S'.
c) Write up the equations for the transformation of the wave vector $\vec{k}$ and the angular frequency ω.
d) Give the equations for the transformation of momentum $\vec{P}$ and energy E of a particle with mass m_0 at rest.

Problem 3 (4 Points)

a) An electron moves with a velocity of $v = 1000\,\mathrm{m/s}$. Calculate the kinetic energy and the de Broglie wavelength of the electron.
b) Calculate the energy of the photons for light with the wavelength $\lambda = 745\,\mathrm{nm}$.
c) Write up the conservation laws of energy and momentum for the collision between an electron and a photon.
d) Give the relation between energy and momentum of a relativistic electron.

Problem 4 (4 Points)

a) Write up the Schrödinger equation for an electron moving within a region with constant potential energy V. The total energy of the electron is $E > V$.
b) Calculate the wave function and momentum of an electron moving along the x-axis.
c) The electron moves within a region with $V = V_1 < E$ in the direction of increasing x values and hits a barrier where the potential energy changes from V_1 to $V_2 < E$. How large is the probability that the electron is reflected?
Hint: The wave function at the interface of the barrier is both continuous and continuously differentiable.

Please note reverse side.

Problem 5 (4 Points)

The angular momentum of an electron results from the addition of the orbital angular momentum $\vec{\ell}$ and its spin $\vec{s}$.

a) Give the quantum numbers and the range of these quantum numbers characterizing the orbital angular momentum and the spin.
b) The total angular momentum $\vec{j}$ results from the addition of $\vec{\ell}$ and $\vec{s}$. Give the quantum numbers and the range of these quantum numbers characterizing the total angular momentum.
c) Write up the relation between the magnetic moment of an electron and the orbital angular momentum. Write up the relation between the magnetic moment and the spin of an electron.
d) Calculate the potential energy of an electron in an homogeneous magnetic field $\vec{B}$ in terms of the quantum numbers of $\vec{\ell}$ and $\vec{s}$.

Problem 6 (4 Points)

The orbital angular momentum $\vec{\ell}$ of the electron and its spin $\vec{s}$ couple to the total angular momentum $\vec{j}$ according to $\xi\vec{\ell}\cdot\vec{s}$.

a) Explain the origin for the coupling of $\vec{\ell}$ and $\vec{s}$.
b) Give the sign of the coupling constant ξ.
c) Calculate the energy difference between the energy of the total angular momentum state with the highest and the smallest j-value, respectively.
d) Calculate the potential energy of an electron in an homogeneous magnetic field $\vec{B}$ in terms of the quantum numbers of $\vec{j}$.

Problem 7 (4 Points)

a) What is the meaning of Pauli's principle?
b) Explain the origin of the electronic band structure of a solid.
c) What is the definition of the Fermi energy?
d) Describe the band structure of a metal, a semiconductor and of an insulator.

Problem 8 (4 Points)

a) What is a phonon?
b) How is it possible to distinguish between acoustic and optical phonons?
c) In an experiment the light of a laser ($\lambda = 623$ nm) is scattered by a phonon at an angle of 90°. Calculate the momentum of the phonon.
d) Give the direction of the phonon with respect to the incident photon, when the phonon is excited by the photon.

Required physical constants:

Velocity of light:	$c = 3 \cdot 10^8\,\mathrm{m/s}$
Mass of the electron at rest:	$m_\mathrm{e} = 500\,\mathrm{keV}/c^2$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\mathrm{eVs}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\mathrm{As}$

Problem 1 (4 Points)

An antenna on board of a satellite emits one short pulse per second. The satellite is moving the velocity of 90 % relative to a receiver.

a) Calculate the frequency of pulses arriving at the receiver.
b) A second antenna is placed 2 m behind the first antenna in the direction of the motion of the satellite. Calculate the time interval between two pulses recorded by the receiver, when the pulses are emitted simultaneously by the two antennae on board of the satellite.
c) The mass of the satellite at rest is 1 kg. Calculate the energy which is a least necessary to accelerate the satellite from zero to 90 % of the velocity of light?
d) Calculate the time which is necessary for a power station with an output power of 1 GW to provide this amount of energy.

a) During the time $\Delta t = 1\,\mathrm{s}$ covers the satellite the distance

$$\Delta\ell = \Delta t \cdot 0.9\,\mathrm{c}$$

and the additional time for the next pulse in the frame of the satellite to reach the receiver is

$$\delta t = \frac{\Delta\ell}{c}.$$

The time between two pulses measured in the frame of the satellite is

$$\Delta t_0 = \Delta t + \delta t = \Delta t(1 + 0.9).$$

The time between two pulses measured by the receiver at rest is due to time dilation

$$\Delta t_\mathrm{R} = \Delta t_0 \frac{1}{\sqrt{1 - 0.9^2}}$$

or

$$\Delta t_\mathrm{R} = \Delta t \frac{(1 + 0.9)}{\sqrt{1 - 0.9^2}} = \Delta t \sqrt{\frac{1 + 0.9}{1 - 0.9}}.$$

The frequency $\nu_\mathrm{R} = 1/\Delta t_\mathrm{R}$ is

$$\nu_\mathrm{R} = 1\,\mathrm{Hz}\sqrt{\frac{1 - 0.9}{1 + 0.9}} = 0.23\,\mathrm{Hz}.$$

b) Due to length contraction in the frame of the receiver covers the pulse of the second antenna the additional distance

$$\Delta\ell = 2\,\mathrm{m}\sqrt{1 - (\mathrm{v/c})^2}$$

and the additional time for the pulse of the second antenna is

$$\Delta t = \frac{\Delta\ell}{c} = 2\,\mathrm{m}\sqrt{1 - (\mathrm{v/c})^2}/\mathrm{c} = 2\,\mathrm{m}\sqrt{1 - 0.9^2}/3 \cdot 10^8\,\mathrm{m/s} = 0.29 \cdot 10^{-8}\,\mathrm{s}.$$

c) The total energy of the satellite is

$$E = mc^2.$$

The kinetic energy is

$$E = (m - m_0)c^2 = \frac{m_0 c^2}{\sqrt{1 - 0,9^2}} - m_0 c^2$$
$$= 1\,\text{kg} \cdot (3 \cdot 10^8\,\text{m/s})^2 \cdot 1,294 = 11,65 \cdot 10^{16}\,\text{Ws}$$

d) To produce this amount of energy a power station needs the time

$$t = \frac{11,65 \cdot 10^{16}\,\text{Ws}}{10^9\,\text{W}} = 11,65 \cdot 10^7\,\text{s} = 1348\,\text{d} = 3,7\,\text{a}.$$

Problem 2 (4 Points)

Two frames of reference S and S' move with the relative velocity $\vec{v}$ with respect to each other.

a) Write up the equations for the transformation of the coordinates $\vec{r}$ and t between frame S and S'.
b) Give the equations for the transformation of the velocity $\vec{v}$ of a particle in frame S and S'.
c) Write up the equations for the transformation of the wave vector $\vec{k}$ and angular frequency ω.
d) Give the equations for the transformation of energy and momentum of a particle with mass m_0 at rest.

a) Moves the frame of reference S' along the x-axis in the direction of increasing x-values, then reads the Lorentz transformation with $\gamma = 1/\sqrt{1 - v^2/c^2}$

$$
\begin{aligned}
x' &= \gamma(x - vt)\\
y' &= y\\
z' &= z\\
t' &= \gamma\left(t - \frac{v}{c^2}x\right).
\end{aligned}
$$

b) The transformation of the velocity is

$$
\begin{aligned}
v'_x &= \frac{dx'}{dt'} = \frac{dx - v dt}{dt - \frac{v}{c^2}dx} = \frac{v_x - v}{1 - \frac{v v_x}{c^2}}\\
v'_y &= \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{v_y}{\gamma(1 - \frac{v v_x}{c^2})}\\
v'_z &= \frac{dz'}{dt'} = \frac{dz}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{v_z}{\gamma(1 - \frac{v v_x}{c^2})}
\end{aligned}
$$

c) With respect to the transformation one has to note the correspondence $\vec{r} \mathrel{\hat{=}} c\vec{k}$ und $ct \mathrel{\hat{=}} \omega$ and the Lorentz transformation becomes

$$
\begin{aligned}
k_x{}' &= \gamma(k_x - \frac{v}{c^2}\omega)\\
k_y{}' &= k_y\\
k_z{}' &= k_y\\
\omega' &= \gamma(\omega - v k_x).
\end{aligned}
$$

d) Due to $\vec{P} = \hbar\vec{k}$ and $E = \hbar\omega$ the Lorentz transformation reads

$$
\begin{aligned}
{P_x}' &= \gamma(P_x - \frac{v}{c^2}E) \\
{P_y}' &= P_y \\
{P_z}' &= P_y \\
E' &= \gamma(E - vP_x).
\end{aligned}
$$

Problem 3 (4 Points)

a) An electron moves with a velocity of $v = 1000\,\text{m/s}$. Calculate the kinetic energy and the de Broglie wavelength of the electron.
b) Calculate the energy of the photons for light with the wavelength of $\lambda = 745\,\text{nm}$.
c) Write up the conservation laws of energy and momentum for the collision between an electron and a photon.
d) Give the relation between energy and momentum of a relativistic electron.

a) The kinetic energy of the electron is

$$E_{\text{kin}} = \frac{1}{2}\text{m}v^2 = \frac{1}{2}5 \cdot 10^5\,\text{eV}\left(\frac{1000}{3 \cdot 10^8}\right)^2 = 2.78 \cdot 10^{-6}\,\text{eV}.$$

The de Broglie wave length of the electron is

$$\lambda = \frac{h}{P} = \frac{h}{m_\text{e}v} = \frac{4,14 \cdot 10^{-15}\,\text{eVs}(3 \cdot 10^8\,\text{m/s})^2}{5 \cdot 10^5\,\text{eV}\,1000\,\text{m/s}} = 7,45 \cdot 10^{-7}\,\text{m}.$$

b) The energy of the photon is

$$E = h\nu = h\frac{c}{\lambda} = 4.14 \cdot 10^{-15}\,\text{eVs}\frac{3 \cdot 10^8\,\text{m/s}}{745 \cdot 10^{-9}\,\text{m}} = 1.67\,\text{eV}.$$

c) Conservation of energy

$$E_\text{e} + E_\gamma = E'_\text{e} + E'_\gamma,$$

conservation of momentum

$$\vec{P}_\text{e} + \vec{P}_\gamma = \vec{P}'_\text{e} + \vec{P}'_\gamma.$$

E, E' und $\vec{P}$, $\vec{P}'$ denotes energy and momentum before and after the collision.

d) The energy-momentum relation is

$$E_\text{e}^2 = c^2\vec{P}_\text{e}^{\,2} + m_\text{e}^2c^4.$$

Problem 4 (4 Points)

a) Write up the Schrödinger equation for an electron moving within a region with constant potential energy V. The total energy of the electron is $E > V$.
b) Calculate the wave function and momentum of an electron moving along the x-axis.
c) The electron moves with the total energy E within a region with $V = V_1 < E$ in the direction of increasing x values and hits a barrier where the potential energy changes from V_1 to $V_2 < E$. How large is the probability that the electron is reflected?
Hint: The wave function at the interface of the barrier is both continuous and continuously differentiable.

a) Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2\nabla^2}{2m_\mathrm{e}} + \mathrm{V}\right)\psi$$

b) The wave function is

$$\psi = \psi_0 \mathrm{e}^{\mathrm{i}(Px-Et)/\hbar}.$$

The momentum becomes with

$$E = \frac{P^2}{2m_\mathrm{e}} + \mathrm{V} \rightarrow P = \sqrt{2m_\mathrm{e}(E-V)}.$$

c) The condition on the interface $V_1 \rightarrow V_2$ is

$$\psi_\mathrm{i} + \psi_\mathrm{r} = \psi_\mathrm{t}.$$

Thereby denotes

$$\psi_\mathrm{i} = \psi_0 \mathrm{e}^{\mathrm{i}(P_1x-Et)/\hbar}$$
$$\psi_\mathrm{r} = R\psi_0 \mathrm{e}^{\mathrm{i}(-P_1x-Et)/\hbar}$$
$$\psi_\mathrm{t} = T\psi_0 \mathrm{e}^{\mathrm{i}(P_2x-Et)/\hbar}$$

the incident, reflected and transmitted wave. Due to the condition that the wave function is continuous and continuously differentiable on the interface one gets the usual formula for the reflection coefficient

$$R = \frac{P_1 - P_2}{P_1 + P_2}$$

and the probability for reflection is R^2 with $P_1 = \sqrt{2m_\mathrm{e}(E-V_1)}$ and $P_2 = \sqrt{2m_\mathrm{e}(E-V_2)}$.

Problem 5 (4 Points)

The angular momentum of an electron results from the addition of the orbital angular momentum $\vec{\ell}$ and its spin $\vec{s}$.

a) Give the quantum numbers and the range of these quantum numbers characterizing the orbital angular momentum and the spin.
b) The total angular momentum $\vec{j}$ results from the addition of $\vec{\ell}$ and $\vec{s}$. Give the quantum numbers and the range of these quantum numbers characterizing the total angular momentum.
c) Write up the relation between the magnetic moment of an electron and the orbital angular momentum. Write up the relation between the magnetic moment and the spin of an electron.
d) Calculate the potential energy of an electron in an homogeneous magnetic field $\vec{B}$ in terms of the quantum numbers of $\vec{\ell}$ and $\vec{s}$.

a) The orbital angular momentum $\vec{\ell}$ is characterized by the quantum number of $\vec{\ell}^2$ which takes the values

$$\ell = 0, 1, 2, ...$$

and the quantum number m of the z-component ℓ_z of the angular momentum with the values

$$|m| = 0, 1, 2, ... \leq \ell.$$

The square of the spin of an electron $\vec{s}^2$ has the quantum number $s = 1/2$ and the quantum number of the z-component s_z takes the values $m_s = \pm 1/2$.

b) The total angular momentum $\vec{j} = \vec{\ell} + \vec{s}$ is characterized by the quantum number j for $\vec{j}^2$ with

$$j = \ell - 1/2 \text{ or } j = \ell + 1/2$$

and the quantum number m_j for the z-component j_z with the values

$$|m_j| = 1/2, 3/2, ... \leq j.$$

c) The relation between the magnetic moment and the orbital angular momentum is

$$\vec{\mu}_\ell = -\mu_\mathrm{B} \frac{\vec{\ell}}{\hbar}.$$

Thereby denotes μ_B Bohr's magneton.
The relation between the magnetic moment and the spin of the electron is

$$\vec{\mu}_s = -g_\mathrm{e} \mu_\mathrm{B} \frac{\vec{s}}{\hbar}.$$

Thereby denotes g_e the g-factor of the electron with $g_\mathrm{e} = 2,002319...$

d) The potential energy of an electron in a magnetic field $\vec{B}$-Feld is

$$E_{\mathrm{pot}} = \mu_{\mathrm{B}} \left(\frac{\vec{\ell}}{\hbar} + g_{\mathrm{e}} \frac{\vec{s}}{\hbar} \right) \cdot \vec{B}.$$

When the direction of the magnetic field denotes the z-direction one gets

$$E_{\mathrm{pot}} = \mu_{\mathrm{B}} \left(m_{\ell} + g_{\mathrm{e}} m_{s} \right) \cdot B.$$

Problem 6 (4 Points)

The orbital angular momentum $\vec{\ell}$ of the electron and its spin $\vec{s}$ couple to the total angular momentum $\vec{j}$ according to $\xi\vec{\ell}\cdot\vec{s}$.

a) Explain the origin for the coupling of $\vec{\ell}$ and $\vec{s}$.
b) Give the sign of the coupling constant ξ.
c) Calculate the energy difference between the energy of the total angular momentum state with the highest and the smallest j-value, respectively.
d) Calculate the potential energy of an electron in an homogeneous magnetic field $\vec{B}$ in terms of the quantum numbers of $\vec{j}$.

a) The magnetic moment due to the spin of the electron orients in the magnetic field caused by the nucleus due to the orbital motion of the nucleus around the electron.

b) The sign of the coupling constant ξ is negative, i.e. spin and orbital angular momentum orient in opposite directions for minimal energy.

c) With $\vec{\ell}\cdot\vec{s} = \frac{1}{2}(\vec{j}^2 - \vec{\ell}^2 - \vec{s}^2)$ one gets for the energy $E_{j,\ell,s}$

$$E_{j,\ell,s} = \frac{\xi}{2}\hbar^2(j(j+1) - \ell(\ell+1) - s(s+1))$$

and

$$E_{j=\ell+1/2,\ell,s} - E_{j=\ell-1/2,\ell,s} = \frac{\xi}{2}\hbar^2\left((\ell+\frac{1}{2})(\ell+\frac{3}{2}) - (\ell-\frac{1}{2})(\ell+\frac{1}{2})\right)$$
$$= \frac{\xi}{2}\hbar^2(\ell+\frac{1}{2})(\ell+2).$$

d) The potential energy in an homogeneous magnetic field of an electron characterized by j and ℓ is

$$E_{\ell,j,m_j} = g_j\mu_B B m_j$$

with the g_j-factor of the electron.

($g_j = \frac{3j(j+1)-\ell(\ell+1)+3/4}{2j(j+1)}$, and $g_e = 2$)

Problem 7 (4 Points)

a) What is the meaning of Pauli's principle?
b) Explain the origin of the electronic band structure of a solid.
c) What is the definition of the Fermi energy?
d) Describe the band structure of a metal, a semiconductor and of an insulator.

a) Each quantum state can be occupied at most by one electron.

b) In a solid there are wave functions which are no longer localized on one atom. The difference between the energies of these quantum states decreases with the increasing number of atoms so that finally the discrete energy levels of single atoms merge into a continuous band of energy.

c) The Fermi energy denotes in a solid the highest energy of quantum states occupied by electrons.

d)
- The Fermi energy in a metal is located within an energy band, so that small energies are sufficient to excite an electron from one quantum state to a neighboring quantum state. Therefore a small amount of energy is sufficient to shift the electron density in space and the solid can conduct electric charge.
- All quantum states up to the highest energy of a band are occupied in a semi-conductor at $T \to 0$. When the energy gap to the next higher energy band is small, thermal energies are sufficient to excite electrons to quantum states of the next higher energy band. Then again a small amount of energy is sufficient to excite an electron from one quantum state to a neighboring quantum state and thermally activated conductivity of the semi-conductor results.
- Similar to a semi-conductor all quantum states up to the highest energy of a band are occupied in an insulator. In contrast to a semi-conductor is the energy gap to the next higher energy band in an insulator so large, that thermal activation of electrons can be neglected. Therefore it is not possible to change the occupation of quantum states by a small amount of energy and the electron density is fixed in space. The solid is an insulator.

Problem 8 (4 Points)

a) What is a phonon?
b) How is it possible to distinguish between acoustic and optical phonons?
c) In an experiment the light of a laser ($\lambda = 623$ nm) is scattered by a phonon at an angle of 90°. Calculate the momentum of the phonon.
d) Give the direction of the phonon with respect to the incident photon, when the phonon is excited by the photon.

a) Phonons are quantum particles forming the waves resulting from lattice vibrations.

b) An acoustic phonon belongs to a dispersion branch with

$$\lim_{q \to 0} \omega_q(q) = 0,$$

whereas an optical phonon belongs to a dispersion branch with

$$\lim_{q \to 0} \omega_q(q) \neq 0,$$

c) The momentum of a photon of the laser beam is

$$p_\gamma = \frac{h}{\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}}{623 \cdot 10^{-9}\,\text{m}} = 6,64 \cdot 10^{-9}\,\text{eVs/m}.$$

Since the energy of the photon $E = cp_\gamma$, i.e.

$$E/hc = (1/623 \cdot 10^{-9})\,\text{m}^{-1} = 16000\,\text{cm}^{-1}$$

is very much large than any phonon energy only the direction of the momentum is changed due to the scattering and the momentum of the phonon is

$$q = \sqrt{2}p_\gamma.$$

d) Since the angle between the momenta of the incident and scattered photon is 90°, the angle between the momenta of the incident photon and the excited phonon is 135°.

Problem 1 (4 Points)

An elementary particle moves with 90% of the speed of light 10 km through the atmosphere of the earth.

a) What is measured by an observer at rest on earth for the time the elementary particle needs to pass this distance?
b) Calculate this distance in a frame of reference moving with the particle.
c) Which time is necessary in that frame of reference to pass this distance?
d) Describe an experiment to test this effect experimentally.

Problem 2 (4 Points)

The neutrons in the bath of a reactor are assumed to be in thermal equilibrium with the moderating water at $T = 300\,\mathrm{K}$.

a) Calculate the neutron velocity v_{max} in the maximum of this velocity distribution.
b) Calculate the de Broglie wavelength of the neutrons with this velocity v_{max}.
c) Neutrons with this de Broglie wavelength pass a crystal lattice with a lattice constant $d = 3.0 \cdot 10^{-10}\,\mathrm{m}$. Calculate the angle of observation for the 1^{st} diffraction maximum with respect to the incident neutron beam.
d) What is the effect of the thermal motion of the atoms in the crystal on the diffraction of the neutrons?

Problem 3 (4 Points)

In the decay $^{60}\mathrm{Co} \rightarrow {}^{60}\mathrm{Ni}$ two photons with the energy 1.17 MeV and 1.33 MeV are emitted.

a) Calculate the wavelengths corresponding to the energies of these photons, respectively
b) Assume that the photons are scattered at electrons at rest. Calculate the energies of the scattered photons.
c) Calculate for both incident photon energies i) the smallest photon energy and ii) the highest electron energy after scattering, respectively

Problem 4 (4 Points)

a) Give the temperature on the surface of the Sun, if the maximum of the electromagnetic spectrum is measured at $\lambda = 500\,\mathrm{nm}$.
b) Give the intensity of the electromagnetic radiation of the Sun on Earth.
c) Calculate the temperature of the Earth. Assume that the Earth is a black body and a sphere formed by a perfect thermal conductor.

Problem 5 (4 Points)

a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.

d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

Problem 6 (4 Points)

There are two electrons in the neutral Helium atom. The excited states with lowest energy result from the excitation of one electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the energy level scheme, when the excited electron is lifted into a state with the quantum number $n = 2$. Denote the energy levels and explain the difference between para- and ortho-Helium. Neglect the influence of spin-orbit coupling and include the ground state in the sketch.
b) Explain the energy difference between the corresponding energy levels of the para- and ortho-Helium.
c) Give reason for the energy difference between states with the angular momentum $L = 0$ and $L = 1$.
d) Indicate in the energy level scheme of a) the allowed dipole transitions and give the corresponding selection rules.

Problem 7 (4 Points)

A crystal is formed by N unit cells each containing n atoms.

a) What is a phonon?
b) Give the number of acoustic and optic phonon branches, respectively.
c) Give the number of the phonon $\vec{q}$-modes.
d) The heat capacity of an insulating crystal is proportional to T^3 for $T \to 0$. Why?
e) Explain the meaning of Brillouin and Raman scattering.

Problem 8 (4 Points)

For a crystal lattice the solutions of the Schrödinger equation in the independent electron approximation are Bloch waves.

a) Explain the basic properties of Bloch waves.
b) Calculate the energy of a free electron described by a simple plane wave in a region of constant potential energy V.
c) When the electron energy $E(k)$ is independent of the direction of the $\vec{k}$ vector, the electrons occupy at $T = 0$ the Fermi sphere. Calculate the radius k_F of the Fermi sphere in the free electron approximation for N electrons.
d) Calculate the Fermi energy $E_F = E(k_F)$ for Cu in the free-electron approximation.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\mathrm{m/s}$
Mass of the electron:	$m_e = 500\,\mathrm{keV}/c^2$
Mass of the neutron:	$m_n = 939\,\mathrm{MeV}/c^2$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\mathrm{eVs}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\mathrm{As}$
Compton wavelength:	$\lambda_C = 2.426 \cdot 10^{-12}\,\mathrm{m}$

Stefan-Boltzmann's constant:	$\sigma = 5.67 \cdot 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$
Wien's constant:	$b = 2.9\,\mathrm{mm} \cdot \mathrm{K}$
Boltzmann constant:	$k_\mathrm{B} = 8.6 \cdot 10^{-5}\mathrm{eV/K}$
Rydberg unit of energy:	$\mathrm{Ry} = 13.6\,\mathrm{eV}$
Avogadro's number:	$N_\mathrm{A} = 6 \cdot 10^{23}\,\mathrm{mol^{-1}}$
Radius of the Earth:	$r_\mathrm{E} = 6.37 \cdot 10^{6}\,\mathrm{m}$
Radius of the Sun:	$r_\mathrm{S} = 6.96 \cdot 10^{8}\,\mathrm{m}$
Distance Sun-Earth:	$r_\mathrm{SE} = 1.5 \cdot 10^{11}\,\mathrm{m}$
Density of Copper:	$\rho_\mathrm{Cu} = 9\,\mathrm{g/cm^3}$
Molar mass of Cu:	$m_\mathrm{Cu} = 63.5\,\mathrm{g/mol}$

Problem 1 (4 Points)

An elementary particle moves with 90% of the speed of light 10 km through the atmosphere of the earth.

a) What is measured by an observer at rest on earth for the time the elementary particle needs to pass this distance?
b) Calculate this distance in a frame of reference moving with the particle.
c) Which time is necessary in that frame of reference to pass this distance?
d) Describe an experiment to test this effect experimentally.

a) Time for 10 km:

$$t_0 = 10 \cdot 10^3\,\mathrm{m}/(0.9 \cdot\ 3 \cdot 10^8\,\mathrm{m/s}) = 3.7 \cdot 10^{-5}\,\mathrm{s}$$

b) In the frame of the elementray particle moves the distance of 10 km with the velocity $-0.9 \cdot c$. The resulting length contraction is

$$\ell(v) = 10\,\mathrm{km}\sqrt{1-0.9^2} = 10\,\mathrm{km} \cdot 0.436 = 4.36\,\mathrm{km}.$$

c) The time necessary for the passage in the frame of the elementary partilce is

$$t_e = \frac{\ell(v)}{0.9 \cdot\ 3 \cdot 10^8\,\mathrm{m/s}} = t_0\sqrt{1-0.9^2} = 3.7 \cdot 10^{-5}\,\mathrm{s}\,0.436 = 1.6 \cdot 10^{-5}\,\mathrm{s}$$

d) Instead of the 16 μs needed by the particle to pass the distance an observer at rest measures 37 μs. The effect of time dilation, i.e.

$$t(v) = t_0 = \frac{t_e}{\sqrt{1-0.9^2}}$$

can be tested by using the life time of the elementray particle as a reference (e.g. the life time of the μ^- with $\tau_0 = 2.2 \cdot 10^{-6}$ s in its rest frame of reference). Starting with N_0 particles $N_1 = N_0 e^{-t/\tau}$ arrive after the passage of 10 km. The life time of the moving particle can be measured by counting the starting and arriving particles

$$\tau(v) = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

Problem 2 (4 Points)

The neutrons in the bath of a reactor are assumed to be in thermal equilibrium with the moderating water at $T = 300\,\mathrm{K}$.

a) Calculate the neutron velocity $\mathrm{v}_{\mathrm{max}}$ in the maximum of this velocity distribution.
b) Calculate the de Broglie wavelength of the neutrons with this velocity $\mathrm{v}_{\mathrm{max}}$.
c) Neutrons with this de Broglie wavelength pass a crystal lattice with a lattice constant $d = 3.0 \cdot 10^{-10}\,\mathrm{m}$. Calculate the angle of observation for the 1st diffraction maximum with respect to the incident neutron beam.
d) What is the effect of the thermal motion of the atoms in the crystal on the diffraction of the neutrons?

a) The kinetic energy is $E_{\mathrm{kin}} = k_{\mathrm{B}}T$.

$$\mathrm{v}_{\mathrm{max}} = c\sqrt{\frac{2k_{\mathrm{B}}T}{m_{\mathrm{n}}c^2}} = c\sqrt{\frac{2 \cdot 8.6 \cdot 10^{-5}\,\mathrm{eVK^{-1}} \cdot 300\,\mathrm{K}}{939 \cdot 10^6\,\mathrm{eV}}} = 7.4 \cdot 10^{-6} \cdot c = 2.22\,\mathrm{km/s}.$$

b) The de Broglie wavelenth is

$$\lambda = \frac{h}{p_{\mathrm{n}}} = \frac{\mathrm{c}^2 h}{m_{\mathrm{n}}c^2\,\mathrm{v}_{\mathrm{max}}} = \frac{3 \cdot 10^8\,\mathrm{ms^{-1}} 4.14 \cdot 10^{-15}\,\mathrm{eVs}}{939 \cdot 10^6\,\mathrm{eV}\, 7.4 \cdot 10^{-6}} = 1.8 \cdot 10^{-10}\,\mathrm{m}.$$

c) Bragg's law $n\lambda = 2d\sin\theta_n$

$$\sin\theta_1 = \frac{\lambda}{2d} = \frac{1.8 \cdot 10^{-10}}{2 \cdot 3 \cdot 10^{-10}\,\mathrm{m}} = 0.3 \rightarrow \theta_1 = 17.5°$$

The angle with respect to the incident neutron beam is $\pm 2\theta_1 \hat{=} \pm 35°$.

d) In addition to elastic scattering the absorption and emission of phonons leads to inelastic scattering and therefore to additional peaks in the diffraction pattern of the scattered neutrons.

Problem 3 (4 Points)

In the decay ^{60}Co $\rightarrow$ ^{60}Ni two photons with the energy 1.17 MeV and 1.33 MeV are emitted.

a) Calculate the wavelengths corresponding to the energies of these photons, respectively
b) Assume that the photons are scattered at electrons at rest. Calculate the energies of the scattered photons.
c) Calculate for both incident photon energies i) the smallest photon energy and ii) the highest electron energy after scattering, respectively

a) The wavelenth for 1.17 MeV is

$$E_\lambda = \frac{hc}{\lambda} \rightarrow \lambda = \frac{hc}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\, 3 \cdot 10^{8}\,\text{m/s}}{1.17 \cdot 10^{6}\,\text{eV}} = 10.6 \cdot 10^{-13}\,\text{m}.$$

and for 1.33 MeV

$$\lambda = \frac{hc}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\, 3 \cdot 10^{8}\,\text{m/s}}{1.33 \cdot 10^{6}\,\text{eV}} = 9.3 \cdot 10^{-13}\,\text{m}.$$

b) The increase of the wavelenth (i.e. the reduction of energy) due to the scattering on a free electron at rest is

$$\lambda' - \lambda = \lambda_C(1 - \cos\theta) \rightarrow \lambda' = \lambda + \lambda_C(1 - \cos\theta)$$

and the energy of the scattered photon is

$$E_{\lambda'} = \frac{hc}{\lambda'} = \frac{hc}{\lambda + \lambda_C(1 - \cos\theta)} \rightarrow E_{\lambda'}^{-1} = E_\lambda^{-1} + E_C^{-1}(1 - \cos\theta)$$

with $E_C = hc/\lambda_C = 4.14 \cdot 10^{-15}\,\text{eVs}\, 3 \cdot 10^{8}\,\text{ms}^{-1}/2.426 \cdot 10^{-12}\text{m} = 5,1 \cdot 10^{5}\,\text{eV}$

c) i) The smallest photon energy results from back-scattering $\theta = 180°$. With

$$E_{\text{min}}^{-1} = E_\lambda^{-1} + 2E_C^{-1}$$

one gets for 1.17 MeV

$$E_{\text{min}} = \frac{1}{E_\lambda^{-1} + 2E_C^{-1}} = \frac{\text{MeV}}{1.17^{-1} + 2 \cdot 0.51^{-1}} = 0.21\,\text{MeV}$$

and for 1.33 MeV

$$E_{\text{min}} = \frac{1}{E_\lambda^{-1} + 2E_C^{-1}} = \frac{\text{MeV}}{1.33^{-1} + 2 \cdot 0.51^{-1}} = 0.21\,\text{MeV}.$$

ii) The highest energy of the scattered electrons is for 1.17 MeV

$$E_{\text{max}} = 1.17\,\text{MeV} - 0.21\,\text{MeV} = 0.96\,\text{MeV}$$

and for 1.33 MeV

$$E_{\text{max}} = 1.33\,\text{MeV} - 0.21\,\text{MeV} = 1.12\,\text{MeV}.$$

Problem 4 (4 Points)

a) Give the temperature on the surface of the Sun, if the maximum of the electromagnetic spectrum is measured at $\lambda = 500\,\text{nm}$.
b) Give the intensity of the electromagnetic radiation of the Sun on Earth.
c) Calculate the temperature of the Earth. Assume that the Earth is a sphere formed by a perfect thermal conductor.

a) With Wien's displacement law

$$\lambda_{\text{max}} = \frac{b}{T}$$

one gets

$$T = \frac{2.9 \cdot 10^{-3}\,\text{m} \cdot \text{K}}{500 \cdot 10^{-9}\,\text{m}} = 5800\,\text{K}.$$

b) With the Stefan-Boltzmann law is the power of the Sun

$$P = 4\pi r_{\text{S}}^2 \sigma T^4$$

and the intensity on Earth

$$I = \left(\frac{r_{\text{S}}}{r_{\text{SE}}}\right)^2 \sigma T^4 = \left(\frac{6.96 \cdot 10^8\,\text{m}}{1.5 \cdot 10^{11}\,\text{m}}\right)^2 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4} 5800^4\,\text{K}^4$$
$$= (4.64 \cdot 10^{-3})^2 5.67 \cdot 10^{-8}\,\text{Wm}^{-2} 1.1 \cdot 10^{15} = 1340\,\text{W/m}^2.$$

c) The total power on Earth received from the Sun is

$$P_{\text{ES}} = \pi r_{\text{E}}^2 \cdot I.$$

When the Earth behaves like a black body is the total power emitted by the Earth

$$P_{\text{E}} = 4\pi r_{\text{E}}^2 \sigma T_{\text{E}}^4.$$

In equilibrium one has $P_{\text{ES}} = P_{\text{E}}$ and the temperature is

$$\pi r_{\text{E}}^2 \cdot I = 4\pi r_{\text{E}}^2 \sigma T_{\text{E}}^4 \rightarrow I = 4\sigma T_{\text{E}}^4$$

and

$$T_{\text{E}} = \left(\frac{I}{4\sigma}\right)^{1/4} = \left(\frac{1340\,\text{Wm}^{-2}}{4 \cdot 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}}\right)^{1/4} = 277\,\text{K}.$$

Problem 5 (4 Points)

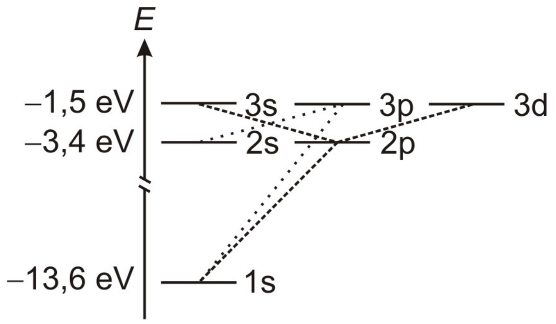
a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.
d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

a) The energy is with the Rydberg unit of energy

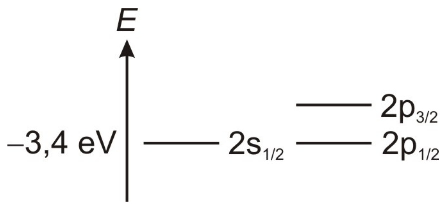
$$E_\text{n} = -\frac{\text{Ry}}{\text{n}^2},$$

i.e. $E_1 = -13.6\,\text{eV}$, $E_2 = -3.4\,\text{eV}$ and $E_3 = -1.5\,\text{eV}$.

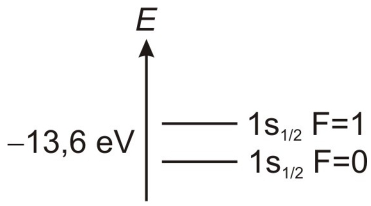
b) The energy level scheme and the allowed electric dipole transitions (dashed lines). The selection rule is $\Delta\ell = \pm 1$.



c) The energy level scheme of the first excited state. The p-orbital is splitted due to the spin-orbit-coupling into a state with the total angular momentum $j = 1/2$ and a state with $j = 3/2$.



d) The ground state of the hydrogen atom is splitted due to the hyperfine interaction with the proton. There are two states with the total angular momentum $F = 0$ and $F = 1$.

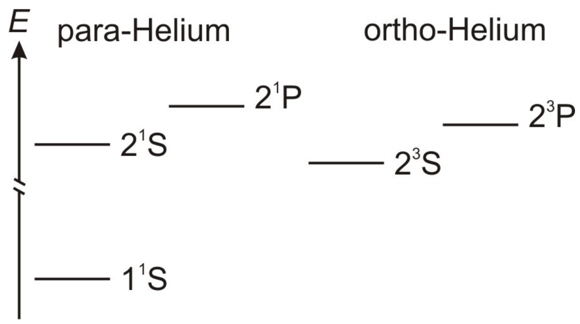


Problem 6 (4 Points)

There are two electrons in the neutral Helium atom. The excited states with lowest energy result from the excitation of one electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the energy level scheme, when the excited electron is lifted into a state with the quantum number n = 2. Denote the energy levels and explain the difference between para- and ortho-Helium. Neglect the influence of spin-orbit coupling and include the ground state in the sketch.
b) Explain the energy difference between the corresponding energy levels of the para- and ortho-Helium.
c) Give reason for the energy difference between states with the angular momentum $L = 0$ and $L = 1$.
d) Indicate in the energy level scheme of a) the allowed dipole transitions and give the corresponding selection rules.

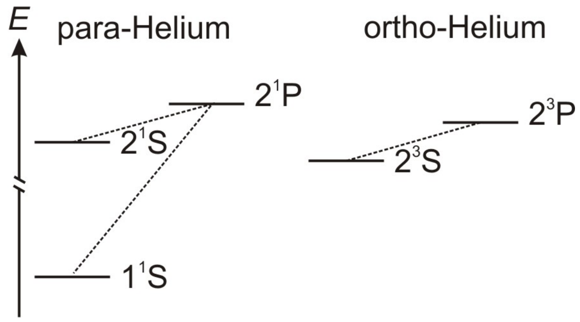
a) Energy level scheme of the [1s2s] and [1s2p] configuration of the helium atom. For para-Helium the spins of the electrons couple to the total spin $S = 0$. For the ortho-Helium the spins couple to $S = 1$.



b) The Coulomb repulsion between electrons is stronger when the spins are aligned antiparallel than when they are aligned parallel. Therefore, the binding energy for the singlet states is smaller than for the triplet states. The effect is called exchange interaction.

c) The screening of the nuclear charge is more effective for the p-electron than for the s-electron. There is no centrifugal potential for a s-electron so that it can even move through the nucleus. Therefore the binding energy is higher for s-electrons than for p-electrons.

d) In addition to $\Delta\ell = \pm 1$ the selection rule is $\Delta S = 0$. The spin quantum number is not changed by electric dipole transitions (dashed lines). The 2^3S is metastable since it cannot decay due to an electric dipole transition to 1^1S ground state.



Problem 7 (4 Points)

A crystal is formed by N unit cells each containing n atoms.

a) What is a phonon?
b) Give the number of acoustic and optic phonon branches, respectively.
c) Give the number of the phonon $\vec{q}$-modes.
d) The heat capacity of an insulating crystal is proportional to T^3 for $T \to 0$. Why?
e) Explain the meaning of Brillouin and Raman scattering.

a) Phonons are quantum particles forming the waves due to the oscillation of atoms in crystals. They carry energy according to the energy dispersion relation of lattice vibrations. The momentum is restricted to the 1$^{\text{st}}$ Brillouin zone due to the periodicity of the crystal lattice.
b) The number of phonon branches equals three times the number n of atoms in the unit cell. There are always three acoustic branches and $3(n-1)$ optical branches.

c) Due to the finite size of the crystal the wave vector modes $\vec{q}$ are discrete and the number of $\vec{q}$ modes equals the number of unit cells within the crystal lattice.

d) For $T \to 0$ only the acoustic phonon modes are thermally excited. The dispersion relation of acoustic phonons

$$\omega(q) = c_S q$$

for $T \to 0$ (c_S denotes the sound velocity) is similar to the dispersion relation of photons.Therefore the total energy of the phonons varies in analogy to the Stefan-Boltzmann law of photons proportional to T^4 and the heat capacity $\frac{\partial E}{\partial T}$ is proportional to T^3.

e) Electrons, neutrons, photon etc. are scattered at periodic the density modulations due to lattice vibrations. This corresponds to the absorption or emission of a phonon by the scattered particle. Since the wave number of photons is very small compared to the size of the reciprocal lattice vectors for visible light only phonons in the range of the Γ-point of the Brillouin zone can be absorbed or emitted. The absorption/emission of acoustic phonons by visible light is called Brillouin scattering whereas the absorption/emission of optical phonons by visible light is called Raman scattering.

Problem 8 (4 Points)

For a crystal lattice the solutions of the Schrödinger equation in the independent electron approximation are Bloch waves.

a) Explain the basic properties of Bloch waves.
b) Calculate the energy of a free electron described by a simple plane wave in a region of constant potential energy V.
c) When the electron energy $E(k)$ is independent of the direction of the $\vec{k}$ vector, the electrons occupy at $T = 0$ the Fermi sphere. Calculate the radius k_F of the Fermi sphere in the free electron approximation for N electrons.
d) Calculate the Fermi energy $E_\mathrm{F} = \mathrm{E}(k_\mathrm{F})$ for Cu in the free-electron approximation.

a) Since the electron density is periodic in the crystal lattice one has

$$|\psi_{\vec{k}}(\vec{r},t)|^2 = |\psi_{\vec{k}}(\vec{r}+\vec{R},t)|^2.$$

The Bloch waves are also periodic in the reciprocal lattice

$$\psi_{\vec{k}}(\vec{r},t) = \psi_{\vec{k}+\vec{G}}(\vec{r},t).$$

$\vec{R}$ is a vector of the Bravais lattice and $\vec{G}$ is a vector of the reciprocal lattice.

b) The wave function of a plane wave is

$$\psi(\vec{r},t) = \psi_0 e^{i(\vec{k}\vec{r}-\omega t)}.$$

With the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial \mathrm{t}} = -\frac{\hbar^2\nabla^2}{2m}\psi + V\psi$$

one gets

$$E = \hbar\omega = \frac{\hbar^2k^2}{2m} + V$$

with $k^2 = (\vec{k})^2$.

c) Each $\vec{k}$ state takes a volume

$$\frac{(2\pi)^3}{V}$$

in the reciprocal lattice. V denotes the volume of the crystal. The number of $\vec{k}$ states in a sphere with radius k is

$$\frac{\frac{4}{3}\pi k^3}{\frac{(2\pi)^3}{V}} = \frac{Vk^3}{6\pi^2}.$$

Since each $\vec{k}$ state can be occupied by two electrons one has

$$N = 2\,\frac{V k_\mathrm{F}^3}{6\pi^2}$$

and $k_\mathrm{F} = \sqrt[3]{3\pi^2 N/V}$.

d) When there is one valence electron per Cu atom the electron density is

$$\frac{N}{V} = N_A \frac{\rho_{Cu}}{m_{Cu}} = 6 \cdot 10^{23} \text{mol}^{-1} \frac{9\text{g} \cdot 10^6 \text{m}^{-3}}{63.5\,\text{gmol}^{-1}} = 8.5 \cdot 10^{28} \text{m}^{-3}.$$

The Fermi wavenumber is

$$k_F = \sqrt[3]{3\pi^2 N/V} = \sqrt[3]{3\pi^2 8.5 \cdot 10^{28} \text{m}^{-3}} = 1.36 \cdot 10^{10}\,\text{m}^{-1}$$

and the Fermi energy

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{c^2 \hbar^2 k_F^2}{2m_e c^2} = \frac{(3 \cdot 10^8 \text{ms}^{-1} 4.14 \cdot 10^{-15} \text{eVs} 1.36 \cdot 10^{10}\,\text{m}^{-1})^2}{(2\pi)^2 2 \cdot 5 \cdot 10^5 \text{eV}} = 7.3\,\text{eV}$$

Problem 1 (4 Points)

K^+ mesons are produced in a height of 10 km above ground. The half-life time of K^+ mesons is $\tau_{1/2} = 8.6 \cdot 10^{-9}$ s.

a) Calculate the minimum velocity of the K^+ mesons, for which at least half of the particles are expected to reach the ground.
b) What is the time of flight t_E of the K^+ mesons in the reference frame of the earth?
c) How is the time of flight t_E related to the half-life time $\tau_{1/2}$ of the K^+ mesons?

Problem 2 (4 Points)

The kinetic energy of cold neutrons is 0.1 meV.

a) Calculate the speed of these neutrons.
b) Calculate the de Broglie wavelength of these neutrons.
c) Spherical viruses form a crystal with the lattice constant $d = 1.0 \cdot 10^{-8}$ m. Calculate for these neutrons the angle under which the 1st order diffraction maximum can be observed with respect to the incident neutron beam.
d) Calculate the length of the corresponding vector of the reciprocal lattice.

Problem 3 (4 Points)

During the decay $^{22}\text{Na} \rightarrow {}^{22}\text{Ne}$ one photon with the energy 1.28 MeV is emitted.

a) Calculate the wavelength of the corresponding electromagnetic wave.
b) Assume that the photons are scattered at electrons at rest. Calculate the energies of the scattered photons.
c) Calculate i) the smallest photon energy and ii) the highest electron energy after scattering.

Problem 4 (4 Points)

On a hot day (30° C) a black car is exposed to the plain sun. The intensity of the sun is $700\,\text{W/m}^2$.

a) What is the temperature of the car in thermal equilibrium when only half of its surface is exposed to the sun? Assume that the car behaves like a black body and neglect the effect of thermal conduction.
b) Sketch the radiation spectrum of a black body.
c) How large is the wavelength of the maximum of the spectrum for the car in thermal equilibrium?
d) Repeat the calculation for a car which reflects 30 % of the incident radiation. How large is the equilibrium temperature of this car?

Problem 5 (4 Points)

a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.

d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

Problem 6 (4 Points)

There are two electrons in the neutral Helium atom. The excited states with lowest energy result from the excitation of one electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the expected energy level scheme and denote the energy levels, when the excited electron is lifted into a state with the quantum number $n = 3$. Ignore spin-orbit coupling.
b) Give reasons for the energy difference between states of different angular momenta L.
c) Explain the energy difference between spin singlet and triplet states.
d) Give the states of problem a), which can be excited from the groundstate by electric dipole transitions.

Problem 7 (4 Points)

a) The vectors of a primitive unit cell of the bcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x}-\vec{y}+\vec{z})$, $\vec{a}_2 = \frac{a}{2}(\vec{y}-\vec{z}+\vec{x})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z}-\vec{x}+\vec{y})$. $\vec{x}$, $\vec{y}$ and $\vec{z}$ denote orthogonal unit vectors. What is the volume of the unit cell?
b) Give the primitive vectors of the corresponding reciprocal lattice.
c) The vectors of a primitive unit cell of the fcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x}+\vec{y})$, $\vec{a}_2 = \frac{a}{2}(\vec{y}+\vec{z})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z}+\vec{x})$. What is the volume of the unit cell?
d) Give the primitive vectors of the corresponding reciprocal lattice.

Problem 8 (4 Points)

a) Calculate the energy of a free electron described by a simple plane wave in a region of constant potential energy V.
b) Calculate the density of conduction electrons of potassium. Hint: The mass of the potassium atom is $39u$ and there is one conduction electron per atom.
c) Calculate the Fermi wavenumber k_F of potassium.
d) Calculate the Fermi energy of potassium.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\text{m/s}$
Mass of the electron:	$m_e = 500\,\text{keV}/c^2$
Mass of the neutron:	$m_n = 939\,\text{MeV}/c^2$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\text{eVs}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\text{As}$
Compton wavelength:	$\lambda_C = 2.426 \cdot 10^{-12}\,\text{m}$
Stefan-Boltzmann's constant:	$\sigma = 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}$
Wien's constant:	$b = 2.9\,\text{mm} \cdot \text{K}$
Boltzmann constant:	$k_B = 8.6 \cdot 10^{-5}\text{eV/K}$

Rydberg unit of energy:	$R = 13.6\,\text{eV}$
Density of potassium:	$\rho_\text{K} = 0.86\,\text{g}/\text{cm}^3$
Atomic mass unit:	$u = 1.66 \cdot 10^{-27}\,\text{kg}$

Problem 1 (4 Points)

K^+ mesons are produced in a height of 10 km above ground. The half-life time of K^+ mesons is $\tau_{1/2} = 8.6 \cdot 10^{-9}$ s.

a) Calculate the minimum velocity of the K^+ mesons, for which at least half of the particles are expected to reach the ground.
b) What is the time of flight t_E of the K^+ mesons in the reference frame of the earth?
c) How is the time of flight t_E related to the half-life time $\tau_{1/2}$ of the K^+ mesons?

a) The velocity of the K^+ mesons should be without relativity at least

$$v = \frac{10 \cdot 10^3\,\text{m}}{8.6 \cdot 10^{-9}\,\text{s}} = 1.16 \cdot 10^{12}\,\text{m/s} >> c.$$

Relativity is necessary to understand why K^+ mesons reach ground. The condition for the time of flight is

$$t = \frac{\ell(v)}{v} < \tau_{1/2},$$

Therefore due to length contraction

$$t = \frac{\ell_0\sqrt{1-(v/c)^2}}{v} < \tau_{1/2},$$

and

$$1 - (v/c)^2 < \left(\frac{v\tau_{1/2}}{\ell_0}\right)^2 = (\text{v/c}^2)\left(\frac{c\tau_{1/2}}{\ell_0}\right)^2$$

and

$$\left(\frac{v}{c}\right) > \frac{1}{\sqrt{1+\left(\frac{c\tau_{1/2}}{\ell_0}\right)^2}},$$

$$\text{v} > \text{c}\left(1 - \frac{1}{2}\left(\frac{3\cdot 10^8\,\text{m/s}\cdot 8.6\cdot 10^{-9}\,\text{s}}{10^4\,\text{m}}\right)^2\right) = \text{c}(1 - 3.125\cdot 10^{-8}).$$

b) The time of flight in the frame of the earth is

$$t_E = \frac{10\,\text{km}}{c} = \frac{10^4\,\text{m}}{3\cdot 10^8\,\text{m/s}} = 3.33\,\text{ms}.$$

c) In the reference frame of the earth is the time of flight smaller than the half-life time of the K^+ mesons, which is prolonged due to the effect of time dilation

$$t_E \leq \tau_{1/2}(v) = \tau_{1/2}\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

Problem 2 (4 Points)

The kinetic energy of cold neutrons is 0.1 meV.

a) Calculate the speed of these neutrons.
b) Calculate the de Broglie wavelength of these neutrons.
c) Spherical viruses form a crystal with the lattice constant $d = 1.0 \cdot 10^{-8}$ m. Calculate for these neutrons the angle under which the 1st order diffraction maximum can be observed with respect to the incident neutron beam.
d) Calculate the length of the corresponding vector of the reciprocal lattice.

a) The speed of the neutrons is

$$E_{\text{kin}} = \frac{1}{2} m_{\text{n}} c^2 \left(\frac{v}{c}\right)^2 \rightarrow v = c\sqrt{\frac{2E_{\text{kin}}}{m_{\text{n}}c^2}} = 3 \cdot 10^8\,\text{m/s}\sqrt{\frac{2 \cdot 0.1 \cdot 10^{-3}\,\text{eV}}{939 \cdot 10^6\,\text{eV}}} = 138.5\,\text{m/s}$$

b) The de Broglie wave length is

$$\lambda = \frac{h}{p} = \frac{h}{mc^2\left(\frac{v}{c^2}\right)} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,(3 \cdot 10^8\,\text{m/s})^2}{939 \cdot 10^6\,\text{eV}\,138.5\,\text{m/s}} = 2.87 \cdot 10^{-9}\,\text{m}.$$

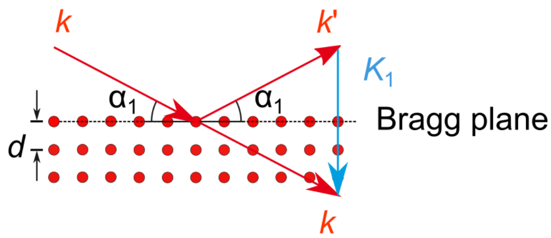
c) With Bragg's law

$$\lambda = 2d \sin\alpha_1 \rightarrow \sin\alpha_1 = \frac{\lambda}{2d} = \frac{2.87 \cdot 10^{-9}\,\text{m}}{10^{-8}\,\text{m}} = 0.287 \rightarrow \alpha_1 = 16.43°$$

and the angle with respect to the incident beam is $2\alpha_1 = 32.9°$.

d) The length of the reciprocal lattice vector is (compare the following sketch, $k = k' = 2\pi/\lambda$)

$$K_1 = \frac{2\pi}{d} = \frac{2\pi}{10^{-8}\,\text{m}} = 2\pi \cdot 10^8\,\text{m}^{-1}.$$



Problem 3 (4 Points)

During the decay $^{22}\text{Na} \to {}^{22}\text{Ne}$ one photon with the energy 1.28 MeV is emitted.

a) Calculate the wavelength of the corresponding electromagnetic wave.
b) Assume that the photons are scattered at electrons at rest. Calculate the energies of the scattered photons.
c) Calculate i) the smallest photon energy and ii) the highest electron energy after scattering.

a) The wavelength for 1.28 MeV is

$$E_\lambda = \frac{hc}{\lambda} \to \lambda = \frac{hc}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^8\,\text{m/s}}{1.28 \cdot 10^6\,\text{eV}} = 9,7 \cdot 10^{-13}\,\text{m}$$

b) The formula for the Compton scattering is

$$\lambda' - \lambda = \lambda_\text{C}(1 - \cos\theta) \to \lambda' = \lambda + \lambda_\text{C}(1 - \cos\theta).$$

λ denotes the wavelength of the incident wave, λ' the wavelength of the scattered wave and θ the scattering angle with respect to the incident beam. The energy of the scattered photon is then

$$E_{\lambda'} = \frac{hc}{\lambda'} = \frac{hc}{\lambda + \lambda_\text{C}(1 - \cos\theta)} \to E_{\lambda'}^{-1} = E_\lambda^{-1} + \text{E}_\text{C}^{-1}(1 - \cos\theta).$$

Thereby E_C denotes

$$E_\text{C} = hc/\lambda_\text{C} = 4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^8\,\text{ms}^{-1}/2.426 \cdot 10^{-12}\text{m} = 5.1 \cdot 10^5\,\text{eV},$$

which is the rest energy of the electron

c) i) The smallest photon energy results from back-scattering i.e. $\theta = 180°$. Therefore

$$E_\text{min}^{-1} = E_\lambda^{-1} + 2\text{E}_\text{C}^{-1}$$

$$E_\text{min} = \frac{1}{E_\lambda^{-1} + 2\text{E}_\text{C}^{-1}} = \frac{\text{MeV}}{1.28^{-1} + 2 \cdot 0.51^{-1}} = 0.21\,\text{MeV}.$$

ii) The highest energy of the scattered electron is then

$$E_\text{max} = 1.28\,\text{MeV} - 0.21\,\text{MeV} = 1.07\,\text{MeV}.$$

Problem 4 (4 Points)

On a hot day (30° C) a black car is exposed to the plain sun. The intensity of the sun is 700 W/m^2.

a) What is the temperature of the car in thermal equilibrium when only half of its surface is exposed to the sun? Assume that the car behaves like a black body and neglect the effect of thermal conduction.
b) Sketch the radiation spectrum of a black body.
c) How large is the wavelength of the maximum of the spectrum for the car in thermal equilibrium?
d) Repeat the calculation for a car which reflects 30 % of the incident radiation. How large is the equilibrium temperature of this car?

a) In equilibrium one can write

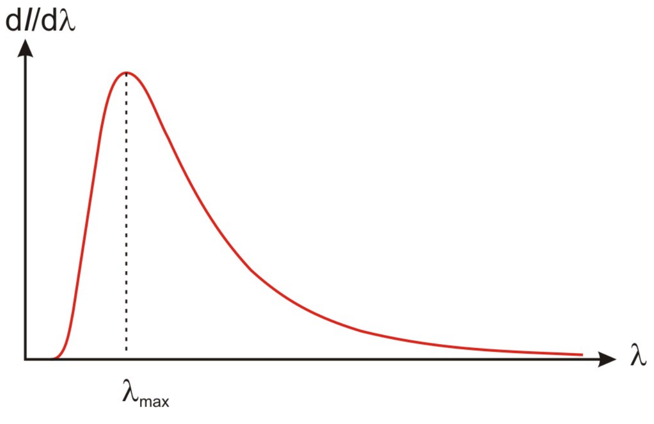
$$700\,\mathrm{Wm^{-2}}\frac{A}{2} = A\sigma(T_C^4 - T_0^4).$$

Thereby T_0 denotes the temperature of the environment and T_C the equilibrium temperature of the car

$$T_C = \left(\frac{350\,\mathrm{Wm^{-2}}}{\sigma} + T_0^4\right)^{1/4} = \left(\frac{350\,\mathrm{Wm^{-2}}}{5.67\cdot 10^{-8}\mathrm{Wm^{-2}K^{-4}}} + (273\,\mathrm{K} + 30\,\mathrm{K})^4\right)^{1/4}$$
$$= 347.6\,\mathrm{K}$$

This is a temperature of 74.6° C.

b) The radiation spectrum of a black body is



$I = P/A$ denotes the intensity of the radiation.

c) With Wien's law λ_{max} becomes

$$\lambda_{max} = \frac{b}{T} = \frac{2.9\,\mathrm{mmK}}{347.6\,\mathrm{K}} = 8.3\cdot 10^{-6}\,\mathrm{m}.$$

d) With problem a)

$$T_C = \left(\frac{0.7\cdot 350\,\mathrm{Wm^{-2}}}{5.67\cdot 10^{-8}\mathrm{Wm^{-2}K^{-4}}} + (273\,\mathrm{K} + 30\,\mathrm{K})^4\right)^{1/4} = 336\,\mathrm{K}$$

the equilibrium temperature is 63° C.

Problem 5 (4 Points)

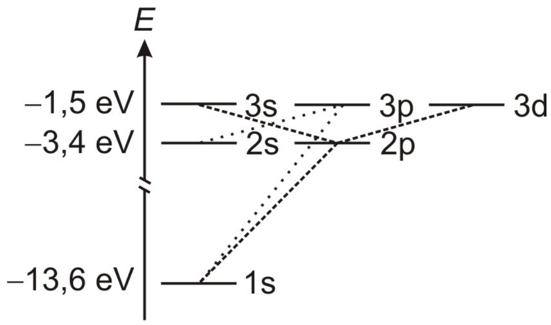
a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.
d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

a) The energy is with the Rydberg unit of energy

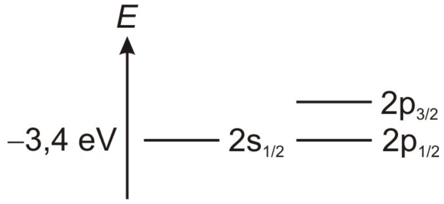
$$E_n = -\frac{R}{n^2},$$

i.e. $E_1 = -13.6\,\text{eV}$, $E_2 = -3.4\,\text{eV}$ and $E_3 = -1.5\,\text{eV}$.

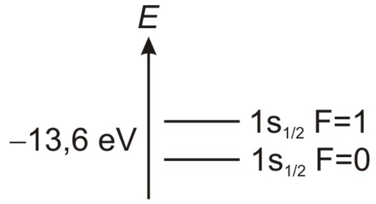
b) The energy level scheme and the allowed electric dipole transitions (dashed lines). The selection rule is $\Delta\ell = \pm 1$.



c) The energy level scheme of the first excited state. The p-orbital is splitted due to the spin-orbit-coupling into a state with the total angular momentum $j = 1/2$ and a state with $j = 3/2$.



d) The ground state of the hydrogen atom splits due to the hyperfine interaction with the proton. The spin of the electron and the proton couple to the total angular momentum $F = 0$ and $F = 1$. The energy splitting corresponds to a wavelength of 21 cm

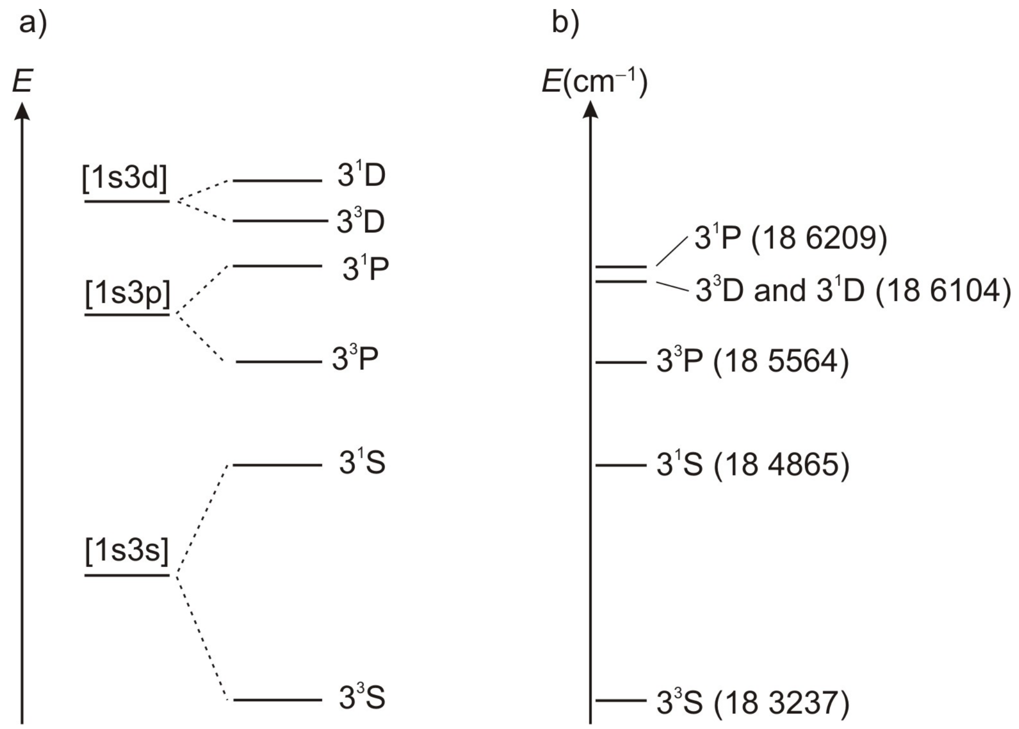


Problem 6 (4 Points)

There are two electrons in the neutral Helium atom. The excited states with lowest energy result from the excitation of one electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the expected energy level scheme and denote the energy levels, when the excited electron is lifted into a state with the quantum number n = 3. Ignore spin-orbit coupling.
b) Give reasons for the energy difference between states of different angular momenta L.
c) Explain the energy difference between spin singlet and triplet states.
d) Give the states of problem a), which can be excited from the groundstate by electric dipole transitions.

a) The following figure a) shows the sketch of the expected energy level scheme[1].



b) The mean distance between the electron cloud and the nucleus increases with increasing orbital angular momentum. Therefore the screening of the nuclear charge by the 1s electron is more effective for higher quantum numbers of the angular momentum of the excited electron.

[1] Figure b) shows the experimental results. The exchange interaction is more effective than the screening of the nuclear charge. Without exchange is the energy of the [1s3d] configuration only slightly higher than the [1s3p] configuration. The exchange interaction of the [1s3d] configuration is very small.

c) In the singlet state the electron-electron repulsion is stronger than in the triplet state, since the two electrons occupy the same orbital and the mean electron-electron distance is smaller than in the triplet state. Therefore the binding energy of the singlet state is reduced. The effect is known as exchange interaction.

d) The election rules for electric dipole transitions are $\Delta S = 0$ and $\Delta\ell = \pm 1$. Therefore only the transition 1^1S $\leftrightarrow$ 3^1P is possible by electric dipole radiation.

Problem 7 (4 Points)

a) The vectors of a primitive unit cell of the bcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x} - \vec{y} + \vec{z})$, $\vec{a}_2 = \frac{a}{2}(\vec{y} - \vec{z} + \vec{x})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z} - \vec{x} + \vec{y})$. $\vec{x}$, $\vec{y}$ and $\vec{z}$ denote orthogonal unit vectors. What is the volume of the unit cell?
b) Give the primitive vectors of the corresponding reciprocal lattice.
c) The vectors of a primitive unit cell of the fcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x} + \vec{y})$, $\vec{a}_2 = \frac{a}{2}(\vec{y} + \vec{z})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z} + \vec{x})$. What is the volume of the unit cell?
d) Give the primitive vectors of the corresponding reciprocal lattice.

a) There are two lattice points in the simple cubic unit cell. Therefore the volume of the primitive unit cell of the bcc lattice is

$$V_{\text{Cell}} = \frac{a^3}{2}.$$

b) The primitive vectors of the reciprocal lattice are

$$\vec{b}_1 = \frac{2\pi}{V_{\text{Cell}}} \vec{a}_2 \times \vec{a}_3,$$

i.e.

$$\vec{b}_1 = \frac{4\pi}{4a} [(\vec{y} - \vec{z} + \vec{x}) \times (\vec{z} - \vec{x} + \vec{y})] = \frac{\pi}{a} (\vec{x} + \vec{z} + \vec{y} + \vec{x} - \vec{y} + \vec{z}) = \frac{2\pi}{a} (\vec{x} + \vec{z})$$

By cyclic permutation one gets

$$\vec{b}_2 = \frac{2\pi}{a} (\vec{y} + \vec{x})$$

and

$$\vec{b}_3 = \frac{2\pi}{a} (\vec{z} + \vec{y})$$

c) There are four lattice points in the simple cubic unit cell. Therefore the volume of the primitive cell of the fcc lattice is

$$V_{\text{Cell}} = \frac{a^3}{4}.$$

d) The primitive vectors of the reciprocal lattice are

$$\begin{aligned} \vec{b}_1 &= \frac{2\pi}{V_{\text{Cell}}} \vec{a}_2 \times \vec{a}_3 \\ &= \frac{2\pi}{a} [(\vec{y} + \vec{z}) \times (\vec{z} + \vec{x})] \\ &= \frac{2\pi}{a} (\vec{x} - \vec{z} + \vec{y}) \end{aligned}$$

and by cyclic permutation

$$\vec{b}_2 = \frac{2\pi}{a}\left(\vec{y} - \vec{x} + \vec{z}\right)$$
$$\vec{b}_3 = \frac{2\pi}{a}\left(\vec{z} - \vec{y} + \vec{x}\right).$$

(Conclusion: The reciprocal lattice of the bcc lattice is a fcc lattice and vice versa.)

Problem 8 (4 Points)

a) Calculate the energy of a free electron described by a simple plane wave in a region of constant potential energy V.
b) Calculate the density of conduction electrons of potassium. Hint: The mass of the potassium atom is $39u$ and there is one conduction electron per atom.
c) Calculate the Fermi wavenumber k_F of potassium.
d) Calculate the Fermi energy of potassium.

a) With the Schrödinger equation is the energy of a plane wave

$$\psi = \psi_0 e^{i(\vec{k}\vec{r}-\omega t)}$$

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\nabla^2}{2m_e}\psi + V\psi \rightarrow E = \hbar\omega(\vec{k}) = \frac{\hbar^2\vec{k}^2}{2m_e} + V$$

b) With the number of atoms N and the density of potassium one gets

$$\rho_K = \frac{m}{V} = \frac{N \cdot 39u}{V}$$

and the electron density is

$$\frac{N}{V} = \frac{\rho_K}{39u} = \frac{0.86 \cdot 10^{-3}\,\mathrm{kg/cm^3}}{39 \cdot 1.66 \cdot 10^{-27}\,\mathrm{kg}} = 1.33 \cdot 10^{22}\,1/\mathrm{cm}^3 = 1.33 \cdot 10^{28}\,1/\mathrm{m}^3.$$

c) The Fermi wavenumber results from

$$\frac{N}{2} = \frac{4\pi k_F^3 V}{3(2\pi)^3} \rightarrow k_F = \left(3\pi^2\frac{N}{V}\right)^{1/3} = \left(3\pi^2 1.33 \cdot 10^{28}\,1/\mathrm{m}^3\right)^{1/3} = 7.3 \cdot 10^9\,\mathrm{m}^{-1}.$$

d) The Fermi energy is

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{h^2 k_F^2}{8\pi^2 m_e} = \frac{(4.14 \cdot 10^{-15}\,\mathrm{eVs})^2(7.3 \cdot 10^9\,\mathrm{m}^{-1})^2(3 \cdot 10^8)^2\,\mathrm{m^2 s^{-2}}}{8\pi^2 \cdot 500 \cdot 10^3\,\mathrm{eV}}$$
$$= 2.1\,\mathrm{eV}.$$

Problem 1 (4 Points)

K^+ mesons are produced in a height of 10 km above ground. The half-life time of K^+ mesons is $\tau_{1/2} = 8.6 \cdot 10^{-9}$ s.

a) Calculate the minimum velocity of the K^+ mesons, for which at least half of the particles are expected to reach the ground.
b) What is the time of flight t_E of the K^+ mesons in the reference frame of the earth?
c) How is the time of flight t_E related to the half-life time $\tau_{1/2}$ of the K^+ mesons?

Problem 2 (4 Points)

The kinetic energy of cold neutrons is 0.1 meV.

a) Calculate the speed of these neutrons.
b) Calculate the de Broglie wavelength of these neutrons.
c) Spherical viruses form a crystal with the lattice constant $d = 1.0 \cdot 10^{-8}$ m. Calculate for these neutrons the angle under which the 1st order diffraction maximum can be observed with respect to the incident neutron beam.
d) Calculate the length of the corresponding vector of the reciprocal lattice.

Problem 3 (4 Points)

During the decay ^{22}Na $\to$ ^{22}Ne one photon with the energy 1.28 MeV is emitted.

a) Calculate the wavelength of the corresponding electromagnetic wave.
b) Assume that the photons are scattered from electrons at rest. Calculate the energies of the scattered photons.
c) Calculate i) the smallest photon energy and ii) the highest kinetic energy of the electron after scattering.

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On a hot day (30° C) a black car is exposed to the plain sun. The intensity of the sun is 700 W/m^2.

a) What is the temperature of the car in thermal equilibrium when only half of its surface is exposed to the sun? Assume that the car behaves like a black body and neglect the effect of thermal conduction.
b) Sketch the radiation spectrum of a black body.
c) How large is the wavelength of the maximum of the spectrum for the car in thermal equilibrium?
d) Repeat the calculation for a car which reflects 30 % of the incident radiation. How large is the equilibrium temperature of this car?

Problem 5 (4 Points)

a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.

d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

Problem 6 (4 Points)

There are two electrons in the neutral Helium atom. The excited states with lowest energy result from the excitation of one electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the expected energy level scheme and denote the energy levels, when the excited electron is lifted into a state with the quantum number $n = 3$. Ignore spin-orbit coupling.
b) Give reasons for the energy difference between states of different angular momenta L.
c) Explain the energy difference between spin singlet and triplet states.
d) Give the states of problem a), which can be excited from the groundstate by electric dipole transitions.

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d) Give the primitive vectors of the corresponding reciprocal lattice.

Problem 8 (4 Points)

a) Calculate the energy of a free electron described by a simple plane wave in a region of constant potential energy V.
b) Calculate the density of conduction electrons of potassium. Hint: The mass of the potassium atom is $39u$ and there is one conduction electron per atom.
c) Calculate the Fermi wavenumber k_F of potassium.
d) Calculate the Fermi energy of potassium.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\mathrm{m/s}$
Mass of the electron:	$m_e = 500\,\mathrm{keV}/c^2$
Mass of the neutron:	$m_n = 939\,\mathrm{MeV}/c^2$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\mathrm{eVs}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\mathrm{As}$
Compton wavelength:	$\lambda_C = 2.426 \cdot 10^{-12}\,\mathrm{m}$
Stefan-Boltzmann's constant:	$\sigma = 5.67 \cdot 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$
Wien's constant:	$b = 2.9\,\mathrm{mm} \cdot \mathrm{K}$
Boltzmann constant:	$k_B = 8.6 \cdot 10^{-5}\mathrm{eV/K}$

Rydberg unit of energy:	$R = 13.6\,\text{eV}$
Density of potassium:	$\rho_\text{K} = 0.86\,\text{g}/\text{cm}^3$
Atomic mass unit:	$u = 1.66 \cdot 10^{-27}\,\text{kg}$

Problem 1 (4 Points)

K^+ mesons are produced in a height of 10 km above ground. The half-life time of K^+ mesons is $\tau_{1/2} = 8.6 \cdot 10^{-9}$ s.

a) Calculate the minimum velocity of the K^+ mesons, for which at least half of the particles are expected to reach the ground.

b) What is the time of flight t_E of the K^+ mesons in the reference frame of the earth?

c) How is the time of flight t_E related to the half-life time $\tau_{1/2}$ of the K^+ mesons?

a) The velocity of the K^+ mesons should be without relativity at least

$$v = \frac{10 \cdot 10^3\,\mathrm{m}}{8.6 \cdot 10^{-9}\,\mathrm{s}} = 1.16 \cdot 10^{12}\,\mathrm{m/s} >> c.$$

Relativity is necessary to understand why K^+ mesons reach ground. The condition for the time of flight is

$$t = \frac{\ell(v)}{v} < \tau_{1/2},$$

Therefore due to length contraction

$$t = \frac{\ell_0\sqrt{1-(v/c)^2}}{v} < \tau_{1/2},$$

and

$$1 - (v/c)^2 < \left(\frac{v\tau_{1/2}}{\ell_0}\right)^2 = (\mathrm{v/c^2})\left(\frac{c\tau_{1/2}}{\ell_0}\right)^2$$

and

$$\left(\frac{v}{c}\right) > \frac{1}{\sqrt{1+\left(\frac{c\tau_{1/2}}{\ell_0}\right)^2}},$$

$$\mathrm{v} > \mathrm{c}\left(1 - \frac{1}{2}\left(\frac{3 \cdot 10^8\,\mathrm{m/s} \cdot 8.6 \cdot 10^{-9}\,\mathrm{s}}{10^4\,\mathrm{m}}\right)^2\right) = \mathrm{c}(1 - 3.125 \cdot 10^{-8}).$$

Remark: It would be better to ask for the minimum energy of the K^+ *mesons. The problem avoids this additional complication:*

$$E = \frac{m_0c^2}{\sqrt{1-(v/c)^2}} > m_0c^2/(c\tau_{1/2}/\ell_0) = 3{,}88 \cdot 10^3\, m_0c^2.$$

b) The time of flight in the frame of the earth is

$$t_E = \frac{10\,\mathrm{km}}{c} = \frac{10^4\,\mathrm{m}}{3 \cdot 10^8\,\mathrm{m/s}} = 33.3\,\mu\mathrm{s}.$$

c) In the reference frame of the earth is the time of flight smaller than the half-life time of the K^+ mesons, which is prolonged due to the effect of time dilation

$$t_E \le \tau_{1/2}(v) = \tau_{1/2}\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

Problem 2 (4 Points)

The kinetic energy of cold neutrons is 0.1 meV.

a) Calculate the speed of these neutrons.
b) Calculate the de Broglie wavelength of these neutrons.
c) Spherical viruses form a crystal with the lattice constant $d = 1.0 \cdot 10^{-8}$ m. Calculate for these neutrons the angle under which the 1st order diffraction maximum can be observed with respect to the incident neutron beam.
d) Calculate the length of the corresponding vector of the reciprocal lattice.

a) The speed of the neutrons is

$$E_{\text{kin}} = \frac{1}{2} m_{\text{n}} c^2 \left(\frac{v}{c}\right)^2 \rightarrow v = c\sqrt{\frac{2E_{\text{kin}}}{m_{\text{n}}c^2}} = 3 \cdot 10^8\,\text{m/s}\sqrt{\frac{0.2 \cdot 10^{-3}\,\text{eV}}{939 \cdot 10^6\,\text{eV}}} = 138.4\,\text{m/s}$$

b) The de Broglie wave length is

$$\lambda = \frac{h}{p} = \frac{h}{mc^2\left(\frac{v}{c^2}\right)} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,(3 \cdot 10^8\,\text{m/s})^2}{939 \cdot 10^6\,\text{eV}\,138.4\,\text{m/s}} = 2.87 \cdot 10^{-9}\,\text{m}.$$

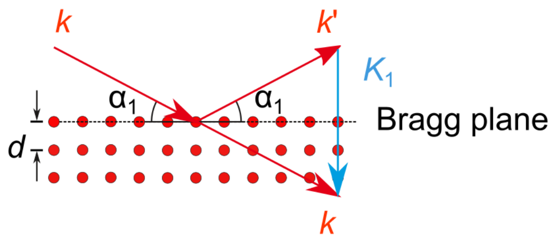
c) With Bragg's law

$$\lambda = 2d\sin\alpha_1 \rightarrow \sin\alpha_1 = \frac{\lambda}{2d} = \frac{2.87 \cdot 10^{-9}\,\text{m}}{10^{-8}\,\text{m}} = 0.287 \rightarrow \alpha_1 = 16.7°$$

and the angle with respect to the incident beam is $2\alpha_1 = 33.4°$.

d) The length of the reciprocal lattice vector is (compare the following sketch, $k = k' = 2\pi/\lambda$)

$$K_1 = \frac{2\pi}{d} = \frac{2\pi}{10^{-8}\,\text{m}} = 2\pi \cdot 10^8\,\text{m}^{-1}.$$



Problem 3 (4 Points)

During the decay $^{22}\text{Na} \rightarrow {}^{22}\text{Ne}$ one photon with the energy 1.28 MeV is emitted.

a) Calculate the wavelength of the corresponding electromagnetic wave.
b) Assume that the photons are scattered at electrons at rest. Calculate the energies of the scattered photons.
c) Calculate i) the smallest photon energy and ii) the highest kinetic energy of the electron after scattering.

a) The wavelength for 1.28 MeV is

$$E_\lambda = \frac{hc}{\lambda} \rightarrow \lambda = \frac{hc}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^8\,\text{m/s}}{1.28 \cdot 10^6\,\text{eV}} = 9,7 \cdot 10^{-13}\,\text{m}$$

b) The formula for the Compton scattering is

$$\lambda' - \lambda = \lambda_\text{C}(1 - \cos\theta) \rightarrow \lambda' = \lambda + \lambda_\text{C}(1 - \cos\theta).$$

λ denotes the wavelength of the incident wave, λ' the wavelength of the scattered wave and θ the scattering angle with respect to the incident beam. The energy of the scattered photon is then

$$E_{\lambda'} = \frac{hc}{\lambda'} = \frac{hc}{\lambda + \lambda_\text{C}(1 - \cos\theta)} \rightarrow E_{\lambda'}^{-1} = E_\lambda^{-1} + \text{E}_\text{C}^{-1}(1 - \cos\theta).$$

Thereby E_C denotes

$$E_\text{C} = hc/\lambda_\text{C} = 4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^8\,\text{ms}^{-1}/2.426 \cdot 10^{-12}\text{m} = 5.1 \cdot 10^5\,\text{eV}.$$

(This is the rest energy of the electron)

c) i) The smallest photon energy results from back-scattering i.e. $\theta = 180°$. Therefore

$$E_\text{min}^{-1} = E_\lambda^{-1} + 2\text{E}_\text{C}^{-1}$$

$$E_\text{min} = \frac{1}{E_\lambda^{-1} + 2\text{E}_\text{C}^{-1}} = \frac{\text{MeV}}{1.28^{-1} + 2 \cdot 0.51^{-1}} = 0.21\,\text{MeV}.$$

ii) The highest kinetic energy of the scattered electron is then

$$E_\text{max} = 1.28\,\text{MeV} - 0.21\,\text{MeV} = 1.07\,\text{MeV}.$$

Problem 4 (4 Points)

On a hot day (30° C) a black car is exposed to the plain sun. The intensity of the sun is $700\,\mathrm{W/m^2}$.

a) What is the temperature of the car in thermal equilibrium when only half of its surface is exposed to the sun? Assume that the car behaves like a black body and neglect the effect of thermal conduction.
b) Sketch the radiation spectrum of a black body.
c) How large is the wavelength of the maximum of the spectrum for the car in thermal equilibrium?
d) Repeat the calculation for a car which reflects 30 % of the incident radiation. How large is the equilibrium temperature of this car?

a) In equilibrium one can write with the Stefan-Boltzmann law

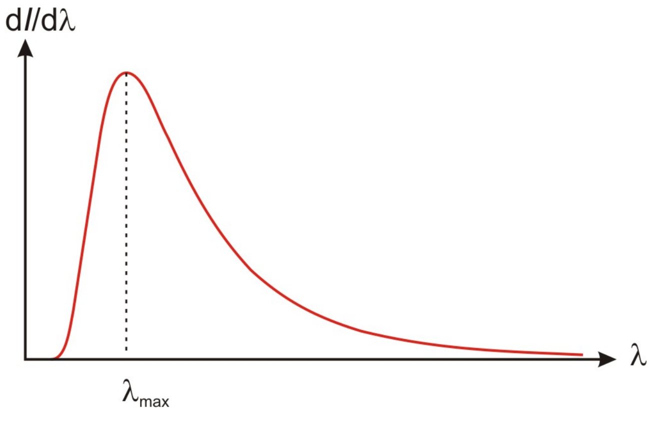
$$700\,\mathrm{Wm^{-2}}\frac{A}{2} = \mathrm{A}\sigma(T_\mathrm{C}^4 - T_0^4).$$

Thereby T_0 denotes the temperature of the environment and T_C the equilibrium temperature of the car

$$T_\mathrm{C} = \left(\frac{350\,\mathrm{Wm^{-2}}}{\sigma} + T_0^4\right)^{1/4} = \left(\frac{350\,\mathrm{Wm^{-2}}}{5.67\cdot 10^{-8}\mathrm{Wm^{-2}K^{-4}}} + (273\,\mathrm{K} + 30\,\mathrm{K})^4\right)^{1/4}$$
$$= 347.6\,\mathrm{K}$$

This is a temperature of 74.6° C.

b) The radiation spectrum of a black body is



$I = P/A$ denotes the intensity of the radiation.

c) With Wien's law λ_max becomes

$$\lambda_\mathrm{max} = \frac{b}{T} = \frac{2.9\,\mathrm{mmK}}{347.6\,\mathrm{K}} = 8.3\cdot 10^{-6}\,\mathrm{m}.$$

d) With problem a)

$$T_\mathrm{C} = \left(\frac{0.7\cdot 350\,\mathrm{Wm^{-2}}}{5.67\cdot 10^{-8}\mathrm{Wm^{-2}K^{-4}}} + (273\,\mathrm{K} + 30\,\mathrm{K})^4\right)^{1/4} = 336\,\mathrm{K}$$

the equilibrium temperature is 63° C.

Problem 5 (4 Points)

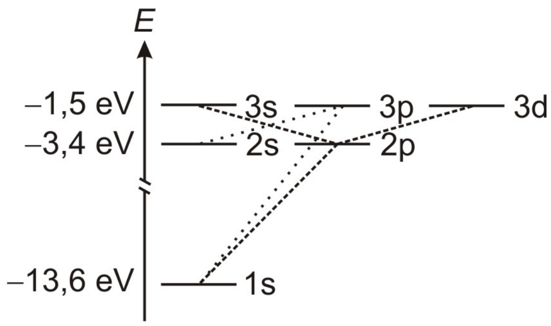
a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.
d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

a) The energy is with the Rydberg unit of energy

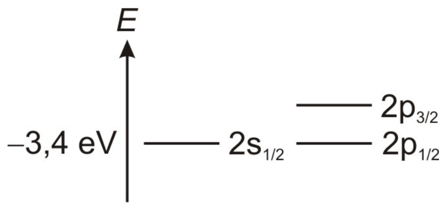
$$E_n = -\frac{R}{n^2},$$

i.e. $E_1 = -13.6\,\text{eV}$, $E_2 = -3.4\,\text{eV}$ and $E_3 = -1.5\,\text{eV}$.

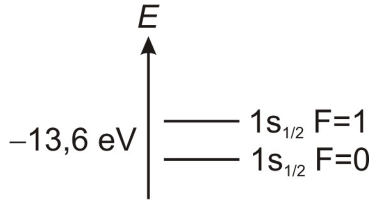
b) The energy level scheme and the allowed electric dipole transitions (dashed lines). The selection rule is $\Delta\ell = \pm 1$.



c) The energy level scheme of the first excited state. The p-orbital is splitted due to the spin-orbit-coupling into a state with the total angular momentum $j = 1/2$ and a state with $j = 3/2$.



d) The ground state of the hydrogen atom is splitted due to the hyperfine interaction with the proton. There are two states with the total angular momentum $F = 0$ and $F = 1$.

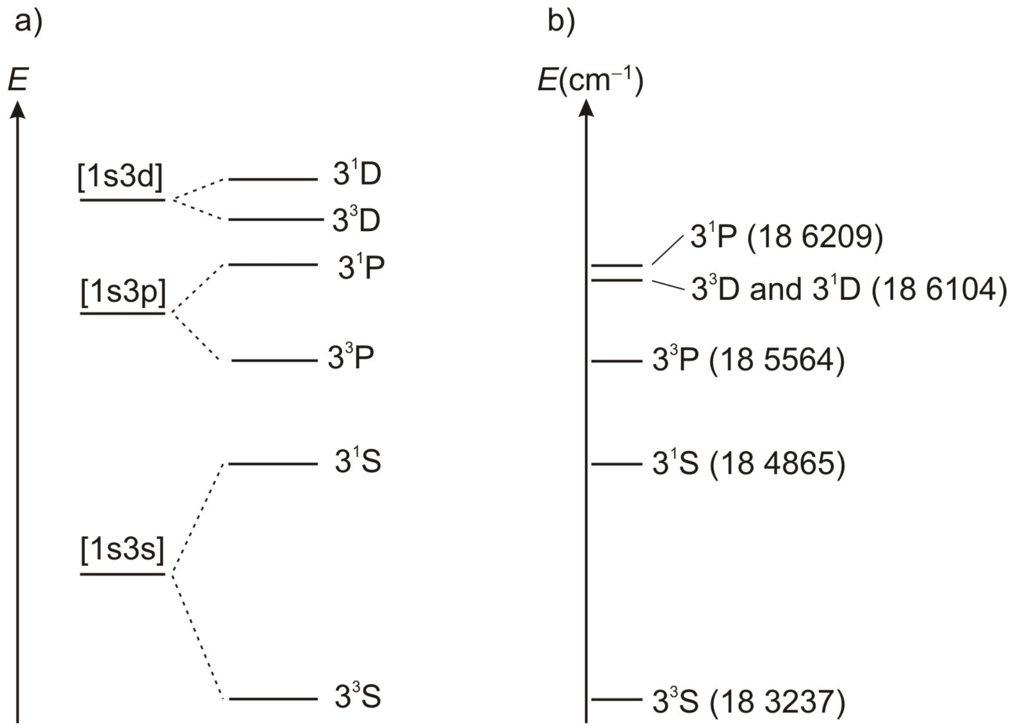


Problem 6 (4 Points)

There are two electrons in the neutral Helium atom. The excited states with lowest energy result from the excitation of one electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the expected energy level scheme and denote the energy levels, when the excited electron is lifted into a state with the quantum number $n = 3$. Ignore spin-orbit coupling.
b) Give reasons for the energy difference between states of different angular momenta L.
c) Explain the energy difference between spin singlet and triplet states.
d) Give the states of problem a), which can be excited from the groundstate by electric dipole transitions.

a) The following figure a) shows the sketch of the expected energy level scheme[2].



[2]Figure b) shows the experimental results. The exchange interaction is more effective than the screening of the nuclear charge. Without exchange is the energy of the [1s3d] configuration only slightly higher than the [1s3p] configuration. The exchange interaction of the [1s3d] configuration is very small.

b) The mean distance between the electron cloud and the nucleus increases with increasing orbital angular momentum. Therefore the screening of the nuclear charge by the 1s electron is more effective for higher quantum numbers of the angular momentum of the excited electron and the binding energy is reduced.

c) In the singlet state the electron-electron repulsion is stronger than in the triplet state, since the two electrons occupy the same orbital and the mean electron-electron distance is smaller than in the triplet state. Therefore the binding energy of the singlet state is reduced. The effect is known as exchange interaction.

d) The selection rules for electric dipole transitions are $\Delta S = 0$ and $\Delta \ell = \pm 1$. Therefore only the transition 1^1S $\leftrightarrow$ 3^1P is possible by electric dipole radiation.

Problem 7 (4 Points)

a) The vectors of a primitive unit cell of the bcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x} - \vec{y} + \vec{z})$, $\vec{a}_2 = \frac{a}{2}(\vec{y} - \vec{z} + \vec{x})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z} - \vec{x} + \vec{y})$. $\vec{x}$, $\vec{y}$ and $\vec{z}$ denote orthogonal unit vectors. What is the volume of the unit cell?
b) Give the primitive vectors of the corresponding reciprocal lattice.
c) The vectors of a primitive unit cell of the fcc lattice with the lattice constant a are $\vec{a}_1 = \frac{a}{2}(\vec{x} + \vec{y})$, $\vec{a}_2 = \frac{a}{2}(\vec{y} + \vec{z})$ and $\vec{a}_3 = \frac{a}{2}(\vec{z} + \vec{x})$. What is the volume of the unit cell?
d) Give the primitive vectors of the corresponding reciprocal lattice.

a) There are two lattice points in the simple cubic unit cell. Therefore the volume of the primitive unit cell of the bcc lattice is

$$V_{\text{Cell}} = \frac{a^3}{2}.$$

Alternatively one can calculate $\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \frac{a^3}{2}$.

b) The primitive vectors of the reciprocal lattice are

$$\vec{b}_1 = \frac{2\pi}{V_{\text{Cell}}} \vec{a}_2 \times \vec{a}_3,$$

i.e.

$$\vec{b}_1 = \frac{4\pi}{4a} \left[(\vec{y} - \vec{z} + \vec{x}) \times (\vec{z} - \vec{x} + \vec{y}) \right] = \frac{\pi}{a} (\vec{x} + \vec{z} + \vec{y} + \vec{x} - \vec{y} + \vec{z}) = \frac{2\pi}{a} (\vec{x} + \vec{z})$$

By cyclic permutation one gets

$$\vec{b}_2 = \frac{2\pi}{a} (\vec{y} + \vec{x})$$

and

$$\vec{b}_3 = \frac{2\pi}{a} (\vec{z} + \vec{y})$$

c) There are four lattice points in the simple cubic unit cell. Therefore the volume of the primitive cell of the fcc lattice is

$$V_{\text{Cell}} = \frac{a^3}{4}.$$

Alternatively one can calculate $\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \frac{a^3}{4}$.

d) The primitive vectors of the reciprocal lattice are

$$\begin{aligned}\vec{b}_1 &= \frac{2\pi}{V_{\text{Cell}}}\vec{a}_2 \times \vec{a}_3 \\ &= \frac{2\pi}{a}\left[(\vec{y}+\vec{z}) \times (\vec{z}+\vec{x})\right] \\ &= \frac{2\pi}{a}\left(\vec{x}-\vec{z}+\vec{y}\right)\end{aligned}$$

and by cyclic permutation

$$\begin{aligned}\vec{b}_2 &= \frac{2\pi}{a}\left(\vec{y}-\vec{x}+\vec{z}\right) \\ \vec{b}_3 &= \frac{2\pi}{a}\left(\vec{z}-\vec{y}+\vec{x}\right).\end{aligned}$$

(Conclusion: The reciprocal lattice of the bcc lattice is a fcc lattice and vice versa.)

Problem 8 (4 Points)

a) Calculate the energy of a free electron described by a simple plane wave in a region of constant potential energy V.
b) Calculate the density of conduction electrons of potassium. Hint: The mass of the potassium atom is $39u$ and there is one conduction electron per atom.
c) Calculate the Fermi wavenumber k_F of potassium.
d) Calculate the Fermi energy of potassium.

a) With the Schrödinger equation is the energy of a plane wave

$$\psi = \psi_0 e^{i(\vec{k}\vec{r}-\omega t)}$$

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\nabla^2}{2m_e}\psi + V\psi \rightarrow E = \hbar\omega(\vec{k}) = \frac{\hbar^2\vec{k}^2}{2m_e} + V$$

b) With the number of atoms N and the density of potassium one gets

$$\rho_K = \frac{m}{V} = \frac{N\cdot 39u}{V}$$

and the electron density is

$$\frac{N}{V} = \frac{\rho_K}{39u} = \frac{0.86\cdot 10^{-3}\,\text{kg/cm}^3}{39\cdot 1.66\cdot 10^{-27}\,\text{kg}} = 1.33\cdot 10^{22}\,1/\text{cm}^3 = 1.33\cdot 10^{28}\,1/\text{m}^3.$$

c) The Fermi wavenumber results from

$$\frac{N}{2} = \frac{4\pi k_F^3 V}{3(2\pi)^3} \rightarrow k_F = \left(3\pi^2\frac{N}{V}\right)^{1/3} = \left(3\pi^2 1.33\cdot 10^{28}\,1/\text{m}^3\right)^{1/3} = 7.3\cdot 10^9\,\text{m}^{-1}.$$

d) The Fermi energy is

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{h^2k_F^2}{8\pi^2 m_e} = \frac{(4.14\cdot 10^{-15}\,\text{eVs})^2(7.3\cdot 10^9\,\text{m}^{-1})^2(3\cdot 10^8)^2\,\text{m}^2\text{s}^{-2}}{8\pi^2\cdot 500\cdot 10^3\,\text{eV}}$$
$$= 2.1\,\text{eV}.$$

Problem 1 (4 Points)

A space ship with a length of $\ell_0 = 30\,\text{m}$ is flying with a velocity of 0.95 c directly towards an observer. Radio signals are emitted on both ends of the space ship.

a) How long is the space ship in the frame of reference of the observer?
b) How long is the time interval between the two radio signals in the frame of the space ship, when both signals reach the observer at the same time?
c) How long is the time interval between the emission of the two radio signals in the frame of the observer?
d) How large is the distance between the places of emission in the frame of the observer?

Problem 2 (4 Points)

a) An electron moves with a velocity of $v = 3000\,\text{m/s}$. Calculate the kinetic energy and the corresponding de Broglie wavelength of the electron.
b) Calculate the energy of the photons for light with the wavelength $\lambda = 248\,\text{nm}$.
c) Write up the conservation laws of energy and momentum for the collision between an electron and a photon.
d) Give the relation between energy and momentum of a relativistic electron.

Problem 3 (4 Points)

Photons with the energy 2 MeV hit electrons at rest.

a) Calculate the wave length of an electromagnetic wave formed by these photons.
b) Give the energy of a photon deflected by 90° from the direction of the incident photon beam.
c) Give the energy of an electron after the collision with this photon.
d) Give the momentum $p = |\vec{p}|$ of this electron.

Problem 4 (4 Points)

a) The effective potential energy of the electron in the hydrogen atom is $V_\ell(r)$. Sketch the effective potential energy and explain the meaning of the quantum number ℓ.
b) Explain the quantum numbers characterizing the radial part of the wave function.
c) Sketch the radial wave functions of the first excited state of the electron.
d) Which additional quantum numbers are necessary to characterize the quantum state of the electron?

Problem 5 (4 Points)

a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.

d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

Problem 6 (4 Points)

There are two valence electrons in the neutral Cadmium atom (groundstate configuration $[\mathrm{Kr}]4\mathrm{d}^{10}5\mathrm{s}^2$). The excited states with lowest energy result from the excitation of one 5s electron into hydrogen-like orbitals while the other electron stays in the ground state.

a) Sketch the expected energy level scheme and denote the energy levels, when the excited electron is lifted into a state with the quantum number $\mathrm{n} = 5$ and $\ell = 1$. Ignore spin-orbit coupling.
b) Explain the energy difference between spin singlet and triplet states.
c) Give the states of problem a), which can be excited from the groundstate by electric dipole transitions.
d) Give reasons for the energy difference between states with the quantum numbers $\mathrm{n} = 5$, $\ell = 1$ and $\mathrm{n} = 5$, $\ell = 2$.

Problem 7 (4 Points)

In a Stern-Gerlach experiment silver atoms (atomic mass 107.9 u) are prepared in their ground-state and propagate in the x-direction with a speed of $v_x = 464\,\mathrm{m/s}$. The beam passes a region with a magnetic field gradient $\mathrm{d}B/\mathrm{d}z = 600\,\mathrm{T/m}$ in the z-direction .

a) Describe the Stern-Gerlach experiment and its result.
b) Determine the maximum acceleration of the silver atoms.
c) What is the maximum distance between the lines observed in the detection plane? Assume that the magnetic field is confined to a region of width $\Delta x = 75\,\mathrm{cm}$ in the direction of the beam. Beyond this region the atoms travel a distance of 1.25 m to the detection plane.

Problem 8 (4 Points)

a) The Fermi energy at $T = 0$ K is given by the formulae: $E_\mathrm{F} = \frac{h^2}{8m_\mathrm{e}}\left(\frac{3N}{\pi V}\right)^{2/3}$. Calculate for metallic lithium the Fermi energy assuming that each lithium atom contributes a single electron to the free electron gas. Lithium has a density of $\rho = 0.534\,\mathrm{g/cm^3}$ and an atomic mass of 6.94 u.
b) The heat capacity of the conduction electrons in the Sommerfeld theory is $C_\mathrm{V} = \left(\frac{\partial E}{\partial T}\right)_V = \frac{\pi^2}{2}\frac{Nk_\mathrm{B}^2}{E_\mathrm{F}}T$. In classical physics is the heat capacity of an ideal gas with N particles $C_\mathrm{V} = \frac{3}{2}Nk_\mathrm{B}$. Calculate the fraction of conduction electrons of lithium contributing to the heat capacity at $T = 300\,\mathrm{K}$.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\mathrm{m/s}$
Mass of the electron:	$m_\mathrm{e} = 500\,\mathrm{keV}/c^2$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\mathrm{eVs}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\mathrm{As}$

Compton wavelength:	$\lambda_C = 2.426 \cdot 10^{-12}$ m
Boltzmann constant:	$k_B = 8.6 \cdot 10^{-5}$eV/K
Rydberg unit of energy:	$R = 13.6$ eV
Bohr's magneton:	$\mu_B = 5.8 \cdot 10^{-5}$ eV/T
Atomic mass unit:	$u = 1.66 \cdot 10^{-27}$ kg

Problem 1 (4 Points)

A space ship with a length of $\ell_0 = 30\,\text{m}$ is flying with a velocity of 0.95 c directly towards an observer. Radio signals are emitted on both ends of the space ship.

a) How long is the space ship in the frame of reference of the observer?
b) How long is the time interval between the two radio signals in the frame of the space ship, when both signals reach the observer at the same time?
c) How long is the time interval between the emission of the two radio signals in the frame of the observer?
d) How large is the distance between the places of emission in the frame of the observer?

a) Length contraction

$$\ell(0.95\text{c}) = \ell_0\sqrt{1-0.95^2} = 30\text{m}\cdot 0.31 = 9.37\,\text{m}$$

(1 Point)

b) The signal from the back of the space ship has to reach the tip, before this signal can be emitted, i.e.

$$c\Delta t_\text{S} = \ell_0$$
$$\Delta t_\text{S} = \frac{\ell_0}{c} = \frac{30\,\text{m}}{3\cdot 10^8\,\text{ms}^{-1}} = 10^{-7}\,\text{s}.$$

(1 Point)

c) In the frame of the observer one has correspondingly

$$c\Delta t_\text{B} = \ell_0\sqrt{1-0.95^2} + 0.95\,\text{c}\cdot\Delta t_\text{B}$$
$$\Delta t_\text{B} = \frac{\ell_0\sqrt{1-0.95^2}}{0.05\,\text{c}} = 6.24\,\frac{\ell_0}{c} = 6.24\,\Delta t_\text{S} = 6.24\cdot 10^{-7}\,\text{s},$$

i.e. the signal of the back has to pass the space ship which moves on the distance $0.95c\Delta t_\text{B}$ during the time Δt_B.

(1 point)

d) The distance between the two places of emission is in the frame of the observer

$$c\Delta t_\text{B} = \frac{\ell(0.95c)}{0.05} = \frac{9.37\,\text{m}}{0.05} = 187.3\,\text{m}.$$

(1 Point)

Problem 2 (4 Points)

a) An electron moves with a velocity of $v = 3000\,\mathrm{m/s}$. Calculate the kinetic energy and the corresponding de Broglie wavelength of the electron.
b) Calculate the energy of the photons for light with the wavelength $\lambda = 248\,\mathrm{nm}$.
c) Write up the conservation laws of energy and momentum for the collision between an electron and a photon.
d) Give the relation between energy and momentum of a relativistic electron.

a) The kinetic energy of the electron is

$$E_{\mathrm{kin}} = \frac{1}{2} m_{\mathrm{e}} v^2 = \frac{1}{2} 5 \cdot 10^5\,\mathrm{eV} \left(\frac{3000}{3 \cdot 10^8} \right)^2 = 2.5 \cdot 10^{-5}\,\mathrm{eV}.$$

The de Broglie wave length of the electron is

$$\lambda = \frac{h}{p} = \frac{h}{m_{\mathrm{e}} v} = \frac{4.14 \cdot 10^{-15}\,\mathrm{eVs} (3 \cdot 10^8\,\mathrm{m/s})^2}{5 \cdot 10^5\,\mathrm{eV}\, 3000\,\mathrm{m/s}} = 2.48 \cdot 10^{-7}\,\mathrm{m}.$$

(1 point)

b) The energy of the photon is

$$E = h\nu = h\frac{c}{\lambda} = 4.14 \cdot 10^{-15}\,\mathrm{eVs} \frac{3 \cdot 10^8\,\mathrm{m/s}}{248 \cdot 10^{-9}\,\mathrm{m}} = 5\,\mathrm{eV}.$$

(1 point)

c) Conservation of energy

$$E_{\mathrm{e}} + E_{\gamma} = E'_{\mathrm{e}} + E'_{\gamma},$$

conservation of momentum

$$\vec{p}_{\mathrm{e}} + \vec{p}_{\gamma} = \vec{p}\,'_{\mathrm{e}} + \vec{p}\,'_{\gamma}.$$

E, E' und $\vec{p}$, $\vec{p}\,'$ denotes energy and momentum before and after the collision.
(1 point)

d) The energy-momentum relation is

$$E_{\mathrm{e}}^2 = c^2 \vec{p}_{\mathrm{e}}^{\,2} + m_{\mathrm{e}}^2 c^4.$$

(1 point)

Problem 3 (4 Points)

Photons with the energy 2 MeV hit electrons at rest.

a) Calculate the wave length of an electromagnetic wave formed by these photons.
b) Give the energy of a photon deflected by 90° from the direction of the incident photon beam.
c) Give the energy of an electron after the collision with this photon.
d) Give the momentum $p = |\vec{p}|$ of this electron.

a) The wavelenth of the electromagnetic wave is

$$E_\lambda = \frac{hc}{\lambda} \to \lambda = \frac{hc}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^8\,\text{m/s}}{2 \cdot 10^6\,\text{eV}} = 6.21 \cdot 10^{-13}\,\text{m}.$$

(1 Point)

b) The increase of the wavelenth (i.e. the reduction of energy) due to the scattering on a free electron at rest is with the formula $\lambda' - \lambda = \lambda_C(1 - \cos\theta)$

$$\begin{aligned} \lambda' &= \lambda + \lambda_C \\ &= (6.21 \cdot 10^{-13} + 2.426 \cdot 10^{-12})\,\text{m} = 3.05 \cdot 10^{-12}\,\text{m}. \end{aligned}$$

The energy of the scattered photon is

$$E_{\lambda'} = \frac{\text{hc}}{\lambda'} = \frac{4.14 \cdot 10^{-15}\text{eVs}\,3 \cdot 10^8\,\text{m/s}}{3.05 \cdot 10^{-12}\,\text{m}} = 4.07 \cdot 10^5\,\text{eV}$$

(1 Point)

c) The kinetic energy of the electron is

$$E_{\text{kin}} = E_\lambda - E_{\lambda'} = 2\,\text{MeV} - 0.4\,\text{MeV} = 1.6\,\text{MeV}$$

and the total energy

$$E_e = E_{\text{kin}} + m_e c^2 = 1.6\,\text{MeV} + 0.5\,\text{MeV} = 2.1\,\text{MeV}.$$

(1 Point)

d) With the formula for the energy of the electron

$$E_e^2 = c^2\vec{p}^2 + m_e^2 c^4$$

is the momentum of the electron

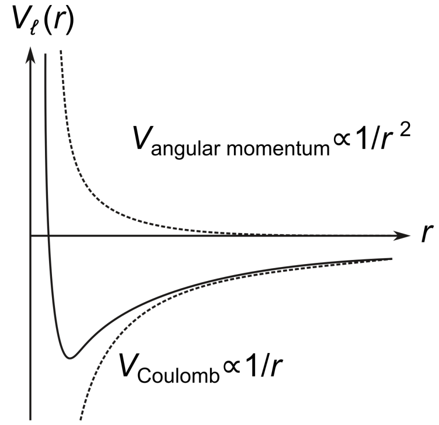
$$\begin{aligned} p &= \frac{1}{c}\sqrt{E_e^2 - m_e^2 c^4} = \frac{1}{c}\sqrt{2.1^2 - 0.5^2}\,\text{MeV} = \frac{1}{c}2.04\,\text{MeV} \\ &= \frac{2.04 \cdot 10^6 \cdot 1.6 \cdot 10^{-19}\,\text{VAs}}{3 \cdot 10^8\,\text{ms}^{-1}} = 1.09 \cdot 10^{-21}\,\frac{\text{kgm}}{\text{s}} \end{aligned}$$

(1 Point)

Problem 4 (4 Points)

a) The effective potential energy of the electron in the hydrogen atom is $V_\ell(r)$. Sketch the effective potential energy and explain the meaning of the quantum number ℓ.
b) Explain the quantum numbers characterizing the radial part of the wave function.
c) Sketch the radial wave functions of the first excited state of the electron.
d) Which additional quantum numbers are necessary to characterize the quantum state of the electron?

a) Sketch of the effective potential



The potential energy of the electron is due to the Coulomb potential in the electric field of the nucleus. In addition there is a contribution due to the centrifugal force. This contribution is proportional to the square of the angular momentum L.

$$V_\ell(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} + \frac{\hbar^2\ell(\ell+1)}{2m_e r^2}.$$

ℓ is the quantum number of L^2

(1 Point)

b) The radial part of the wave function is characterized by the principal quantum number n

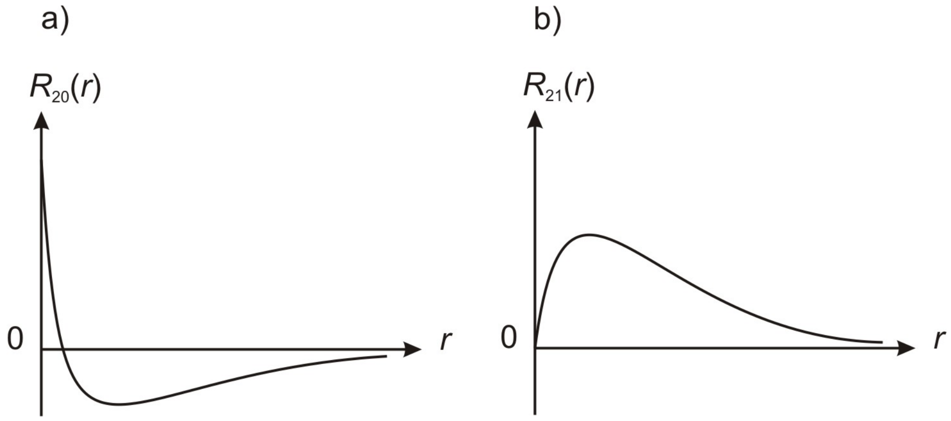
$$E_n = -13.6\,\text{eV}\left(\frac{1}{n}\right)^2$$

and the angular momentum quantum number ℓ

$$L^2 y_{\ell,m} = \hbar^2\ell(\ell+1)y_{\ell,m}.$$

(1 Point)

c) For n $= 2$ there are two radial functions $R_{n=2,\ell=0}$ and $R_{n=2,\ell=1}$.
$R_{n=2,\ell=0}$ starts at $r = 0$ with a finite value, changes the sign and approaches zero for $r \to \infty$.
Due to the centrifugal potential starts $R_{n=2,\ell=1}$ for $r = 0$ at zero and approaches after a maximum zero again for $r \to \infty$.



(1 Point)

d) For the orbital part one needs in addition to n and ℓ

the magnetic quantum number $|m| \leq \ell$ of the angular momentum.

To characterize the spin of the electron one needs the quantum numbers
$s = 1/2$ and $m_s = \pm 1/2$.

s and ℓ add up to the total angular momentum which is characterized by the quantum numbers j and m_j.

(1 Point)

Problem 5 (4 Points)

a) Calculate for the hydrogen atom the energy of the ground state, the first, and the second excited state, ignoring the spin of the electron.
b) Sketch the energy level scheme of these states, denote the states and indicate the allowed electric dipole transitions. Give the corresponding selection rule.
c) Sketch the energy level scheme of the first excited state including now the effect of the spin. Use the spectroscopic notation to denote the energy states.
d) What is the reason for the splitting of the ground state of the hydrogen atom in two energy levels? Sketch the splitting and denote the energy levels.

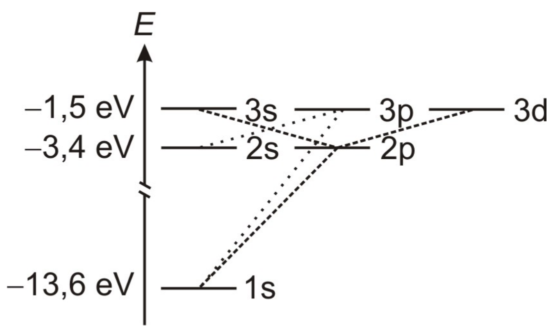
a) The energy is with the Rydberg unit of energy

$$E_\mathrm{n} = -\frac{R}{\mathrm{n}^2},$$

i.e. $E_1 = -13.6\,\mathrm{eV}$, $E_2 = -3.4\,\mathrm{eV}$ and $E_3 = -1.5\,\mathrm{eV}$.

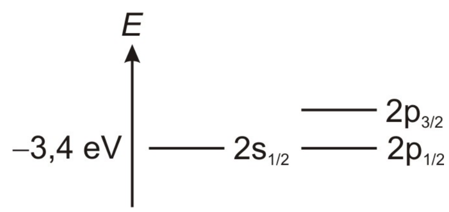
(1 Point)

b) The energy level scheme and the allowed electric dipole transitions (dashed lines). The selection rule is $\Delta\ell = \pm 1$.



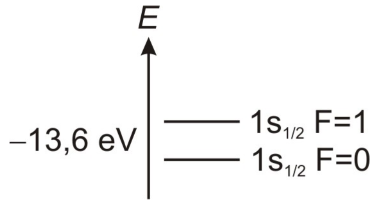
(1 Point)

c) The energy level scheme of the first excited state. The p-orbital is splitted due to the spin-orbit-coupling into a state with the total angular momentum $j = 1/2$ and a state with $j = 3/2$.



(1 Point)

d) The ground state of the hydrogen atom is splitted due to the hyperfine interaction with the proton. There are two states with the total angular momentum $F = 0$ and $F = 1$.



(1 Point)

Problem 6 (4 Points)

There are two valence electrons in the neutral Cadmium atom (groundstate configuration $[\text{Kr}]4\text{d}^{10}5\text{s}^2$). The excited states with lowest energy result from the excitation of one 5s electron into hydrogen-like orbitals while the other electron stays in the ground state.

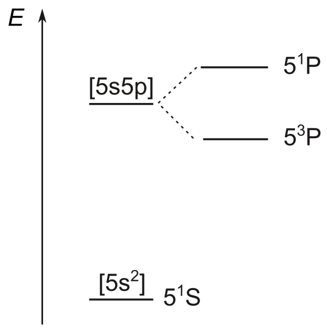
a) Sketch the expected energy level scheme and denote the energy levels, when the excited electron is lifted into a state with the quantum number $\text{n} = 5$ and $\ell = 1$. Ignore spin-orbit coupling.

b) Explain the energy difference between spin singlet and triplet states.

c) Give the states of problem a), which can be excited from the groundstate by electric dipole transitions.

d) Give reasons for the energy difference between states with the quantum numbers $\text{n} = 5$, $\ell = 1$ and $\text{n} = 5$, $\ell = 2$.

a) Sketch of the expected energy level scheme



(1 Point)

b) The binding energy of the singlet state is smaller than the binding energy of the triplet state due to the exchange interaction.

(1 Point)

c) The selection rules for electric dipole transitions are $\Delta S = 0$ and $\Delta\ell = \pm 1$. Therefore only the transition $5^1\text{S} \leftrightarrow 5^1\text{P}$ is possible by electric dipole radiation.

(1 Point)

d) The mean distance between the electron cloud and the nucleus increases with increasing orbital angular momentum. Therefore the screening of the nuclear charge is more effective for higher quantum numbers of the angular momentum and the binding energy is smaller for $\ell = 2$ than for $\ell = 1$. (1 Point)

Problem 7 (4 Points)

In a Stern-Gerlach experiment silver atoms (atomic mass 107.9 u) are prepared in their ground-state and propagate in the x-direction with a speed of $v_x = 464\,\mathrm{m/s}$. The beam passes a region with a magnetic field gradient $\mathrm{d}B/\mathrm{d}z = 600\,\mathrm{T/m}$ in the z-direction .

a) Describe the Stern-Gerlach experiment and its result.
b) Determine the maximum acceleration of the silver atoms.
c) What is the maximum distance between the lines observed in the detection plane? Assume that the magnetic field is confined to a region of width $\Delta x = 75\,\mathrm{cm}$ in the direction of the beam. Beyond this region the atoms travel a distance of 1.25 m to the detection plane.

a) An atomic beam passes in the Stern-Gerlach experiment an inhomogeneous magnetic field, which causes the splitting of the atomic beam. For Ag atoms the beam is splitted in two beams behind the inhomogeneous magnetic field.

(1 Point)

b) The force is

$$F_z = \mu_z \frac{\partial B}{\partial z}.$$

μ_z denotes the magnetic moment in z-direction

$$\mu_z = -g\mu_\mathrm{B} s_z/\hbar$$

with the g-Faktor $g = 2$ and the eigenvalues of the spin s_z: $m_\mathrm{s} = \pm\frac{1}{2}$ results the acceleration

$$a = \frac{F_z}{m_\mathrm{Ag}} = \frac{\mu_\mathrm{B}\frac{\partial B}{\partial z}}{m_\mathrm{Ag}} = \frac{9.27\cdot 10^{-24}\,\mathrm{J/T}\cdot 600\,\mathrm{T/m}}{107.9\cdot 1.66\cdot 10^{-27}\,\mathrm{kg}} = 31\cdot 10^3\,\mathrm{m/s^2}$$

(1 Point)

c) The time of the Ag atoms to pass the inhomogeneous magnetic field is

$$\Delta t = \frac{\Delta x}{v_x} = \frac{0.75\,\mathrm{m}}{464\,\mathrm{m/s}} = 1.6\cdot 10^{-3}\,\mathrm{s}.$$

The shift perpendicular to v_x is

$$\Delta s_1 = \pm\frac{1}{2}at^2 = \pm\frac{1}{2}31\cdot 10^3\,\mathrm{m/s^2}(1.6\cdot 10^{-3}\,\mathrm{s})^2 = \pm 39.7\,\mathrm{mm}$$

and the velocity $v_\perp$

$$v_\perp = \pm a\cdot\Delta t = \pm 31\cdot 10^3\,\mathrm{m/s^2}\,1.6\cdot 10^{-3}\,\mathrm{s} = \pm 49.6\,\mathrm{m/s}.$$

The time to the detection plane is

$$\delta t = \frac{\ell}{v_x} = \frac{1.25\,\mathrm{m}}{464\,\mathrm{m/s}} = 2.7\cdot 10^{-3}\,\mathrm{s}.$$

The shift Δs_2 is

$$\Delta s_2 = v_\perp \delta t = \pm 49.6\,\text{m}/\text{s}\,2.7 \cdot 10^{-3}\,\text{s} = \pm 134\,\text{mm}.$$

The beam splitting is

$$\Delta s = 2(|\Delta s_1| + |\Delta s_2|) = 2 \cdot (39.7 + 134)\,\text{mm} = 347\,\text{mm}.$$

(2 Points)

Problem 8 (4 Points)

a) The Fermi energy at $T = 0$ K is given by the formulae: $E_F = \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3}$. Calculate for metallic lithium the Fermi energy assuming that each lithium atom contributes a single electron to the free electron gas. Lithium has a density of $\rho = 0.534\,\text{g/cm}^3$ and an atomic mass of 6.94 u.

b) The heat capacity of the conduction electrons in the Sommerfeld theory is $C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{\pi^2}{2}\frac{Nk_B^2}{E_F}T$. In classical physics is the heat capacity of an ideal gas with N particles $C_V = \frac{3}{2}Nk_B$. Calculate the fraction of conduction electrons of lithium contributing to the heat capacity at $T = 300\,\text{K}$.

a) The density of Lithium is

$$\rho = \frac{m}{V} = \frac{N \cdot 6.94\,u}{V}$$

and the electron density

$$\frac{N}{V} = \frac{\rho}{6.96\,u} = \frac{0.534 \cdot 10^{-3}\,\text{kg/cm}^3}{6.94 \cdot 1.66 \cdot 10^{-27}\,\text{kg}} = 4.64 \cdot 10^{22}\,\text{cm}^{-3} = 4.64 \cdot 10^{28}\,\text{m}^{-3}.$$

The Fermi energy is

$$E_F = \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{(6.626 \cdot 10^{-34}\,\text{Js})^2}{8 \cdot 9 \cdot 10^{-31}\,\text{kg}}\left(\frac{3 \cdot 4.64 \cdot 10^{28}}{\pi m^3}\right)^{2/3} = 7.64 \cdot 10^{-19}\,\text{J}$$

(2 Points)

b) The ratio $C_{V,\,\text{electrongas}}/C_{V,\,\text{idealgas}}$ is with

$$C_{V,\,\text{Elektronengas}} = \frac{\pi^2}{2}\frac{Nk_B^2}{E_F}T = \frac{\pi^2}{2}\left(\frac{k_B T}{E_F}\right)Nk_B$$

and

$$C_{V,\,\text{idealesGas}} = \frac{3}{2}Nk_B$$

$$\frac{C_{V,\,\text{Elektronengas}}}{C_{V,\,\text{idealesGas}}} = \frac{\frac{\pi^2}{2}\left(\frac{k_B T}{E_F}\right)}{\frac{3}{2}} = \frac{\pi^2}{3}\left(\frac{k_B T}{E_F}\right).$$

With the Fermi energy of Lithium results

$$\frac{C_{V,\,\text{Elektronengas}}}{C_{V,\,\text{idealesGas}}} = \frac{\pi^2}{3}\frac{1.38 \cdot 10^{-23}\,\text{J/K}\,300\,\text{K}}{7.64 \cdot 10^{-19}\,\text{J}} = 0.018.$$

Only $\approx 1.8\,\%$ of the conduction electrons contribute at room temperature to the heat capacity of the electron gas

(2 Points)

Problem 1 (4 Points)

a) Give the definition of entropy.
b) What is the definition of temperature?
c) Write up the formula of Maxwell's velocity distribution function without the prefactor and sketch of the distribution function.
d) Explain the definition of the Kelvin temperature scale.

Problem 2 (4 Points)

The Lorentz-transformation between frame *S* and *S*' is
$t = \gamma(\mathrm{t}' + \frac{V}{c^2}\mathrm{x}')$, $x = \gamma(\mathrm{x}' + Vt')$, $y = y'$, $z = z'$ with $\gamma = \sqrt{1 - V^2/c^2}$.

a) Write up the classical Galilei-transformation for small values of the relative velocity *V*. Sketch the relative motion of frame *S* and *S*'.
b) Write up the Lorentz-transformation of the relativistic wave vector.
c) Calculate the longitudinal Doppler effect starting with the Lorentz transformation.
d) Describe the velocity measurements of streaming gases by means of Laser beams.

Problem 3 (4 Points)

a) The nucleus is formed by protons and neutrons. Write up and explain the general notation of a nucleus.
b) Give the formula for the quantized energy of the hydrogen atom determined by Bohr's model of the atom.
c) Write up the formula for the energy of x-rays according to Moseley.
d) What denotes a K_α-line in x-ray spectroscopy?

Problem 4 (4 Points)

a) Make a sketch of the quantized angular momentum vector. Indicate the length and the z-component of the vector.
b) Write up the eigenvalues equations of the angular momentum operator.
c) Explain, why no values can be assigned to the x- and the y-component of the angular momentum operator.
d) Spherical harmonics are the eigenfunctions of the orbital angular momentum operator. Explain the general structure of the spherical harmonics and write up the spherical harmonics $y_{1,-1}$, $y_{1,0}$ and $y_{1,+1}$. Omit the normalization factors.

Problem 5 (4 Points)

a) Write up the magnetic moment due to the orbit and the spin of the electron.
b) Explain by means of Bohr's model of the atom why spin and orbital angular momentum align antiparallel for electrons in an atom.
c) Write up the Hamilton-operator of the spin-orbit coupling and calculate the energy eigenvalues.
d) For the ground state of Eu^{3+} are the quantum numbers of the orbit and the spin 3, i.e. $\mathrm{L} = 3$ and $\mathrm{S} = 3$. Write up the quantum states of the spin-orbit multiplet of

the ground state starting with the state of lowest energy. Use the atomic notation of quantum states.

Problem 6 (4 Points)

a) Write up the entangled quantum state of two fermions.
b) How many electrons can occupy a quantum state with the orbital angular momentum quantum number $\ell = 1$?
c) Give the entangled states of two p-electrons that differ in the principal quantum number n.
d) Which of these states has the smallest energy, when electrons have the same principal quantum number n?

Problem 7 (4 Points)

a) Sketch the phonon dispersion $\nu(\vec{k})$. Consider one propagation direction and a crystal lattice with one atom in the unit cell.
b) Sketch the phonon dispersion, when there are two atoms within the unit cell and give the typical frequency range of phonon modes.
c) Explain why the phonon modes are confined to the 1st Brillouin zone.
d) There are how many phonon modes in a crystal lattice with n atoms per unit cell?

Problem 8 (4 Points)

a) Write up the Fermi-distribution function and make a sketch for $T = 0$ and $T \neq 0$.
b) Calculate the density of states

$$D(E) = \frac{1}{A}\frac{dN}{dE}$$

of a two-dimensional quasi free electron gas.
c) Compare the heat capacity of a gas formed by atoms or molecules with the heat capacity of a free electron gas. Ignore all prefactors.
d) Plot the energy spectrum of nearly free electrons for waves propagating along the edges of a simple cubic unit cell with lattice parameter a.

Problem 1 (4 Points)

a) What denotes entropy in physics.
b) Give the definition of temperature in physics.
c) Give the Maxwell velocity distribution function without the prefactor and make a sketch of the velocity distribution function.
d) Explain the definition of the Kelvin temperature scale

a)

$$S = k_B \ln \Gamma(E)$$

$\Gamma(E)$ denotes the number of possibilities of a system to realise states with the energy E. (1 Point)

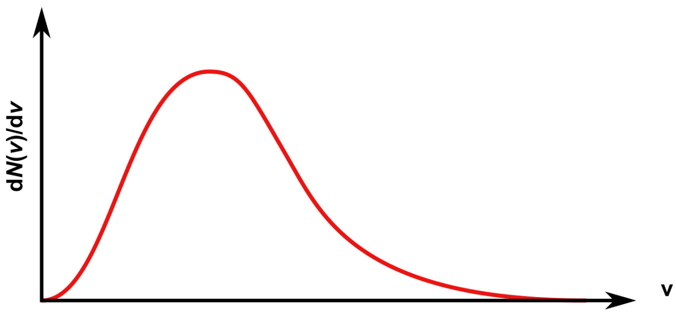
b)

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

(1 Point)

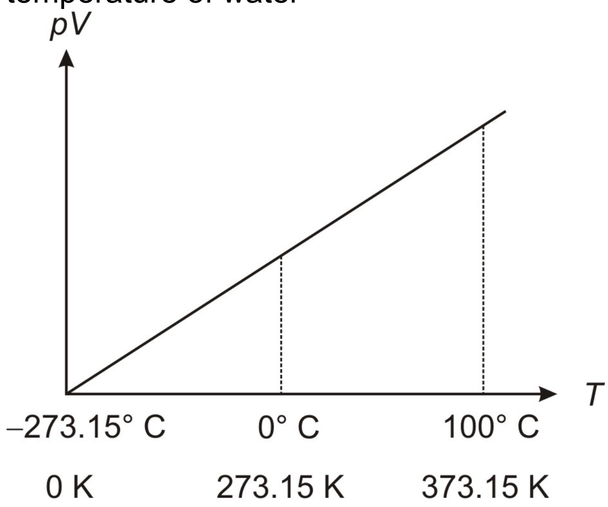
c)

$$\frac{dN(v)}{dv} \propto v^2 e^{-\frac{mv^2}{2k_BT}}$$



(1 Point)

d) With the ideal gas law $pV = Nk_BT$ and the boiling temperature and freezing temperature of water



(1 Point)

Problem 2 (4 Points)

The Lorentz-transformation between frame S and S' is
$t = \gamma(t' + \frac{V}{c^2}x')$, $x = \gamma(x' + Vt')$, $y = y'$, $z = z'$ with $\gamma = \sqrt{1 - V^2/c^2}$.

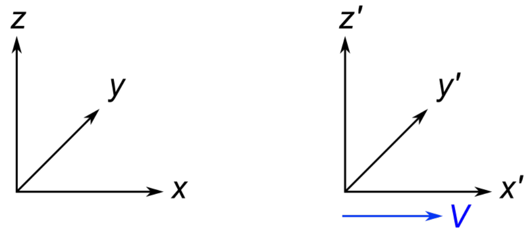
a) Write up the classical Galilei-transformation for small values of the relative velocity V. Sketch the relative motion of frame S and S'.
b) Write up the Lorentz-transformation of the relativistic wave vector.
c) Calculate the longitudinal Doppler effect starting with the Lorentz transformation.
d) Describe the velocity measurements of streaming gases by means of Laser beams.

a) For $V << c$ one gets $\gamma \rightarrow 1$ and $V/c \rightarrow 0$. The classical Galilei-transformation is

$$
\begin{aligned}
t &= t' \\
x &= (x' + Vt') \\
y &= y' \\
z &= z'
\end{aligned}
$$

(1/2 point)

Frame S' moves relative to frame S to the right side



(1/2 point)

b) With the invariant distance $s^2 = (ct) - (\vec{r})^2$ and the corresponding relation for the angular frequency and wave vector $(\omega/c)^2 - (\vec{k})^2 = 0$ one can write for the Lorentz-transformation of the relativistic wave vector

$$
\begin{aligned}
\omega/c &= \gamma(\omega'/c + \frac{V}{c}k'_x) \\
k_x &= \gamma(k'_x + \frac{V}{c}\frac{\omega'}{c}) \\
k_y &= k'_y \\
k_z &= k'_z
\end{aligned}
$$

and $V \parallel x$ and $\gamma = 1/\sqrt{1 - V^2/c^2}$

(1 point)

c) Longitudinal Doppler effect. (Source in frame S' emitting light towards an observer in frame S)

$$k'_x = -\frac{\omega}{c}$$

and with the Lorentz transformation

$$\omega/c = \gamma(\omega'/c - \frac{V}{c^2}\omega')$$
$$\omega = \omega' \frac{1 - \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} = \omega' \sqrt{\frac{1 - \frac{V}{c}}{1 + \frac{V}{c}}}$$

(1 point)

d) The measured frequency shift is determined by the projection of the velocity on the "observation" direction

$$\frac{\omega - \omega_0}{\omega_0} = \pm \frac{v_\parallel}{c}$$

$(+)$: approaching source, $(-)$: receding source.
When measuring the velocity with a laser, the frequency shift of the laser in the reference frame of the gas particle and the frequency shift of the wave emitted from the particle towards the observer contribute to the frequency shift measured in the detector. (1 point)

Problem 3 (4 Points)

a) The nucleus is formed by protons and neutrons. Write up and explain the general notation of a nucleus.
b) Give the formula for the quantized energy of the hydrogen atom neglecting the influence of the electron spin.
c) Write up the formula for the energy of x-rays according to Moseley.
d) What denotes a K_α-line in x-ray spectroscopy?

a)

$$^{A}_{Z}\text{nuclide}$$

Z: number of protons, A number of protons and neutrons.

(1 Point)

b)

$$E_n = -13.6\,\text{eV}\frac{1}{n^2}$$

n=1,2,3 ... (1 Point)

c)

$$h\nu = -13.6\,\text{eV}(Z-\beta)^2(\frac{1}{n^2}-\frac{1}{m^2})$$

$n > m$, β is a screening constant (1 Point)

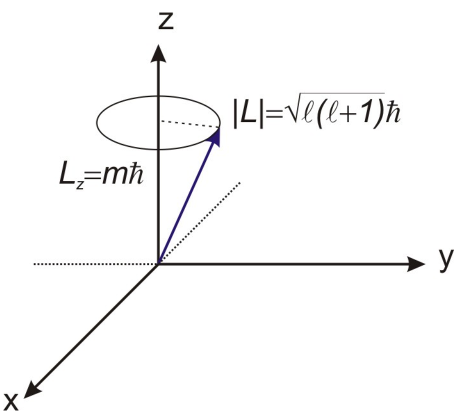
d) K_α denotes the transition $n = 2 \rightarrow 1$

(1 Point)

Problem 4 (4 Points)

a) Make a sketch of the quantized angular momentum vector. Indicate the length and the z-component of the vector.
b) Write up the eigenvalues equations of the angular momentum operator.
c) Explain, why no values can be assigned to the x- and the y-component of the angular momentum operator.
d) Spherical harmonics are the eigenfunctions of the orbital angular momentum operator. Explain the general structure of the spherical harmonics and write up the spherical harmonics $y_{1,-1}$, $y_{1,0}$ and $y_{1,+1}$. Omit the normalization factors.

a) Sketch of the angular momentum



(1 Point)

b) The two eigenvalue equations of the (orbital) angular momentum are

$$\hat{\vec{L}}^2 Y_{\ell,m}(\theta,\varphi) = \ell(\ell+1)\hbar^2 Y_{\ell,m}(\theta,\varphi)$$
$$\hat{L}_z Y_{\ell,m}(\theta,\varphi) = m\hbar Y_{\ell,m}(\theta,\varphi)$$

(1 Point)

or for a general angular momentum $\vec{J}$

$$\hat{\vec{J}}^2 |J,M\rangle = J(J+1)\hbar^2 |J,M\rangle$$
$$\hat{J}_z |J,M\rangle = M\hbar |J,M\rangle$$

c) Due to the uncertainty relation

$$\Delta L_z \Delta\varphi \geq \frac{\hbar}{2}$$

L_z is related to the rotation around the z-axis. When L_z is fixed, i.e. $\Delta L_z = 0$ one gets $\Delta\varphi \to \infty$. Consequently, eigenvalue equations for $\hat{L}_x$ and $\hat{L}_y$ cannot exist.

(1 Point)

d) The spherical harmonics consist of the eigenfunctions of $\hat{L}_z$, i.e. $e^{im\varphi}$ and of a polynomial $P_{\ell,m}(\theta)$, which is formed by $\sin\theta$ and $\cos\theta$ functions. The general structure is

$$Y_{\ell,m}(\theta,\varphi) = P_{\ell,m}(\theta)e^{im\varphi}.$$

(1/2 Point)

$$Y_{1,0} \propto \cos\theta$$
$$Y_{1,\pm1} \propto \sin\theta e^{\pm i\varphi}$$

(1/2 Point)

Problem 5 (4 Points)

a) Write up the magnetic moment due to the orbit and the spin of the electron.
b) Explain by means of Bohr's model of the atom why spin and orbital angular momentum align antiparallel for electrons in an atom.
c) Write up the Hamilton-operator of the spin-orbit coupling and calculate the energy eigenvalues.
d) For the ground state of Eu^{3+} are the quantum numbers of the orbit and the spin 3, i.e. $L = 3$ and $S = 3$. Write up the quantum states of the spin-orbit multiplet of the ground state starting with the state of lowest energy. Use the atomic notation of quantum states.

a) The magnetic moment of the orbit is

$$\vec{\mu}_L = -\mu_B \frac{\vec{L}}{\hbar}$$

the magnetic moment of the electron spin is

$$\vec{\mu}_s = -g\mu_B \frac{\vec{s}}{\hbar}$$

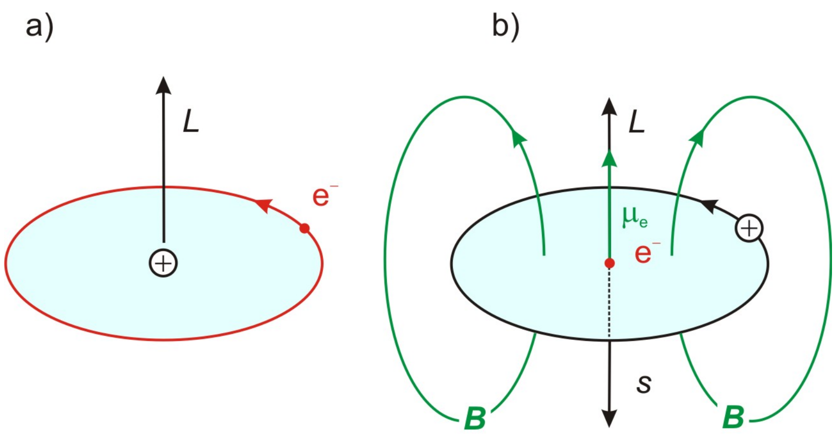
with the *g*-factor $g \approx 2$

(1 Point)

and Bohr's magneton

$$\mu_B = \frac{e\hbar}{2m}.$$

b) The following figure a) shows the motion of an electron around the nulceus in Bohr's model of the atom. Figure b) shows the motion of the nucleus around the electron in the frame of reference attached to the electron. According to Ampere's law induces the motion of the nucleus a magnetic $\vec{B}$-field (rule of the right hand) in which the intrinsic magnetic moment of the electron orients.



(1 Point)

c)

$$\hat{H}_{Ls} = \xi \frac{\hat{\vec{L}}\hat{\vec{s}}}{\hbar^2}$$

(1/2 Point)

and the energy eigenvalues are with $\vec{J}^2 = (\vec{L} + \vec{s})^2 = \vec{L}^2 + \vec{s}^2 + 2\vec{L}\vec{s}$

$$E_{JLs} = \frac{\xi}{2}(J(J+1) - L(L+1) - S(S+1))$$

(1/2 Point)

d) The notation of atomic states is

$$^{2S+1}L_J$$

The states of the ground multiplet are with increasing energy $^7F_0$, $^7F_1$, $^7F_2$, $^7F_3$, $^7F_4$, $^7F_5$, $^7F_6$.

(1 Point)

Problem 6 (4 Points)

a) Write up the entangled quantum state of two fermions.
b) How many electrons can occupy a quantum state with the orbital angular momentum quantum number $\ell = 1$?
c) Give the entangled states of two p-electrons that differ in the principal quantum number n.
d) Which of these states has the smallest energy, when electrons have the same principal quantum number n?

a) The entangled state of two fermions has to be antisymmetric with respect to the interchange of the particles

$$|a\rangle_1 |b\rangle_2 - |a\rangle_2 |b\rangle_1$$

$|a\rangle \neq |b\rangle$ denote the quantum states occupied by the two fermions.

(1 Point)

b) The quantum numbers of the electrons have to differ at least for one quantum number. There a p-orbital can be occupied at most by 6 electrons. The quantum number which have to differ are $m = 0, \pm 1$ and $m_s = \pm 1/2$.

(1 Point)

c) $^1S_0$, $^3S_1$, $^1P_1$, $^3P_{0,1,2}$, $^1D_2$, $^3D_{1,2,3}$

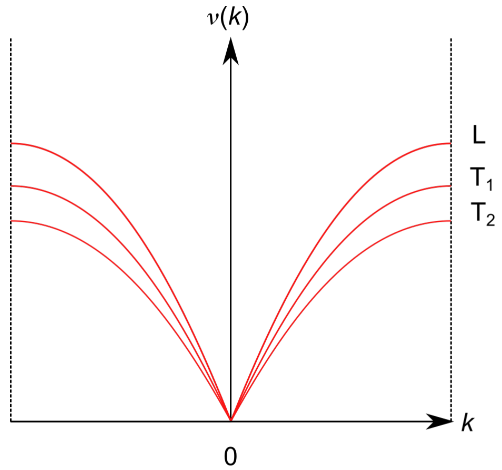
(1 Point)

d) According to Hund's rule is the ground state a $^3P_0$. (1 Point)

Problem 7 (4 Points)

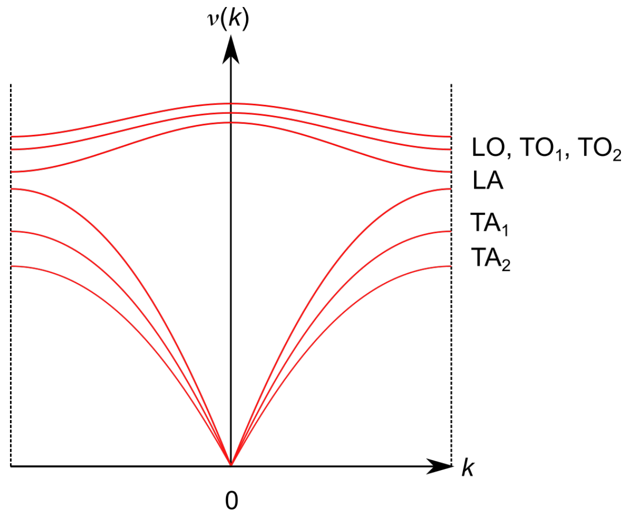
a) Sketch the phonon dispersion $\nu(\vec{k})$. Consider one propagation direction and a crystal lattice with one atom in the unit cell.
b) Sketch the phonon dispersion, when there are two atoms within the unit cell and give the typical frequency range of phonon modes.
c) Explain why the phonon modes are confined to the 1st Brillouin zone.
d) There are how many phonon modes in a crystal lattice with *n* atoms per unit cell?

a) There are two acoustic transversal modes T_1, T_2 and an acoustic longitudinal mode.



(1 Point)

b) There are three acoustic and three optical branches



(1/2 Point)

The phonon frequencies ν range between zero and some THz.

(1/2 Point)

c) The smallest wavelength is two times the distance *a* between two similar atoms in neighbouring unit cells. Therefore the wave numbers range between zero and π/a and the phonon modes are located within the 1st Brillouin zone.

(1 Point)

d) The number of $\vec{k}$ states equals the number N of unit cells within the crystal. The total number of phonon modes is therefore $3nN$. n denotes the number of atoms within the unit cell.

(1 Point)

Problem 8 (4 Points)

a) Write up the Fermi-distribution function and make a sketch for $T = 0$ and $T \neq 0$.
b) Calculate the density of states

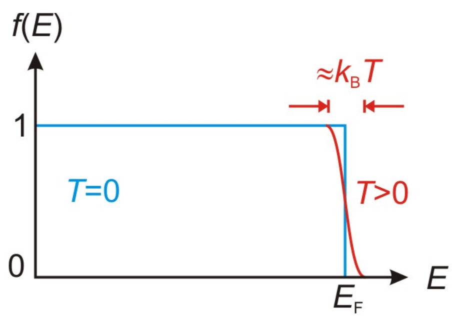
$$D(E) = \frac{1}{A}\frac{\mathrm{d}N}{\mathrm{d}E}$$

of a two-dimensional quasi free electron gas.
c) Compare the heat capacity of a classical gas formed by atoms with a quasi free electron gas.
d) Plot the energy spectrum of nearly free electrons for waves propagating along the edges of a simple cubic unit cell with lattice parameter *a*.

a) Fermi-distribution function

$$f(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}.$$

μ denotes the chemical potential with $\mu \to E_F$ for $T \to 0$.



(1 Point)

b) The number of quantum states in *k*-space is

$$dN = \frac{2\pi k dk}{(2\pi)^2/A}$$

A denotes the area occupied by the two-dimensional electron gas. With the kinetic energy $E = \hbar^2 k^2/2m$ one gets

$$dE = \frac{kdk}{m}$$

and

$$dN = \frac{A}{(2\pi)^2} 2\pi m dE \quad \to \quad D(E) = \frac{m}{(2\pi)}$$

(1 Point)

c) The heat capacity of a gas formed by atoms or molecules is

$$c_V \propto k_B$$

Heat capacity of a quasi free electron gas

$$c_V \propto \left(\frac{k_B T}{E_F}\right) k_B.$$

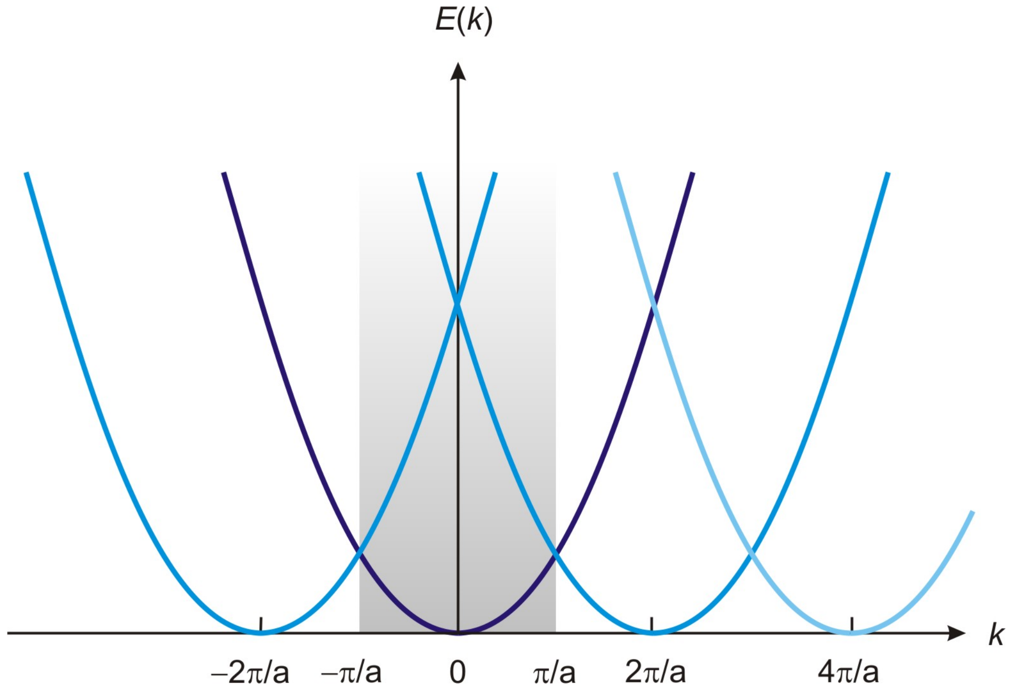
k_B denotes Boltzmann's constant and E_F the Fermi energy.

(1 Point)

d) The energy of a quasi free electron is

$$E = \frac{\hbar^2 k^2}{2m}$$

The energy of electrons in a crystal lattice is periodic in the reciprocal lattice, i.e.



(1 Point)

Problem 1 (4 Points)

a) Give the definition of entropy and temperature.
b) Give the formula of Maxwell's velocity distribution function (without prefactors) and sketch the distribution function.
c) Explain the definition of the Kelvin temperature scale.

Problem 2 (4 Points)

a) Calculate the transition energy between the states with the quantum numbers $n = 2$ and 1 for the hydrogen atom.
b) Calculate the wavelength of the resulting radiation.
c) Give the total number of the quantum states with the principal quantum numbers $n = 2$ and $n = 1$, respectively.
d) Calculate the wavelength of the K_α-line of tungsten (atomic number $Z = 74$).

Problem 3 (4 Points)

A crystal lattice results from the repetition of primitive unit cells which are formed by three linear independent vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$. For the fcc lattice of copper, these vectors are $\vec{a}_1 = \frac{a}{2}\left(\vec{e}_x + \vec{e}_z\right)$, $\vec{a}_2 = \frac{a}{2}\left(\vec{e}_y + \vec{e}_x\right)$ and $\vec{a}_3 = \frac{a}{2}\left(\vec{e}_z + \vec{e}_y\right)$. $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ denote orthogonal unit vectors and $a = 3.61 \cdot 10^{-10}$ m denotes the lattice parameter.

a) Give the definition of the basis vectors $\vec{b}_{i=1,2,3}$ of the reciprocal lattice.
b) Calculate the basis vector $\vec{b}_1$ of copper.
c) Calculate the distance between the lattice planes for the vector $\vec{b}_1$.
d) Calculate for an incident X-ray with the wavelength $\lambda = 0.35$ nm the angles of constructive interference due to diffraction on these lattice planes.

Problem 4 (4 Points)

a) The maximum of the thermal radiation spectrum of the Sun is observed for the wavelength $\lambda = 525$ nm. Calculate the temperature on the surface of the Sun.
b) Calculate the total radiation power of the Sun.
c) For the ESA satellite 'solar orbiter' the radius of the orbit around the Sun is $d = 77 \cdot 10^6$ km. Assume a spherical shape of the satellite and calculate its equilibrium temperature.

Problem 5 (4 Points)

a) The redshift is defined as $z = (\lambda' - \lambda_0)/\lambda_0$. λ_0 denotes the wavelength in the reference frame of the source. The red-shift observed for the cluster of galaxies BAS11 is $z = 0.07$. Calculated the relative velocity between Earth and BAS11.
b) Calculate the speed of He atoms moving at the temperature of $T = 300$ K with the kinetic energy $k_B T$. The atomic mass number of the He atoms is $A = 4$.
c) He atoms recede with the velocity calculated in b) from a laser (wavelength $\lambda = 623.8$ nm) at an angle of 60° with respect to the laser beam. Calculate the wavelength of the laser in the reference frame of the He atoms.
d) The He atoms scatter the light without changing its frequency and recede from an observer who measures the scattered light at an angle of 90° with respect

to the laser beam. Calculate the redshift of the scattered light in the reference frame of the observer.

Problem 6 (4 Points)

a) Which quantum numbers determine the spin of the electron?
b) Sketch the vector diagram of the spin.
c) Calculate the eigenvalues of the spin.
d) Calculate the radius of an electron assuming a homogenous rotating sphere with the mass and the angular momentum of an electron. Assume also that the highest speed on the surface of the sphere is the speed of light. Hint: $L = I\omega$ and the moment of inertia of a sphere is $I = 2mr^2/5$.

Problem 7 (4 Points)

a) How does the Hamilton function transform into the Hamiltonian?
b) Give the most general form of the Schrödinger equation.
c) Give the Schrödinger equation of the hydrogen atom ignoring relativistic effects.
d) Give the Schrödinger equation of the hydrogen atom in a strong magnetic field including the spin of the electron.

Problem 8 (4 Points)

a) Calculate the conduction electron density $n = N/V$ of Copper. The mass density of Copper is $\rho_{\mathrm{Cu}} = 8.96\,\mathrm{g/cm^3}$ and the mass number $A = 63.55$.
Hint: There is one conduction electron per Copper atom.
b) Calculate the Fermi wave number of Copper.
c) Calculate the Fermi energy of Copper.
d) Calculate the Fermi velocity of Copper, i.e. the classical speed of electrons moving with the kinetic energy $E_{\mathrm{kin}} = E_{\mathrm{Fermi}}$.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\mathrm{m/s}$
Boltzmann constant:	$k_B = 8.6 \cdot 10^{-5}\,\mathrm{eV/K} = 1.38 \cdot 10^{-23}\,\mathrm{J/K}$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\mathrm{eVs} = 6.62 \cdot 10^{-34}\,\mathrm{Ws^2}$
Stefan-Boltzmann's constant:	$\sigma = 5.67 \cdot 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$
Wien's constant:	$b = 2.9\,\mathrm{mm \cdot K}$
Atomic mass unit:	$u = 1.66 \cdot 10^{-27}\,\mathrm{kg}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\mathrm{As}$
Mass of the electron:	$m_e = 500\,\mathrm{keV}/c^2 = 9 \cdot 10^{-31}\,\mathrm{kg}$
Rydberg unit of energy:	$R = 13.6\,\mathrm{eV}$
Radius of the Sun:	$r_S = 6.96 \cdot 10^8\,\mathrm{m}$

Problem 1 (4 Points)

a) Give the definition of entropy and temperature.
b) Give the formula of Maxwell's velocity distribution function (without prefactors) and sketch the distribution function.
c) Explain the definition of the Kelvin temperature scale.

a)

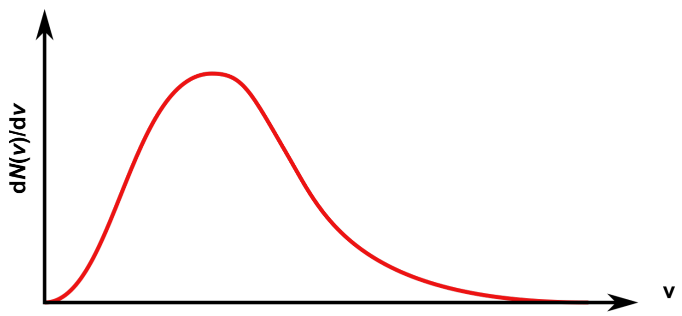
$$S = k_B \ln \Gamma(E)$$

$\Gamma(E)$ denotes the number of possibilities of a system to realise states with the energy E.
The temperature is

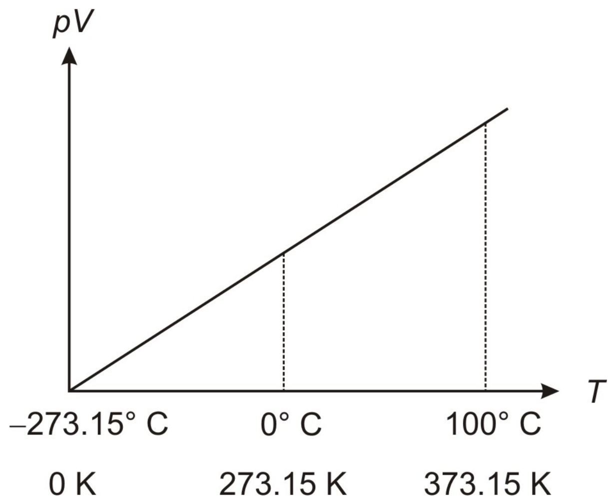
$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

b)

$$\frac{dN(v)}{dv} \propto v^2 e^{-mv^2/2k_B T}$$



c) With the ideal gas law $pV = Nk_B T$ and the boiling temperature and freezing temperature of water



Problem 2 (4 Points)

a) Calculate the transition energy between the states with the quantum numbers $n = 2$ and 1 for the hydrogen atom.
b) Calculate the wavelength of the resulting radiation.
c) Give the total number of the quantum states with the principal quantum numbers $n = 2$ and $n = 1$, respectively.
d) Calculate the wavelength of the K_α-line of tungsten (atomic number $Z = 74$).

a) The energy difference between the states with the quantum numbers $n = 2$ and $n = 1$ is

$$E_2 - E_1 = -R\left(\frac{1}{2^2} - 1\right) = -13.6\,\text{eV}\left(-\frac{3}{4}\right) = 10.2\,\text{eV}$$

b) With Planck's law

$$E = h\nu = \frac{hc}{\lambda} \quad \rightarrow \quad \lambda = \frac{hc}{E_2 - E_1} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^{8}\,\text{ms}^{-1}}{10.2\,\text{eV}} = 122\,\text{nm}$$

c) The ground state is 1s and there are two quantum state with the spin quantum number $m_s = \pm 1/2$. The excited states are 2s and 2p. For 2s there are two quantum state with the spin quantum number $m_s = \pm 1/2$ and for 2p there are six quantum state with the spin quantum number $m_s = \pm 1/2$ and the angular momentum quantum number $m = 0, \pm 1$.
d) The K_α line denotes the transitions between the quantum states $n = 1$ and $n = 2$ for an atom with many electrons. The screening constant of the nuclear charge is $\beta = 1$. The energy difference is

$$\begin{aligned} E_2 - E_1 &= -R(Z-1)^2\left(\frac{1}{2^2} - 1\right) = -13.6\,\text{eV}(74-1)^2\left(-\frac{3}{4}\right) \\ &= 10.2 \cdot 73^2\,\text{eV} = 54.36\,\text{keV}. \end{aligned}$$

The wavelength is

$$\lambda = \frac{hc}{E_2 - E_1} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^{8}\,\text{ms}^{-1}}{54.36\,\text{keV}} = 0.23 \cdot 10^{-10}\,\text{m}$$

Problem 3 (4 Points)

A crystal lattice results from the repetition of primitive unit cells which are formed by three linear independent vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$. For the fcc lattice of copper, these vectors are $\vec{a}_1 = \frac{a}{2}\left(\vec{e}_x + \vec{e}_z\right)$, $\vec{a}_2 = \frac{a}{2}\left(\vec{e}_y + \vec{e}_x\right)$ and $\vec{a}_3 = \frac{a}{2}\left(\vec{e}_z + \vec{e}_y\right)$. $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ denote orthogonal unit vectors and $a = 3.61 \cdot 10^{-10}$ m denotes the lattice parameter.

a) Give the definition of the basis vectors $\vec{b}_{i=1,2,3}$ of the reciprocal lattice.
b) Calculate the basis vector $\vec{b}_1$ of copper.
c) Calculate the distance between the lattice planes for the vector $\vec{b}_1$.
d) Calculate for an incident X-ray with the wavelength $\lambda = 0.35$ nm the angles of constructive interference due to diffraction on these lattice planes.

a) With $\vec{a}_i\,\vec{b}_j = 2\pi\delta_{ij}$ the basis vectors of the reciprocal lattice are

$$\vec{b}_1 = \frac{2\pi}{V_{Cell}}\vec{a}_2 \times \vec{a}_3 \qquad \text{and cyclic permutations}$$

V_{Cell} denotes the volume of the unit cell

$$V_{Cell} = \vec{a}_1(\vec{a}_2 \times \vec{a}_3)$$

b)

$$\vec{b}_1 = \frac{2\pi}{V_{Cell}}\left(\frac{a}{2}\left(\vec{e}_y + \vec{e}_x\right)\right) \times \left(\frac{a}{2}\left(\vec{e}_z + \vec{e}_y\right)\right)$$

with $\vec{e}_y \times \vec{e}_z = \vec{e}_x$, $\vec{e}_x \times \vec{e}_z = -\vec{e}_y$ und $\vec{e}_x \times \vec{e}_y = \vec{e}_z$

$$\vec{b}_1 = \frac{\pi a^2}{2V_{Cell}}(\vec{e}_x - \vec{e}_y + \vec{e}_z)$$

and

$$V_{Cell} = \frac{a^3}{8}\left(\vec{e}_x + \vec{e}_z\right)\left(\vec{e}_x - \vec{e}_y + \vec{e}_z\right) = \frac{a^3}{4}$$

$$\vec{b}_1 = \frac{2\pi}{a}(\vec{e}_x - \vec{e}_y + \vec{e}_z)$$

c) The distance between the lattice planes d is

$$|\vec{b}_1| = \frac{2\pi}{d} \quad \rightarrow \quad \frac{2\pi}{d} = \frac{2\pi}{a}\sqrt{(\vec{e}_x - \vec{e}_y + \vec{e}_z)(\vec{e}_x - \vec{e}_y + \vec{e}_z)} = \frac{2\pi}{a}\sqrt{3}$$

$$\rightarrow \quad d = \frac{a}{\sqrt{3}} = \frac{3.61 \cdot 10^{-10}\,\text{m}}{\sqrt{3}} = 2.1 \cdot 10^{-10}\,\text{m}$$

d) With the Bragg-condition for constructive interference

$$m\lambda = 2d\sin\alpha_m \quad \rightarrow \quad \sin\alpha_m = m\frac{0.35\cdot 10^{-9}\,\mathrm{m}}{2\cdot 2.1\cdot 10^{-10}\,\mathrm{m}} = m\cdot 0.833$$

$$m = 1: \quad \alpha_1 = 54.4°$$

Problem 4 (4 Points)

a) The maximum of the thermal radiation spectrum of the Sun is observed for the wavelength $\lambda = 525$ nm. Calculate the temperature on the surface of the Sun.
b) Calculate the total radiation power of the Sun.
c) The radius of the orbit of the ESA satellite 'solar orbiter' around the Sun is $d = 77 \cdot 10^6$ km. Assume a spherical shape of the satellite and calculate its equilibrium temperature.

a) With Wien's displacement law is the temperature of the surface of the Sun

$$\lambda_{max} = \frac{2.9\,\text{mmK}}{T} \quad \rightarrow \quad T = \frac{2.9 \cdot 10^{-3}\,\text{mK}}{525 \cdot 10^{-9}\,\text{m}} = 5524\,\text{K}$$

b) With the law of Stefan-Boltzmann and the radius of the Sun

$$\begin{aligned} P &= A\sigma T^4 = 4\pi r_S{}^2 \sigma T^4 \\ &= 4\pi \cdot (696340 \cdot 10^3)^2\,\text{m}^2 \cdot 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4} \cdot 5524^4\,\text{K}^4 \\ &= 3.22 \cdot 10^{26}\,\text{W} \end{aligned}$$

c) The intensity of the Sun in a distance d from the centre is

$$I = \frac{P}{4\pi d^2}$$

The satellite receives the power (r denotes the radius of the satellite)

$$P_{\text{in}} = \pi r^2 \cdot I$$

and emits the power

$$P_{\text{out}} = 4\pi r^2 \sigma T^4$$

With the equilibrium condition $P_{\text{in}} = P_{\text{out}}$ results the temperature of the satellite

$$T = \left(\frac{\pi r^2 \cdot I}{4\pi r^2 \sigma}\right)^{1/4} = \left(\frac{2\pi r^2 \cdot P}{4\pi r^2 \sigma 4\pi d^2}\right)^{1/4} = \left(\frac{P}{16\pi d^2 \sigma}\right)^{1/4}$$

$$\left(\frac{P}{16\pi d^2 \sigma}\right)^{1/4} = \left(\frac{3.18 \cdot 10^{26}\,\text{W}}{16\pi (77 \cdot 10^9\,\text{m})^2 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}}\right)^{1/4} = 372\,\text{K}$$

Problem 5 (4 Points)

a) The redshift is defined as $z = (\lambda' - \lambda_0)/\lambda_0$. λ_0 denotes the wavelength in the reference frame of the source. The red-shift observed for the cluster of galaxies BAS11 is $z = 0.07$. Calculated the relative velocity between Earth and BAS11.
b) Calculate the speed of He atoms moving at the temperature of $T = 300$ K with the kinetic energy $k_B T$. The atomic mass number of the He atoms is $A = 4$.
c) He atoms recede with the velocity calculated in b) from a laser (wavelength $\lambda = 623.8$ nm) at an angle of 60° with respect to the laser beam. Calculate the wavelength of the laser in the reference frame of the He atoms.
d) The He atoms scatter the light without changing its frequency and recede from an observer who measures the scattered light at an angle of 90° with respect to the laser beam. Calculate the redshift of the scattered light in the reference frame of the observer.

a) The longitudinal Doppler effect for a receding source is

$$\lambda' = \lambda_0 \sqrt{\frac{1 + \left(\frac{v}{c}\right)}{1 - \left(\frac{v}{c}\right)}}$$

and one gets for the red-shift

$$\mathrm{z} + 1 = \sqrt{\frac{1 + \left(\frac{v}{c}\right)}{1 - \left(\frac{v}{c}\right)}} \quad \rightarrow \quad (\mathrm{z} + 1)^2 = \frac{1 + \left(\frac{v}{c}\right)}{1 - \left(\frac{v}{c}\right)} = 1{,}07^2 = 1{,}14$$

and with

$$1 + \left(\frac{v}{c}\right) = 1{,}14 - 1{,}14\left(\frac{v}{c}\right) \quad \rightarrow \quad 2{,}14\left(\frac{v}{c}\right) = 0{,}14$$

the velocity of BAS11

$$v = c\frac{0{,}14}{2{,}14} = 0{,}065\, c$$

b) The speed of the Helium-atoms with the energy $k_B T$ is

$$\frac{1}{2} m_{\mathrm{He}} v^2 = k_B T \rightarrow \mathrm{v} = \sqrt{\frac{2 k_B T}{m_{\mathrm{He}}}} = \sqrt{\frac{2 \cdot 1.38 \cdot 10^{-23}\,\mathrm{JK^{-1}} \cdot 300\,\mathrm{K}}{4 \cdot 1.66 \cdot 10^{-27}\,\mathrm{kg}}} = 1117\,\mathrm{ms^{-1}}$$

c) Since $v << c$ the wavelength of the laser in the frame of the receding He-atoms is

$$\frac{\lambda_{\mathrm{He}} - \lambda_0}{\lambda_0} = \frac{v}{c}\cos 60° \quad \rightarrow \quad \lambda_{\mathrm{He}} = \lambda_0\left(1 + \frac{v}{c}\cos 60°\right)$$

$$\lambda_{\mathrm{He}} = 623.8\,\mathrm{nm}\left(1 + 0.5 \cdot \frac{1117\,\mathrm{ms^{-1}}}{3 \cdot 10^8\,\mathrm{ms^{-1}}}\right) = 623.8\,\mathrm{nm}\left(1 + 1.86 \cdot 10^{-6}\right)$$

d) The shift of the wavelength in the frame of the observer is

$$\lambda_{\mathrm{observer}} = \lambda_{\mathrm{He}}(1 + \frac{\mathrm{v}}{\mathrm{c}}\cos 30°) = \lambda_0\left(1 + \frac{v}{c}\cos 60°\right)\left(1 + \frac{\mathrm{v}}{\mathrm{c}}\cos 30°\right)$$

and with (the quadratic term $(v/c)^2$ can be neglected)

$$\lambda_{\text{observer}} - \lambda_0 = \lambda_0 \frac{v}{c}(\cos 60° + \cos 30°) = 623.8\,\text{nm}\,1.86 \cdot 10^{-6}(\cos 60° + \cos 30°)$$

is the red-shift in the frame of the observer

$$z = 1.86 \cdot 10^{-6}(\cos 60° + \cos 30°) = 2.54 \cdot 10^{-6}$$

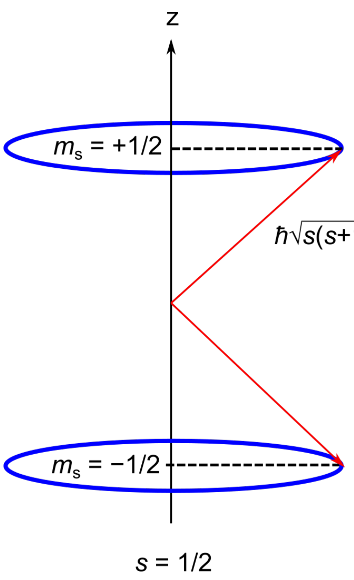
Problem 6 (4 Points)

a) Which quantum numbers determine the spin of the electron?
b) Sketch the vector diagram of the spin.
c) Calculate the eigenvalues of the spin.
d) Calculate the radius of an electron assuming a homogenous rotating sphere with the mass and the angular momentum of an electron. Assume also that the highest speed on the surface of the sphere is the speed of light. Hint: $L = I\omega$ and the moment of inertia of a sphere is $I = 2mr^2/5$.

a) The spin is determined by the quantum number $s = 1/2$ and the quantum numbers $m_s = \pm 1/2$.

(1 point)

b) Sketch of the spin-vector: The length of the vector is determined by the quantum number $s = 1/2$ and the projection of the z-axis by the quantum numbers $m_s = \pm 1/2$.



c) The eigenvalue of $\vec{s}^{\,2}$ is

$$\hbar^2\, s(s+1) = \left(\frac{4.14 \cdot 10^{-15}\,\text{eVs}}{2\pi}\right)^2 \frac{3}{4} = 0.33 \cdot 10^{-30}\,(\text{eVs})^2$$

The eigenvalues of s_z are

$$\hbar m_s = \pm\frac{1}{2}\frac{4.14 \cdot 10^{-15}\,\text{eVs}}{2\pi} = \pm 0.33 \cdot 10^{-15}\,\text{eVs}$$

d) The angular velocity is

$$\omega = \frac{2\pi}{T} \quad \text{and with the period} \quad T = 2\pi\,\frac{r}{c} \quad \text{one gets} \quad \omega = \frac{c}{r}$$

When for the angular momentum the projection of the spin on the z-axis is used, i.e.

$$\frac{\hbar}{2} = I \cdot \omega = \frac{2m_\mathrm{e}r^2}{5} \cdot \frac{c}{r}$$

one gets

$$r = \frac{5\hbar}{4m_\mathrm{e}c} = \frac{5\hbar c}{4m_\mathrm{e}c^2} = \frac{5 \cdot 4.14 \cdot 10^{-15}\,\mathrm{eVs}\,3 \cdot 10^{8}\,\mathrm{ms^{-1}}}{4 \cdot 2\pi\,500 \cdot 10^{3}\,\mathrm{eV}} = 0.5 \cdot 10^{-12}\,\mathrm{m}$$

When the length of the angular momentum vector is used $\hbar\sqrt{s(s+1)}$ one gets instead

$$r' = r\,\frac{\hbar\sqrt{s(s+1)}}{\hbar/2} = r \cdot \sqrt{3} = 0.866 \cdot 10^{-12}\,\mathrm{m}$$

This values for the radius of an electron are several orders of magnitude larger than the radius of a nucleus ($\approx 10^{-15}$ m). Therefore the electron is certainly not a rotating sphere.

Problem 7 (4 Points)

a) How does the Hamilton function transform into the Hamiltonian?
b) Give the most general form of the Schrödinger equation.
c) Give the Schrödinger equation of the hydrogen atom ignoring relativistic effects.
d) Give the Schrödinger equation of the hydrogen atom in a strong magnetic field including the spin of the electron.

a) The momentum has to be replaced in the Hamilton function by the momentum operator

$$\vec{p} \quad \to \quad -\mathrm{i}\hbar\nabla$$

b) The most general version of the Schrödinger equation is

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\psi$$

Thereby $\hat{\mathcal{H}}$ denotes the hamiltonian. ψ denotes a quantum state. This can be a wave function but also e.g. the quantum state of a spin $|s, m_\mathrm{s}\rangle$

c)

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\right)\psi$$

d)

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} + \mu_\mathrm{B}\frac{\hat{\vec{L}}}{\hbar}\vec{B} + 2\mu_\mathrm{B}\frac{\hat{\vec{s}}}{\hbar}\vec{B}\right)\psi$$

The magnetic moments of spin and orbital angular momentum precess independent from each other around the magnetic field (Paschen-Back effect). The spin-orbit coupling can be neglected.

Problem 8 (4 Points)

a) Calculate the conduction electron density $n = N/V$ of Copper. The mass density of Copper is $\rho_{\mathrm{Cu}} = 8.96\,\mathrm{g/cm^3}$ and the mass number $A = 63.55$.
Hint: There is one conduction electron per Copper atom.

b) Calculate the Fermi wave number of Copper.

c) Calculate the Fermi energy of Copper.

d) Calculate the Fermi velocity of Copper, i.e. the classical speed of electrons moving with the kinetic energy $E_{\mathrm{kin}} = E_{\mathrm{Fermi}}$.

a) The density of Copper is, when there are N Cu-atoms

$$\rho_{\mathrm{Cu}} = \frac{m}{V} = \frac{NAu}{V} = \frac{N \cdot 63.55u}{V}$$

and the density of conduction electrons is

$$\frac{N}{V} = \frac{\rho_{\mathrm{Cu}}}{63.55u} = \frac{8.96 \cdot 10^{-3}\,\mathrm{kg/cm^3}}{63.55 \cdot 1.66 \cdot 10^{-27}\,\mathrm{kg}} = 8.5 \cdot 10^{22}\,\mathrm{cm^{-3}} = 8.5 \cdot 10^{28}\,\mathrm{m^{-3}}.$$

b) The Fermi-wave number results, when the volume of the Fermi-sphere is divided by the volume of a k-state. Each k-state can be occupied with two electrons. With

$$N = 2\frac{\frac{4\pi k_{\mathrm{F}}^3}{3}}{\frac{(2\pi)^3}{V}}$$

one gets

$$k_{\mathrm{F}} = \left(3\pi^2\frac{N}{V}\right)^{1/3} = \left(3\pi^2 8.5 \cdot 10^{28}\,1/\mathrm{m^3}\right)^{1/3} = 13.6 \cdot 10^{9}\,\mathrm{m^{-1}}.$$

c) The Fermi-energy is

$$E_{\mathrm{F}} = \frac{\hbar^2 k_{\mathrm{F}}^2}{2m_{\mathrm{e}}} = \frac{(6.62 \cdot 10^{-34}\,\mathrm{Ws^2} \cdot 13.6 \cdot 10^{9}\,\mathrm{m^{-1}})^2}{2(2\pi)^2\,9 \cdot 10^{-31}\,\mathrm{kg}} = 11.4 \cdot 10^{-19}\,\mathrm{Ws} = 7.1\,\mathrm{eV}$$

d) Die Fermi-Geschwindigkeit ist

$$v_{\mathrm{F}} = \frac{\hbar k_{\mathrm{F}}}{m_{\mathrm{e}}} = \frac{\hbar k_{\mathrm{F}} c^2}{m_{\mathrm{e}} c^2} = \frac{4.14 \cdot 10^{-15}\,\mathrm{eVs} \cdot 13.6 \cdot 10^{9}\,\mathrm{m^{-1}}(3 \cdot 10^{8}\,\mathrm{ms^{-1}})^2}{2\pi \cdot 500 \cdot 10^{3}\,\mathrm{eV}} = 0.16 \cdot 10^{7}\,\mathrm{ms^{-1}}$$

Problem 1 (4 Points)

The two stars of the binary star system β-Aurigae have almost the same mass and orbit the common centre of gravity. During the time of observation one star moves at a speed of 100 km/s directly towards the Earth whereas the other star recedes from Earth at this speed.

a) Calculate the Doppler shift of the red Balmer line ($\lambda = 657$ nm) due to the rotation of the two stars around the centre of gravity.
b) The red Balmer line splits in two components due to the rotation. Calculate the energy difference between the two components in units of eV.

The microwave beam of a radar trap (frequency $\nu_0 = 30$ GHz) is oriented under 45° towards the approaching traffic.

c) Calculate the frequency shift of the microwave measured in a car approaching the radar trap with a velocity of 70 km/h.
d) A fraction of the radiation is reflected back to the radar trap. Calculate the frequency shift which is measured by the receiver.

Problem 2 (4 Points)

a) What does the acronym Laser mean?
b) Which condition has to be fulfilled so that Laser light can be created?
c) Explain by means of a sketch how the condition for generating Laser light can be met with a four-energy level scheme.
d) How does the four-level scheme of a continuous-wave Laser differ from that of a pulsed Laser?

Problem 3 (4 Points)

Electrons are accelerated from rest by a voltage of 2 MV.

a) Calculate the total energy of the electrons.
b) Calculate the relativistic mass of the electrons.
c) Calculate the momentum of the electrons.
d) Calculate the wavelength of the electron beam.

Problem 4 (4 Points)

a) Give the formula for the quantized energy of the hydrogen atom.
b) Calculate the shortest wavelength of the Lyman and the Balmer series, respectively.
c) What denotes the K_α-line in x-ray spectroscopy?
d) In the X-ray spectrum of an element, the K_α-line has the wavelength $\lambda = 8.36 \cdot 10^{-10}$ m. What is the atomic number Z of the element?

Problem 5 (4 Points)

a) Give the quantum numbers which characterise an orbital of the hydrogen atom.
b) Which values are possible for the angular momentum quantum number ℓ of the orbitals with the principal quantum number $n = 4$?
c) Sketch the radial wave functions with the quantum numbers $n = 4$, $\ell = 0$ and $n = 4$, $\ell = 3$.

d) Give reasons for your sketches.

Problem 6 (4 Points)

The atomic number of Rb (Rubidium) is $Z = 37$. Consider the ground state of the atom.

a) Make a table which includes all occupied orbitals and occupation numbers.
b) The smallest excitation energy is observed for the transition 5s $\leftrightarrow$ 5p. The transition is split into a doublet. Give the full atomic notation for both transitions.
c) The *nuclear* spin of ^{85}Rb is $I = 5/2$. Give the quantum numbers F of the total angular momentum of the ground state and the two excited states.
d) Give the selection rules for electric dipole transitions between these quantum states.

Problem 7 (4 Points)

a) The momentum of a phonon in a solid is called a crystal momentum. Explain the difference between the crystal momentum and the momentum of Newtonian mechanics.
b) In inelastic neutron scattering, the neutrons can emit and absorb phonons. Write down the energy and momentum conservation laws for the emission of a phonon by a neutron.
c) What is the difference between Brillouin and Raman scattering?
d) Sketch an experimental setup for measuring Raman scattering as well as the vector diagram of the momentum conservation law.

Problem 8 (4 Points)

a) Calculate the density $n = N/V$ of conduction electrons of Rubidium. The mass density of Rb is $\rho_{\text{Rb}} = 1.53\,\text{g/cm}^3$ and the mass number $A = 85.47$.
Hint: There is one conduction electron per Rb atom.
b) Calculate the Fermi wave number k_{F}.
c) Calculate the Fermi energy E_{F}.
d) Calculate the Fermi velocity v_{F} of Rb.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\text{m/s}$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\text{eVs} = 6.62 \cdot 10^{-34}\,\text{Ws}^2$
Atomic mass unit:	$u = 1.66 \cdot 10^{-27}\,\text{kg}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\text{As}$
Mass of the electron:	$m_{\text{e}} = 500\,\text{keV}/c^2 = 9 \cdot 10^{-31}\,\text{kg}$
Rydberg unit of energy:	$R = 13.6\,\text{eV}$

Problem 1 (4 Points)

The two stars of the binary star system β-Aurigae have almost the same mass and orbit the common centre of gravity. During the time of observation one star moves at a speed of 100 km/s directly towards the Earth whereas the other star recedes from Earth at this speed.

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The microwave beam of a radar trap (frequency $\nu_0 = 30$ GHz) is oriented under 45° towards the approaching traffic.

c) Calculate the frequency shift of the microwave measured in a car approaching the radar trap with a velocity of 70 km/h.
d) A fraction of the radiation is reflected back to the radar trap. Calculate the frequency shift which is measured by the receiver.

a) Doppler shift of the approaching and receding star

$$\lambda = \lambda_0 \sqrt{\frac{1 \mp (v/c)}{1 \pm (v/c)}} \quad \xrightarrow{v<<c} \quad \lambda_0(1 \mp (v/c))$$

The wavelength becomes smaller for the approaching star and larger for the receding star

$$\Delta\lambda = \lambda - \lambda_0 = \mp\lambda_0 \frac{v}{c} = \mp 657\,\text{nm} \cdot \frac{10^5 \text{ms}^{-1}}{3 \cdot 10^8\,\text{ms}^{-1}} = \mp 0.219\,\text{nm}$$

(1 Point)

b) Energy difference between the two shifted spectral lines

$$\Delta E = h\Delta\nu$$

e.g. with $\nu = c/\lambda$ is

$$\Delta\nu = -\frac{c}{\lambda^2}\Delta\lambda$$

and the energy difference of the splitting is

$$\Delta E = 2 \cdot \frac{hc}{\lambda_0^2} \cdot \Delta\lambda = 2 \cdot \frac{4.14 \cdot 10^{-15}\,\text{eVs}\; 3 \cdot 10^8\,\text{ms}^{-1}}{(657 \cdot 10^{-9})^2\,\text{m}^2} \cdot 0.219 \cdot 10^{-9}\,\text{m} = 1.26\text{ meV}$$

(1 Point)

Remark: The effect is very much larger than the splitting due to the spin-orbit coupling of the hydrogen atom of about 40 μeV.

c) The frequency measured in the approaching car is increased according to

$$\nu = \nu_0(1 + \frac{v}{c}\cos\alpha),$$

d.h.

$$\begin{aligned}\nu =& \nu_0 + \nu_0\frac{v}{c}\cos\alpha = 30\,\text{GHz} + 30 \cdot 10^9\,\text{Hz}\frac{70 \cdot 10^3\,\text{m}}{3600\,\text{s} \cdot 3 \cdot 10^8\,\text{ms}^{-1}}\cos 45°\\ =& 30\,\text{GHz} + 1375\,\text{Hz}\end{aligned}$$

(1 Point)

d) The frequency of the reflected wave measured with the receiver is also enhanced according to

$$\nu' = \nu(1 + \frac{v}{c}\cos\alpha) = \nu_0(1 + \frac{v}{c}\cos\alpha)^2 = \nu_0(1 + 2\frac{v}{c}\cos\alpha) = 30\,\text{GHz} + 2750\,\text{Hz}$$

(1 Point)

Remark: The term $\nu_0(\frac{v}{c}\cos\alpha)^2 = 12.6 \cdot 10^{-7}$ Hz can be neglected.

Problem 2 (4 Points)

a) What does the acronym Laser mean?
b) Which condition has to be fulfilled so that Laser light can be created?
c) Explain by means of a sketch how the condition for generating Laser light can be met with a four-energy level scheme.
d) How does the four-level scheme of a continuous-wave Laser differ from that of a pulsed Laser?

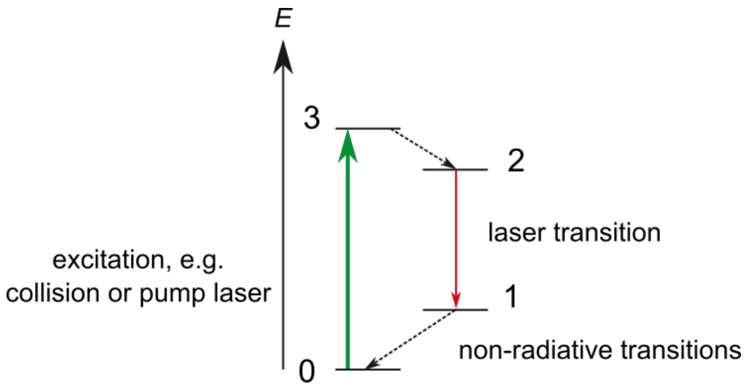
a) LASER: light amplification by stimulated emission of radiation

(1 Point)

b) Population inversion must be achieved, i.e. there must be more electrons in an excited energy level than in an energy level with a smaller energy.

(1 Point)

c) The atoms are excited from the ground state into energy level 3. With the helium-neon laser, this happens through the collision of helium and neon atoms. Usually a second laser is used to pump the transition between the ground state and energy level 3. From energy level 3 the atoms can relax to energy level 2. As a result of the pumping process, the energy level 2 is occupied more than it corresponds to the thermal equilibrium. Population inversion can be achieved between the energy level 2 and an excited energy level 1.



(1 Point)

d) In the case of a continuous laser, the pumping rate between the ground state and the energy level 2 is smaller than the relaxation rate between the energy level 1 and the ground state. Therefore, the population inversion between energy levels 2 and 1 can be achieved permanently.

In the case of a pulsed laser, the pumping rate between the ground state and the energy level 2 is larger than the relaxation rate between the energy level 1 and the ground state. After a short time, the stimulated emission leads to equal population of energy levels 1 and 2, and the amplification of light by stimulated emission of radiation is no longer possible.

(1 Point)

Problem 3 (4 Points)

Electrons are accelerated from rest by a voltage of 2 MV.

a) Calculate the total energy of the electrons.
b) Calculate the relativistic mass of the electrons.
c) Calculate the momentum of the electrons.
d) Calculate the wavelength of the electron beam.

a) The total energy is the sum of rest energy and kinetic energy, i.e.

$$E = m_0c^2 + eU$$

U denotes the acceleration voltage

$$E = 500\,\text{keV} + 2\text{MeV} = 2.5\,\text{MeV}$$

(1 Point)

b) The relativistic mass is

$$E = mc^2$$

i.e.

$$m = \frac{E}{c^2} = 2.5\,\text{MeV}/c^2 = \frac{2.5 \cdot 10^6 \cdot 1.6 \cdot 10^{-19}\,\text{VAs}}{(3 \cdot 10^8\,\text{ms}^{-1})^2} = 4.4 \cdot 10^{-30}\,\text{kg}$$

(1 Point)

c) The momentum is with

$$E^2 = m_0^2c^4 + c^2p^2 \quad \rightarrow \quad p = \frac{1}{c}\sqrt{E^2 - m_0^2c^4}$$

$$p = \frac{1}{c}\sqrt{2.5^2 - 0.5^2}\,\text{MeV} = 2.45\,\text{MeV}/c = \frac{2.45 \cdot 10^6 \cdot 1.6 \cdot 10^{-19}\,\text{VAs}}{3 \cdot 10^8\,\text{ms}^{-1}}$$
$$= 1.33 \cdot 10^{-21}\,\text{kgms}^{-1}$$

(1 Point)

(or with $v \approx c$ and $p \approx mc = 2.5\,\text{MeV}/c$)

d) The de Broglie wave length is

$$\lambda = \frac{h}{p} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^8\,\text{ms}^{-1}}{2.45 \cdot 10^6\,\text{eV}} = 5.07 \cdot 10^{-13}\,\text{m}$$

(1 Point)

Problem 4 (4 Points)

a) Give the formula for the quantized energy of the hydrogen atom.
b) Calculate the shortest wavelength of the Lyman and the Balmer series, respectively.
c) What denotes the K_α-line in x-ray spectroscopy?
d) In the X-ray spectrum of an element, the K_α-line has the wavelength $\lambda = 8.36 \cdot 10^{-10}$ m. What is the atomic number Z of the element?

a) Energy of the hydrogen atom

$$E_n = -\frac{R}{n^2} \qquad n = 1, 2, ...$$

(1 Point)

b) The energy difference between two energy levels E_n and E_m is

$$\Delta E_{n,m} = |E_n - E_m| = R\left|\frac{1}{n^2} - \frac{1}{m^2}\right| = R\frac{|n^2 - m^2|}{n^2m^2}$$

and the wavelength for a transition between E_n and E_m is

$$\lambda_{n,m} = \frac{ch}{\Delta E_{n,m}} = \frac{ch}{R}\frac{n^2m^2}{|n^2 - m^2|} = \frac{3 \cdot 10^8\,\mathrm{ms^{-1}} \cdot 4.14 \cdot 10^{-15}\,\mathrm{eVs}}{13.6\,\mathrm{eV}}\frac{n^2m^2}{|n^2 - m^2|}$$
$$= 91.3\,\mathrm{nm}\frac{n^2m^2}{|n^2 - m^2|}$$

the smallest wavelength is due to the emission of a photon between $(m \to \infty) \to n$ is

$$\lambda_{\infty,n} = 91.3\,\mathrm{nm} \cdot n^2$$

- Lyman series $n = 1$

$$\lambda_{\infty,1} = 91.3\,\mathrm{nm}$$

- Balmer series $n = 2$

$$\lambda_{\infty,2} = 91.3\,\mathrm{nm} \cdot 2^2 = 365\,\mathrm{nm}$$

(1 Point)

c) The K_α-line denotes the transition $n = 2 \to 1$.

(1 Point)

d) According to Moseley's law is the energy of the transition (The nuclear charge Z is screened by the remaining 1s electron)

$$\Delta \mathrm{E}_{2,1} = |E_2 - E_1| = R(Z-1)^2\left|\frac{1}{2^2} - 1\right|$$

and the wavelength of the K_α-line is

$$\lambda_{21} = \frac{91.3\,\mathrm{nm}}{(Z-1)^2} \cdot \frac{2^2}{2^2 - 1} = \frac{91.3\,\mathrm{nm}}{(Z-1)^2} \cdot \frac{4}{3}$$

The atomic number of the element is

$$(Z-1) = \sqrt{\frac{91.3\,\text{nm}}{\lambda_{21}} \cdot \frac{4}{3}} = \sqrt{\frac{91.3 \cdot 10^{-9}\,\text{m}}{8.36 \cdot 10^{-10}\,\text{m}} \cdot \frac{4}{3}} = 12$$

(1 Point)

Remark: Z=13 is the atomic number of Aluminium.

Problem 5 (4 Points)

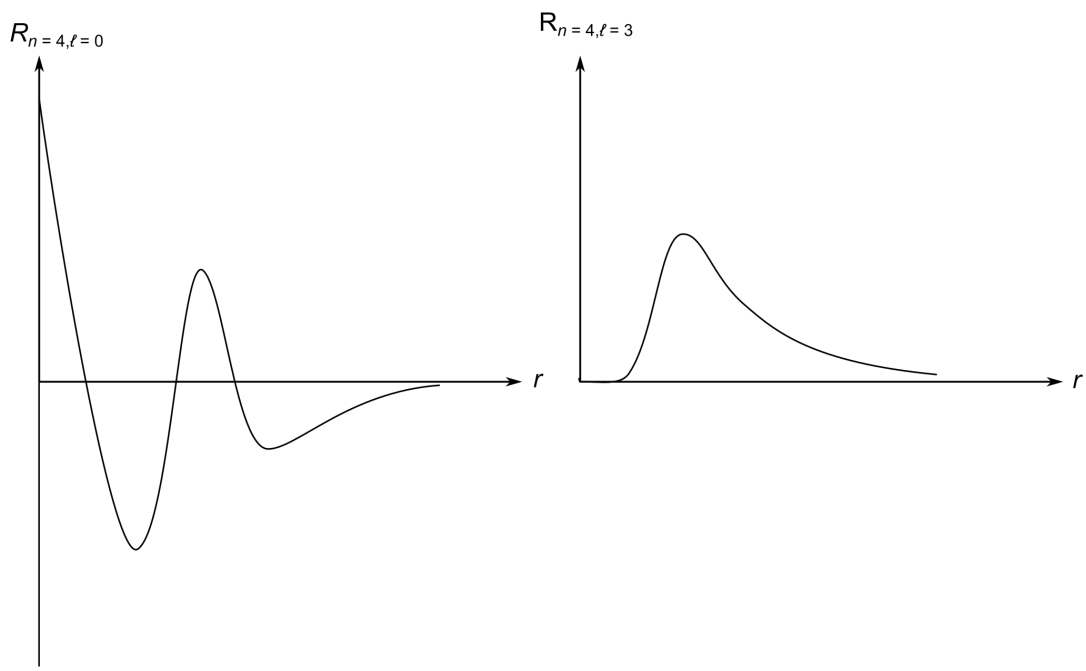
a) Give the quantum numbers which characterise an orbital of the hydrogen atom.
b) Which values are possible for the angular momentum quantum number ℓ of the orbitals with the principal quantum number $n = 4$?
c) Sketch the radial wave functions with the quantum numbers $n = 4$, $\ell = 0$ and $n = 4$, $\ell = 3$.
d) Give reasons for your sketches.

a) There are the
- principal quantum number n
- the quantum numbers of the orbital angular momentum ℓ and m
- and the quantum numbers of the electron spin $s = 1/2$ und $m_s = \pm 1/2$

(1 Point)

b) It is $\ell < n$, i.e. $\ell = 0, 1, 2, 3$ (1 point)

c) Radial wave functions



(1 point)

d)
- $n = 4$, $\ell = 0$:
  - No orbital angular momentum: therefore only radial motion
  - No centrifugal force: therefore $R_{n=4,\ell=0}(r = 0) \neq 0$
  - All kinetic energy is due to the radial motion: therefore many zeros of the wave function which increase the slope of R and thereby the contribution to the kinetic energy
- $n = 4$, $\ell = 3$
  - Maximal orbital angular momentum: therefore strong centrifugal force and $R_{n=4,\ell=0}(r = 0) = 0$
  - Smallest radial kinetic energy: therefore no zeros of the radial wave function (the number of zeros is $n - 1 - \ell$)

(1 Point)

Problem 6 (4 Points)

The atomic number of Rb (Rubidium) is $Z = 37$. Consider the ground state of the atom.

a) Make a table which includes all occupied orbitals and occupation numbers.
b) The smallest excitation energy is observed for the transition 5s $\leftrightarrow$ 5p. The transition is split into a doublet. Give the full atomic notation for both transitions.
c) The *nuclear* spin of ^{85}Rb is $I = 5/2$. Give the quantum numbers F of the total angular momentum of the ground state and the two excited states.
d) Give the selection rules for electric dipole transitions between these quantum states.

a) Table of occupied orbitals according to the periodic table of the elements

orbital	occupation number	sum
1s	2	2
2s	2	4
2p	6	10
3s	2	12
3p	6	18
4s	2	20
3d	10	30
4p	6	36
5s	1	37

(1 Point)

b) The two transitions are
- $5^2s_{1/2} \leftrightarrow 5^2p_{1/2}$
- $5^2s_{1/2} \leftrightarrow 5^2p_{3/2}$

(1 Point)

c) Total angular momentum quantum number due to the hyperfine interaction
- $^2s_{1/2}$: $F = 2, 3$
- $^2p_{1/2}$: $F = 2, 3$
- $^2s_{3/2}$: $F = 1, 2, 3, 4$

(1 Point)

d) The selection rules for electric dipole transitions are
- $\Delta\ell = \pm 1$
- $\Delta j = 0, \pm 1$ (not $0 \leftrightarrow 0$)
- $\Delta F = 0, \pm 1$ (not $0 \leftrightarrow 0$)

(1 Point)

Problem 7 (4 Points)

a) The momentum of a phonon in a solid is called a crystal momentum. Explain the difference between the crystal momentum and the momentum of Newtonian mechanics.

b) In inelastic neutron scattering, the neutrons can emit and absorb phonons. Write down the energy and momentum conservation laws for the emission of a phonon by a neutron.

c) What is the difference between Brillouin and Raman scattering?

d) Sketch an experimental setup for measuring Raman scattering as well as the vector diagram of the momentum conservation law.

a) In contrast to the momentum of Newtonian mechanics, the crystal momentum is not unique. Any vector of the reciprocal lattice can be added without changing the oscillatory motion of the atoms of the crystal lattice. The momenta of the phonons are therefore restricted to a primitive unit cell of the reciprocal lattice.
(1 Point)

b) Energy conservation law

$$E(\vec{k}) = E(\vec{k}\,') + \hbar\omega(\vec{q}) \quad \text{with} \quad E(\vec{k}) = \frac{(\hbar\vec{k})^2}{2m_{\text{n}}}$$

Momentum conservation law

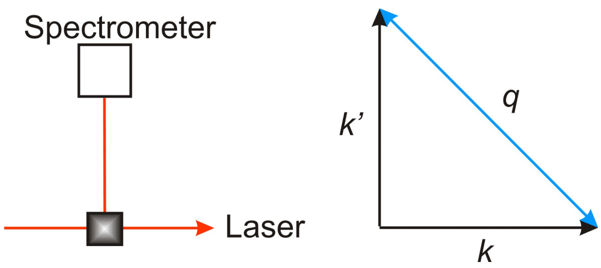
$$\hbar\vec{k} = \hbar\vec{k}\,' + \hbar\vec{q} + \hbar\vec{K}$$

$\hbar\vec{K}$ denotes a vector of the reciprocal lattice (1 Point)

c) - Brillouin scattering describes the scattering of photons of visible light by acoustic phonons
- Raman scattering describes the scattering of photons of visible light by optical phonons

(1 Point)

d) Sketch of the experimental setup and the momentum conservation law



(1 point)

Problem 8 (4 Points)

a) Calculate the density $n = N/V$ of conduction electrons of Rubidium. The mass density of Rb is $\rho_{\mathrm{Rb}} = 1.53\,\mathrm{g/cm^3}$ and the mass number $A = 85.47$.
Hint: There is one conduction electron per Rb atom.

b) Calculate the Fermi wave number k_F.

c) Calculate the Fermi energy E_F.

d) Calculate the Fermi velocity v_F of Rb.

a) Density of conduction electrons

$$\rho_{\mathrm{Rb}} = \frac{NAu}{\mathrm{V}} \quad \rightarrow \quad \frac{N}{V} = n = \frac{\rho_{\mathrm{Rb}}}{Au} = \frac{1.53 \cdot 10^{-3}\,\mathrm{kg} \cdot 10^{6}\,\mathrm{m^{-3}}}{85.47 \cdot 1.66 \cdot 10^{-27}\,\mathrm{kg}} = 1.08 \cdot 10^{28}\,\mathrm{m^{-3}}$$

(1 Point)

b) The Fermi wave number is the radius of the Fermi sphere

$$N = 2\frac{4\pi k_\mathrm{F}^3/3}{(2\pi)^3/V} \quad \rightarrow \quad \frac{N}{V}3\pi^2 = k_\mathrm{F}^3$$

$$k_\mathrm{F} = (3\pi^2 n)^{(1/3)} = (3\pi^2 \cdot 1.08 \cdot 10^{28}\,\mathrm{m^{-3}})^{(1/3)} = 6.84 \cdot 10^{9}\,\mathrm{m^{-1}}$$

(1 Point)

c) The Fermi energy is

$$E_\mathrm{F} = \frac{\hbar^2 k_\mathrm{F}^2}{2m_\mathrm{e}} = \frac{(4.14 \cdot 10^{-15}\,\mathrm{eVs})^2 \cdot (6.84 \cdot 10^{9}\,\mathrm{m^{-1}})^2 \cdot (3 \cdot 10^{8}\,\mathrm{ms^{-1}})^2}{(2\pi)^2 \cdot 2 \cdot 500 \cdot 10^3\,\mathrm{eV}} = 1.83\,\mathrm{eV}$$

(1 Point)

d) The Fermi velocity is

$$E_\mathrm{F} = \frac{1}{2}m_\mathrm{e}v_\mathrm{F}^2 \quad \rightarrow \quad v_\mathrm{F} = \sqrt{\frac{2E_\mathrm{F}}{m_\mathrm{e}}} = c\sqrt{\frac{2E_\mathrm{F}}{m_\mathrm{e}c^2}} = c\sqrt{\frac{2 \cdot 1.83\,\mathrm{eV}}{500 \cdot 10^3\,\mathrm{eV}}} = c \cdot 2.7 \cdot 10^{-3}$$

$$v_\mathrm{F} = 3 \cdot 10^{8}\,\mathrm{ms^{-1}} \cdot 2.7 \cdot 10^{-3} = 8.1 \cdot 10^{5}\,\mathrm{ms^{-1}}$$

(1 Point)

Problem 1 (4 Points)

The two stars of the binary star system β-Aurigae have almost the same mass and orbit the common centre of gravity. During the time of observation one star moves at a speed of 100 km/s directly towards the Earth whereas the other star recedes from Earth at this speed.

a) Calculate the Doppler shift of the red Balmer line ($\lambda = 657$ nm) due to the rotation of the two stars around the centre of gravity.
b) The red Balmer line splits in two components due to the rotation. Calculate the energy difference between the two components in units of eV.

The microwave beam of a radar trap (frequency $\nu_0 = 30$ GHz) is oriented under 45° towards the approaching traffic.

c) Calculate the frequency shift of the microwave measured in a car approaching the radar trap with a velocity of 70 km/h.
d) A fraction of the radiation is reflected back to the radar trap. Calculate the frequency shift which is measured by the receiver.

Problem 2 (4 Points)

a) What does the acronym Laser mean?
b) Which condition has to be fulfilled so that Laser light can be created?
c) Explain by means of a sketch how the condition for generating Laser light can be met with a four-energy level scheme.
d) How does the four-level scheme of a continuous-wave Laser differ from that of a pulsed Laser?

Problem 3 (4 Points)

Electrons are accelerated from rest by a voltage of 2 MV.

a) Calculate the total energy of the electrons.
b) Calculate the relativistic mass of the electrons.
c) Calculate the momentum of the electrons.
d) Calculate the wavelength of the electron beam.

Problem 4 (4 Points)

a) Give the formula for the quantized energy of the hydrogen atom.
b) Calculate the shortest wavelength of the Lyman and the Balmer series, respectively.
c) What denotes the K_α-line in x-ray spectroscopy?
d) In the X-ray spectrum of an element, the K_α-line has the wavelength λ=1.2 nm. What is the atomic number Z of the element?

Problem 5 (4 Points)

a) Give the quantum numbers which characterise an orbital of the hydrogen atom.
b) Which values are possible for the angular momentum quantum number ℓ of the orbitals with the principal quantum number $n = 5$?
c) Sketch the radial wave functions with the quantum numbers $n = 5$, $\ell = 0$ and $n = 5$, $\ell = 4$.

d) Give reasons for your sketches.

Problem 6 (4 Points)

The atomic number of Cs (Cesium) is $Z = 55$. Consider the ground state of the atom.

a) Make a table which includes all occupied orbitals and occupation numbers.
b) The smallest excitation energy is observed for the transition 6s $\leftrightarrow$ 6p. The transition is split into a doublet. Give the full atomic notation for both transitions.
c) The *nuclear* spin of ^{133}Cs is $I = 7/2$. Give the quantum numbers F of the total angular momentum of the ground state and the two excited states.
d) Give the selection rules for electric dipole transitions between these quantum states.

Problem 7 (4 Points)

a) The momentum of a phonon in a solid is called a crystal momentum. Explain the difference between the crystal momentum and the momentum of Newtonian mechanics.
b) In inelastic neutron scattering, the neutrons can emit and absorb phonons. Write down the energy and momentum conservation laws for the emission of a phonon by a neutron.
c) What is the difference between Brillouin and Raman scattering?
d) Sketch an experimental setup for measuring Raman scattering as well as the vector diagram of the momentum conservation law.

Problem 8 (4 Points)

a) Calculate the density $n = N/V$ of conduction electrons of Cesium. The mass density of Cs is $\rho_{Cs} = 1.9\,\mathrm{g/cm^3}$ and the mass number $A = 132.9$.
Hint: There is one conduction electron per Cs atom.
b) Calculate the Fermi wave number k_F.
c) Calculate the Fermi energy E_F.
d) Calculate the Fermi velocity v_F of Cs.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\mathrm{m/s}$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\mathrm{eVs} = 6.62 \cdot 10^{-34}\,\mathrm{Ws^2}$
Atomic mass unit:	$u = 1.66 \cdot 10^{-27}\,\mathrm{kg}$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\mathrm{As}$
Mass of the electron:	$m_e = 500\,\mathrm{keV}/c^2 = 9 \cdot 10^{-31}\,\mathrm{kg}$
Rydberg unit of energy:	$R = 13.6\,\mathrm{eV}$

Problem 1 (4 Points)

The two stars of the binary star system β-Aurigae have almost the same mass and orbit the common centre of gravity. During the time of observation one star moves at a speed of 100 km/s directly towards the Earth whereas the other star recedes from Earth at this speed.

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c) Calculate the frequency shift of the microwave measured in a car approaching the radar trap with a velocity of 70 km/h.
d) A fraction of the radiation is reflected back to the radar trap. Calculate the frequency shift which is measured by the receiver.

a) Doppler shift of the approaching and receding star

$$\lambda = \lambda_0 \sqrt{\frac{1 \mp (v/c)}{1 \pm (v/c)}} \quad \xrightarrow{v<<c} \quad \lambda_0 (1 \mp (v/c))$$

The wavelength becomes smaller for the approaching star and larger for the receding star

$$\Delta\lambda = \lambda - \lambda_0 = \mp \lambda_0 \frac{v}{c} = \mp 657\,\text{nm} \cdot \frac{10^5 \text{ms}^{-1}}{3 \cdot 10^8\,\text{ms}^{-1}} = \mp 0.219\,\text{nm}$$

(1 Point)

b) Energy difference between the two shifted spectral lines

$$\Delta E = h\Delta\nu$$

e.g. with $\nu = c/\lambda$ is

$$\Delta\nu = -\frac{c}{\lambda^2}\Delta\lambda$$

and the energy difference of the splitting is

$$\Delta E = 2 \cdot \frac{hc}{\lambda_0^2} \cdot \Delta\lambda = 2 \cdot \frac{4.14 \cdot 10^{-15}\,\text{eVs}\, 3 \cdot 10^8\,\text{ms}^{-1}}{(657 \cdot 10^{-9})^2\,\text{m}^2} \cdot 0.219 \cdot 10^{-9}\,\text{m} = 1.26 \text{ meV}$$

(1 Point)

Remark: The effect is very much larger than the splitting due to the spin-orbit coupling of the hydrogen atom of about 40 μeV.

c) The frequency measured in the approaching car is increased according to

$$\nu = \nu_0(1 + \frac{v}{c}\cos\alpha),$$

d.h.

$$\begin{aligned}\nu =&\nu_0 + \nu_0\frac{v}{c}\cos\alpha = 30\,\text{GHz} + 30\cdot 10^9\,\text{Hz}\frac{70\cdot 10^3\,\text{m}}{3600\,\text{s}\cdot 3\cdot 10^8\,\text{ms}^{-1}}\cos 45°\\ =&30\,\text{GHz} + 1375\,\text{Hz}\end{aligned}$$

(1 Point)

d) The frequency of the reflected wave measured with the receiver is also enhanced according to

$$\nu' = \nu(1 + \frac{v}{c}\cos\alpha) = \nu_0(1 + \frac{v}{c}\cos\alpha)^2 = \nu_0(1 + 2\frac{v}{c}\cos\alpha) = 30\,\text{GHz} + 2750\,\text{Hz}$$

(1 Point)

Remark: The term $\nu_0(\frac{v}{c}\cos\alpha)^2 = 12.6\cdot 10^{-7}$ Hz can be neglected.

Problem 2 (4 Points)

a) What does the acronym Laser mean?
b) Which condition has to be fulfilled so that Laser light can be created?
c) Explain by means of a sketch how the condition for generating Laser light can be met with a four-energy level scheme.
d) How does the four-level scheme of a continuous-wave Laser differ from that of a pulsed Laser?

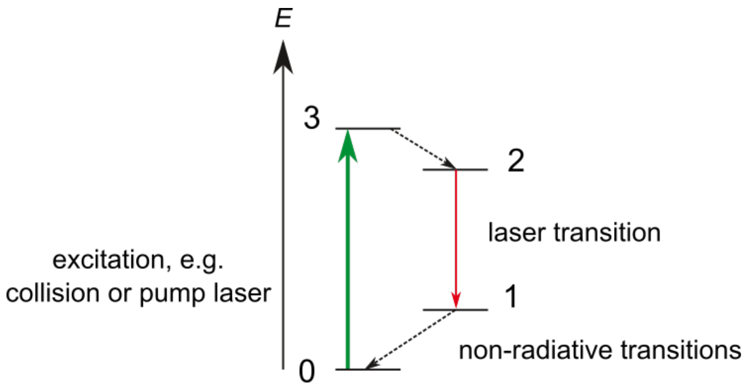
a) LASER: light amplification by stimulated emission of radiation

(1 Point)

b) Population inversion must be achieved, i.e. there must be more electrons in an excited energy level than in an energy level with a smaller energy.

(1 Point)

c) The atoms are excited from the ground state into energy level 3. With the helium-neon laser, this happens through the collision of helium and neon atoms. Usually a second laser is used to pump the transition between the ground state and energy level 3. From energy level 3 the atoms can relax to energy level 2. As a result of the pumping process, the energy level 2 is occupied more than it corresponds to the thermal equilibrium. Population inversion can be achieved between the energy level 2 and an excited energy level 1.



(1 Point)

d) In the case of a continuous laser, the pumping rate between the ground state and the energy level 2 is smaller than the relaxation rate between the energy level 1 and the ground state. Therefore, the population inversion between energy levels 2 and 1 can be achieved permanently.

In the case of a pulsed laser, the pumping rate between the ground state and the energy level 2 is larger than the relaxation rate between the energy level 1 and the ground state. After a short time, the stimulated emission leads to equal population of energy levels 1 and 2, and the amplification of light by stimulated emission of radiation is no longer possible.

(1 Point)

Problem 3 (4 Points)

Electrons are accelerated from rest by a voltage of 2 MV.

a) Calculate the total energy of the electrons.
b) Calculate the relativistic mass of the electrons.
c) Calculate the momentum of the electrons.
d) Calculate the wavelength of the electron beam.

a) The total energy is the sum of rest energy and kinetic energy, i.e.

$$E = m_0 c^2 + eU$$

U denotes the acceleration voltage

$$E = 500\,\text{keV} + 2\text{MeV} = 2.5\,\text{MeV}$$

(1 Point)

b) The relativistic mass is

$$E = mc^2$$

i.e.

$$m = \frac{E}{c^2} = 2.5\,\text{MeV}/c^2 = \frac{2.5 \cdot 10^6 \cdot 1.6 \cdot 10^{-19}\,\text{VAs}}{(3 \cdot 10^8\,\text{ms}^{-1})^2} = 4.4 \cdot 10^{-30}\,\text{kg}$$

(1 Point)

c) The momentum is with

$$E^2 = m_0^2 c^4 + c^2 p^2 \quad \rightarrow \quad p = \frac{1}{c}\sqrt{E^2 - m_0^2 c^4}$$

$$p = \frac{1}{c}\sqrt{2.5^2 - 0.5^2}\,\text{MeV} = 2.45\,\text{MeV}/c = \frac{2.45 \cdot 10^6 \cdot 1.6 \cdot 10^{-19}\,\text{VAs}}{3 \cdot 10^8\,\text{ms}^{-1}}$$
$$= 1.33 \cdot 10^{-21}\,\text{kgms}^{-1}$$

(1 Point)

(or with $v \approx c$ and $p \approx mc = 2.5\,\text{MeV}/c$)

d) The de Broglie wave length is

$$\lambda = \frac{h}{p} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^8\,\text{ms}^{-1}}{2.45 \cdot 10^6\,\text{eV}} = 5.07 \cdot 10^{-13}\,\text{m}$$

(1 Point)

Problem 4 (4 Points)

a) Give the formula for the quantized energy of the hydrogen atom.
b) Calculate the shortest wavelength of the Lyman and the Balmer series, respectively.
c) What denotes the K_α-line in x-ray spectroscopy?
d) In the X-ray spectrum of an element, the K_α-line has the wavelength λ=1.2 nm. What is the atomic number Z of the element?

a) Energy of the hydrogen atom

$$E_n = -\frac{R}{n^2} \qquad n = 1, 2, ...$$

(1 Point)

b) The energy difference between two energy levels E_n and E_m is

$$\Delta E_{n,m} = |E_n - E_m| = R\left|\frac{1}{n^2} - \frac{1}{m^2}\right| = R\frac{|n^2 - m^2|}{n^2 m^2}$$

and the wavelength for a transition between E_n and E_m is

$$\lambda_{n,m} = \frac{ch}{\Delta E_{n,m}} = \frac{ch}{R}\frac{n^2 m^2}{|n^2 - m^2|} = \frac{3 \cdot 10^8\,\mathrm{ms}^{-1} \cdot 4.14 \cdot 10^{-15}\,\mathrm{eVs}}{13.6\,\mathrm{eV}}\frac{n^2 m^2}{|n^2 - m^2|}$$
$$= 91.3\,\mathrm{nm}\frac{n^2 m^2}{|n^2 - m^2|}$$

the smallest wavelength is due to the emission of a photon between $(m \to \infty) \to n$ is

$$\lambda_{\infty,n} = 91.3\,\mathrm{nm} \cdot n^2$$

- Lyman series $n = 1$

$$\lambda_{\infty,1} = 91.3\,\mathrm{nm}$$

- Balmer series $n = 2$

$$\lambda_{\infty,2} = 91.3\,\mathrm{nm} \cdot 2^2 = 365\,\mathrm{nm}$$

(1 Point)

c) The K_α-line denotes the transition $n = 2 \to 1$.

(1 Point)

d) According to Moseley's law is the energy of the transition (The nuclear charge Z is screened by the remaining 1s electron)

$$\Delta \mathrm{E}_{2,1} = |E_2 - E_1| = R(Z - 1)^2\left|\frac{1}{2^2} - 1\right|$$

and the wavelength of the K_α-line is

$$\lambda_{21} = \frac{91.3\,\mathrm{nm}}{(Z - 1)^2} \cdot \frac{2^2}{2^2 - 1} = \frac{91.3\,\mathrm{nm}}{(Z - 1)^2} \cdot \frac{4}{3}$$

The atomic number of the element is

$$(Z-1) = \sqrt{\frac{91.3\,\text{nm}}{\lambda_{21}} \cdot \frac{4}{3}} = \sqrt{\frac{91.3\,\text{nm}}{1.2\,\text{nm}} \cdot \frac{4}{3}} = 10$$

(1 Point)

Remark: Z=11 is the atomic number of Sodium.

Problem 5 (4 Points)

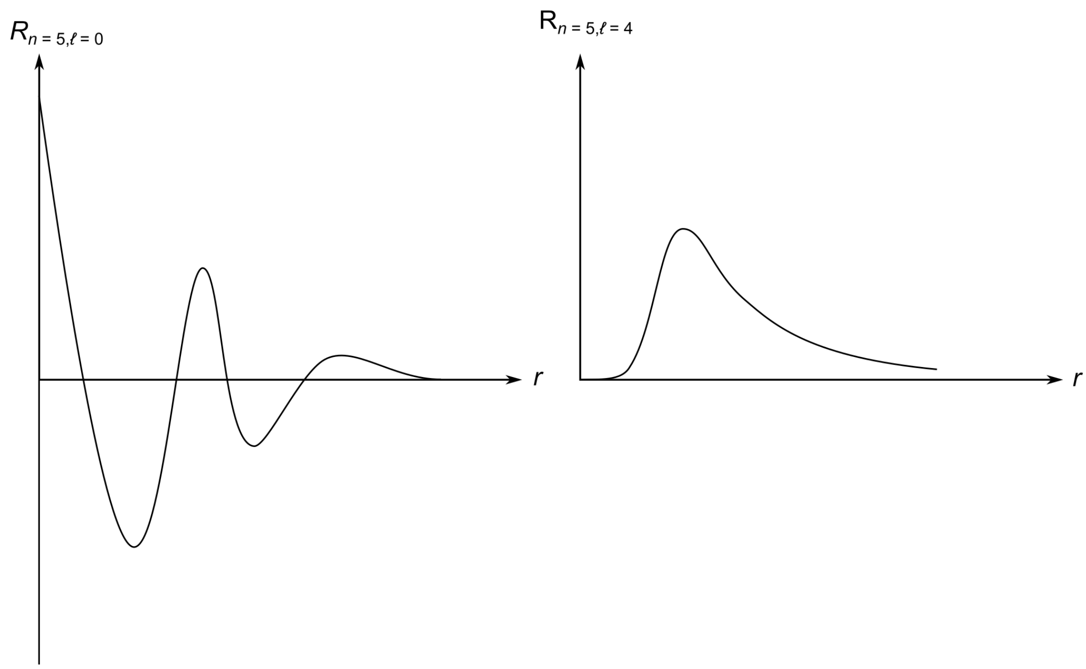
a) Give the quantum numbers which characterise an orbital of the hydrogen atom.
b) Which values are possible for the angular momentum quantum number ℓ of the orbitals with the principal quantum number $n = 5$?
c) Sketch the radial wave functions with the quantum numbers $n = 5$, $\ell = 0$ and $n = 5$, $\ell = 4$.
d) Give reasons for your sketches.

a) There are the
   - principal quantum number n
   - the quantum numbers of the orbital angular momentum ℓ and m
   - and the quantum numbers of the electron spin $s = 1/2$ und $m_s = \pm 1/2$

(1 Point)

b) It is $\ell < n$, i.e. $\ell = 0, 1, 2, 3, 4$ (1 point)

c) Radial wave functions



(1 point)

d)
- $n = 5$, $\ell = 0$:
  - No orbital angular momentum: therefore only radial motion
  - No centrifugal force: therefore $R_{n=5,\ell=0}(r = 0) \neq 0$
  - All kinetic energy is due to the radial motion: therefore $n - 1$ zeros of the wave function increase the slope of R and thereby the contribution to the kinetic energy
- $n = 5$, $\ell = 4$
  - Maximal orbital angular momentum: therefore strong centrifugal force and $R_{n=5,\ell=0}(r = 0) = 0$
  - Smallest radial kinetic energy: therefore no zeros of the radial wave function (the number of zeros is $n - 1 - \ell$)

(1 Point)

Problem 6 (4 Points)

The atomic number of Cs (Cesium) is $Z = 55$. Consider the ground state of the atom.

a) Make a table which includes all occupied orbitals and occupation numbers.
b) The smallest excitation energy is observed for the transition 6s $\leftrightarrow$ 6p. The transition is split into a doublet. Give the full atomic notation for both transitions.
c) The *nuclear* spin of ^{133}Cs is $I = 7/2$. Give the quantum numbers F of the total angular momentum of the ground state and the two excited states.
d) Give the selection rules for electric dipole transitions between these quantum states.

a) Table of occupied orbitals according to the periodic table of the elements

orbital	occupation number	sum
1s	2	2
2s	2	4
2p	6	10
3s	2	12
3p	6	18
4s	2	20
3d	10	30
4p	6	36
5s	2	38
4d	10	48
5p	6	54
6s	1	55

(1 Point)

b) The two transitions are
- $6^2s_{1/2} \leftrightarrow 6^2p_{1/2}$
- $6^2s_{1/2} \leftrightarrow 6^2p_{3/2}$

(1 Point)

c) Total angular momentum quantum number due to the hyperfine interaction
- $^2s_{1/2}$: $F = 3, 4$
- $^2p_{1/2}$: $F = 3, 4$
- $^2s_{3/2}$: $F = 2, 3, 4, 5$

(1 Point)

d) The selection rules for electric dipole transitions are

- $\Delta\ell = \pm 1$
- $\Delta j = 0, \pm 1$ (not $0 \leftrightarrow 0$)
- $\Delta F = 0, \pm 1$ (not $0 \leftrightarrow 0$)

(1 Point)

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d) Sketch an experimental setup for measuring Raman scattering as well as the vector diagram of the momentum conservation law.

a) In contrast to the momentum of Newtonian mechanics, the crystal momentum is not unique. Any vector of the reciprocal lattice can be added without changing the oscillatory motion of the atoms of the crystal lattice. The momenta of the phonons are therefore restricted to a primitive unit cell of the reciprocal lattice.
(1 Point)

b) Energy conservation law

$$E(\vec{k}) = E(\vec{k}') + \hbar\omega(\vec{q}) \quad \text{with} \quad E(\vec{k}) = \frac{(\hbar\vec{k})^2}{2m_\text{n}}$$

Momentum conservation law

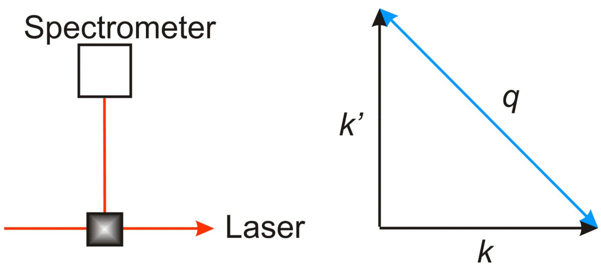
$$\hbar\vec{k} = \hbar\vec{k}' + \hbar\vec{q} + \hbar\vec{K}$$

$\hbar\vec{K}$ denotes a vector of the reciprocal lattice (1 Point)

c) - Brillouin scattering describes the scattering of photons of visible light by acoustic phonons
- Raman scattering describes the scattering of photons of visible light by optical phonons

(1 Point)

d) Sketch of the experimental setup and the momentum conservation law



(1 point)

Problem 8 (4 Points)

a) Calculate the density $n = N/V$ of conduction electrons of Cesium. The mass density of Cs is $\rho_{\text{Cs}} = 1.9\,\text{g}/\text{cm}^3$ and the mass number $A = 132.9$.
Hint: There is one conduction electron per Cs atom.
b) Calculate the Fermi wave number k_F.
c) Calculate the Fermi energy E_F.
d) Calculate the Fermi velocity v_F of Cs.

a) Density of conduction electrons

$$\rho_{\text{Cs}} = \frac{NAu}{\text{V}} \quad \rightarrow \quad \frac{N}{V} = n = \frac{\rho_{\text{Cs}}}{Au} = \frac{1.9 \cdot 10^{-3}\,\text{kg} \cdot 10^6\,\text{m}^{-3}}{132.9 \cdot 1.66 \cdot 10^{-27}\,\text{kg}} = 0.86 \cdot 10^{28}\,\text{m}^{-3}$$

(1 Point)

b) The Fermi wave number is the radius of the Fermi sphere

$$N = 2\frac{4\pi k_\text{F}^3/3}{(2\pi)^3/V} \quad \rightarrow \quad \frac{N}{V}3\pi^2 = k_\text{F}^3$$

$$k_\text{F} = (3\pi^2 n)^{(1/3)} = (3\pi^2 \cdot 0.86 \cdot 10^{28}\,\text{m}^{-3})^{(1/3)} = 6.34 \cdot 10^9\,\text{m}^{-1}$$

(1 Point)

c) The Fermi energy is

$$E_\text{F} = \frac{\hbar^2 k_\text{F}^2}{2m_\text{e}} = \frac{(4.14 \cdot 10^{-15}\,\text{eVs})^2 \cdot (6.34 \cdot 10^9\,\text{m}^{-1})^2 \cdot (3 \cdot 10^8\,\text{ms}^{-1})^2}{(2\pi)^2 \cdot 2 \cdot 500 \cdot 10^3\,\text{eV}} = 1.57\,\text{eV}$$

(1 Point)

d) The Fermi velocity is

$$E_\text{F} = \frac{1}{2}m_\text{e}v_\text{F}^2 \quad \rightarrow \quad v_\text{F} = \sqrt{\frac{2E_\text{F}}{m_\text{e}}} = c\sqrt{\frac{2E_\text{F}}{m_\text{e}c^2}} = c\sqrt{\frac{2 \cdot 1.57\,\text{eV}}{500 \cdot 10^3\,\text{eV}}} = c \cdot 2.5 \cdot 10^{-3}$$

$$v_\text{F} = 3 \cdot 10^8\,\text{ms}^{-1} \cdot 2.5 \cdot 10^{-3} = 7.5 \cdot 10^5\,\text{ms}^{-1}$$

(1 Point)

Problem 1 (4 Points)

a) Write down the formula for the relativistic momentum and explain the quantities involved.
b) A particle moves with the velocity $v_y = 0.5\,c$ relative to a stationary observer. Calculate the momentum of the particle in units of m_0c. Here, m_0 denotes the rest mass of the particle.
c) A second observer moves perpendicularly to the velocity of the particle with velocity $v_x = 0.5\,c$. Make a sketch of the frames of reference and write down the Lorentz transformations between the two frames of reference.
d) Calculate the momentum of the particle in the frame of reference of the moving observer parallel and perpendicular to his velocity.

Problem 2 (4 Points)

a) Write down the Schrödinger equation for an electron moving in the electric field of a proton.
b) Write down the energy eigenvalues of the electron.
c) Which quantum numbers determine the energy eigenvalues and the wave functions of the electron if the spin of the electron is not considered.
d) What are the numerical values of these quantum numbers in the ground state and in the first excited state of the electron?

Problem 3 (4 Points)

a) Write down the equations that define the spin of the electron and explain the quantities used.
b) What are the values of the quantum numbers of the electron spin?
c) What is the cause of the so-called spin-orbit coupling?
d) Write down the Hamiltonian of the spin-orbit coupling and give its eigenvalues for the excited states of the hydrogen atom with principal quantum number $n = 2$.

Problem 4 (4 Points)

The nucleus of helium contains two protons and can therefore bind two electrons.

a) In the ground state, the two electrons occupy the 1s orbital. Give the quantum numbers and the term symbol for the ground state of the helium atom.
b) The lowest energy excitation occurs when an electron is excited into the 2s orbital. Give the quantum numbers and term symbols of this electron configuration.
c) Sketch the energy level scheme of these excited states.
d) How does the exchange interaction manifest itself in this energy level scheme? Give reasons for your answer.

Problem 5 (4 Points)

a) Write down the energy and momentum of a phonon and explain the quantities used.

b) Why is the energy of the phonons a periodic function in the reciprocal lattice?
c) The momentum of a phonon is a crystal momentum. What is the difference between a crystal momentum and the momentum in Newtonian mechanics?
d) How many phonon modes are there in a crystal lattice with n atoms per primitive unit cell?

Problem 6 (4 Points)

The work function of metallic sodium is $W_A = 2.28\,\text{eV}$ and the Fermi energy is $E_F = 3.24\,\text{eV}$. Metallic Sodium is illuminated with light of wavelength $\lambda = 200\,\text{nm}$.

a) What is the largest value of the kinetic energy of the photoelectrons?
b) What is the smallest value of the kinetic energy of the photoelectrons?
c) Calculate the cut-off wavelength for which the photoelectric effect can be observed.
d) All photoelectrons of a certain energy are measured with a spectrometer. Sketch the spectrum of the photoelectrons as a function of the kinetic energy of the photoelectrons.

Problem 7 (4 Points)

a) What is the Fermi energy?
b) For sodium, the electron in the 3s orbital is the valence electron. These electrons are the conduction electrons in metallic sodium. Calculate the Fermi wavenumber of sodium.
c) Give the definition of the density of states of conduction electrons and explain the quantities used.
d) Sketch the density of states that results for a quasi-free electron gas.

Hint: The density of sodium is $\rho = 0.968\,\text{gcm}^{-3}$ and the atomic mass is 22.98 u

Problem 8 (4 Points)

Conduction electrons can move quasi freely in a simple cubic lattice with the lattice constant $a = 3.6 \cdot 10^{-10}\,\text{m}$.

a) Sketch the band structure in the reduced and in the periodic zone scheme along the Γ-X direction in case the influence of the periodic crystal potential is negligibly small.
b) The energy zero point is in the minimum of the first band. Calculate the highest energy of the 1st and 2nd bands, when the mass of the charge carriers is $m_e = 500\,\text{keV}/c^2$.
c) Sketch the band structure in the reduced zone scheme along the Γ-X direction in case the influence of the periodic crystal potential is not negligibly small.
d) Mark the energy ranges in the band structure in which the charge carriers can be assigned a positive charge. Justify your sketch.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8\,\text{m/s}$
Planck's constant:	$h = 4.14 \cdot 10^{-15}\,\text{eVs} = 6.62 \cdot 10^{-34}\,\text{Ws}^2$

Atomic mass unit:	$u = 1.66 \cdot 10^{-27}\,\mathrm{kg}$
Rydberg unit of energy:	$R = 13.6\,\mathrm{eV}$

Problem 1 (4 Points)

a) Write down the formula for the relativistic momentum and explain the quantities involved.
b) A particle moves with the velocity $v_y = 0.5\,c$ relative to a stationary observer. Calculate the momentum of the particle in units of $m_0 c$. Here, m_0 denotes the rest mass of the particle.
c) A second observer moves perpendicularly to the velocity of the particle with velocity $v_x = 0.5\,c$. Make a sketch of the frames of reference and write down the Lorentz transformations between the two frames of reference.
d) Calculate the momentum of the particle in the frame of reference of the moving observer parallel and perpendicular to his velocity.

a) The formula for relativistic momentum is

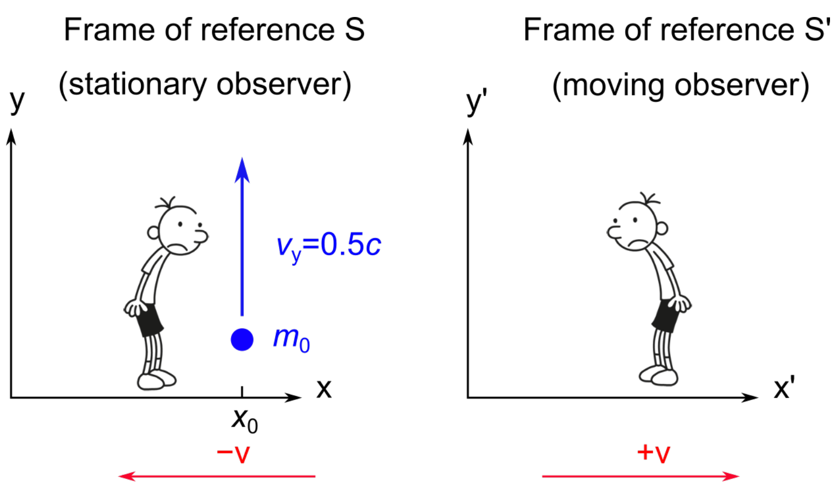
$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- $\vec{v}$ bzw. $v = |\vec{v}|$ is the velocity of the mass m_0
- m_0 is the rest mass of the particle

b) Momentum of the particle

$$p = \frac{m_0 0.5c}{\sqrt{1 - 0.5^2}} = 0.577\, m_0 c$$

c) Sketch of the frames of reference



The Lorentz transformations are

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y' \quad \text{and} \quad t = \frac{t' + (v/c^2)x'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y' = y \quad \text{and} \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

d) With the Lorentz transformation, the velocity of the particle parallel to the velocity of the observer is

$$v'_{\mathrm{x}} = \frac{\mathrm{d}x'}{\mathrm{d}t'} = -v = -0.5c$$

According to the Lorentz transformations, $\mathrm{d}x' = \frac{-v\mathrm{d}t}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\mathrm{d}t' = \frac{\mathrm{d}t}{\sqrt{1-\frac{v^2}{c^2}}}$.

The velocity of the particle perpendicular to the velocity of the observer is

$$v'_{\mathrm{y}} = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \sqrt{1-\frac{v^2}{c^2}}\mathrm{v}_{\mathrm{y}} = \sqrt{1-0.5^2}\,0,5\,c = 0.433\,c$$

The square of the particle velocity v in the reference system of the moving observer is

$$v'^2 = (v'_{\mathrm{x}})^2 + (v'_{\mathrm{y}})^2 = (0.5^2 + 0.433^2)c^2 = 0.438\,c^2$$

The momentum of the particle in the reference frame of the moving observer is

$$p'_{\mathrm{x}} = -\frac{m_0 0.5c}{\sqrt{1-0.438}} = -0.667\,m_0 c$$

and

$$p'_{\mathrm{y}} = \frac{m_0 0.433c}{\sqrt{1-0.438}} = 0.577\,m_0 c = p_{\mathrm{y}}$$

Problem 2 (4 Points)

a) Write down the Schrödinger equation for an electron moving in the electric field of a proton.
b) Write down the energy eigenvalues of the electron.
c) Which quantum numbers determine the wave functions of the electron if the spin of the electron is not considered.
d) What are the numerical values of these quantum numbers in the ground state and in the first excited state of the electron?

a) Schrödinger equation for an electron in the electric field of a proton

$$i\hbar\frac{\partial\psi}{\partial t} = \left\{-\frac{\hbar^2\nabla^2}{2m_e} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\right\}\psi$$

b) The energy eigenvalues are

$$E_n = -\frac{R}{n^2} \quad \text{and} \quad n = 1, 2, 3, ...$$

c) The wave functions are determined by
- the principal quantum number n
- the orbital angular momentum quantum number ℓ
- and the magnetic quantum number m

d) The quantum numbers have the following numerical values in the ground state
- $n = 1$
- $\ell = 0$
- $m = 0$

and in the first excited state
- $n = 2$
- $\ell = 0$
- $m = 0$

and
- $n = 2$
- $\ell = 1$
- $m = 0, \pm 1$

Problem 3 (4 Points)

a) Write down the equations that define the spin of the electron and explain the quantities used.
b) What are the values of the quantum numbers of the electron spin?
c) What is the cause of the so-called spin-orbit coupling?
d) Write down the Hamiltonian of the spin-orbit coupling and give its eigenvalues for the excited states of the hydrogen atom with principal quantum number $n = 2$.

a) The eigenvalue equations of the electron spin are

$$\begin{aligned}\hat{\vec{s}}^2 |s, m_s\rangle =& s(s+1)\hbar^2 |s, m_s\rangle \\ \hat{s}_z |s, m_s\rangle =& m\hbar |s, m_s\rangle\end{aligned}$$

- $\hat{\vec{s}}^2$ squared spin operator
- $\hat{s}_z$ z component of the spin operator
- $|s, m_s\rangle$ Dirac notation of the eigenstates of the spin operator
- s and m_s Quantum numbers of the spin operator

b) The quantum numbers of the spin operator have the values
- $s = 1/2$
- $m_s = \pm 1/2$

c) The magnetic moment of the electron spins aligns in the magnetic field caused by the orbital motion of the electron.

d) The Hamiltonian of the spin-orbit coupling is

$$\hat{H}_{s,\ell} = \frac{\xi}{\hbar^2} \hat{\vec{s}} \cdot \hat{\vec{\ell}}$$

The energy eigenvalues of the Hamiltonian are

$$E_{s,\ell,j} = \frac{\xi}{2}(j(j+1) - s(s+1) - \ell(\ell+1))$$

The first excited state of the hydrogen atom consists of an s orbital ($\ell = 0$) and a p orbital ($\ell = 1$)
- The energy eigenvalue for the s orbital is $E_{s,\,\ell=0,\,j=s} = 0$
- Is the total angular momentum for the p orbital $j = 1/2$, then the energy eigenvalue is

$$E_{s,\,\ell=1,\,j=1/2} = \frac{\xi}{2}(-2) = -\xi$$

- or if $j = 3/2$ then

$$E_{s,\,\ell=1,\,j=3/2} = \frac{\xi}{2}((3/2)(5/2) - (1/2)(3/2) - 2) = +\frac{\xi}{2}$$

Problem 4 (4 Points)

The nucleus of helium contains two protons and can therefore bind two electrons.

a) In the ground state, the two electrons occupy the 1s orbital. Give the quantum numbers and the term symbol for the ground state of the helium atom.

b) The lowest energy excitation occurs when an electron is excited into the 2s orbital. Give the quantum numbers and term symbols of this electron configuration.

c) Sketch the energy level scheme of these excited states.

d) How does the exchange interaction manifest itself in this energy level scheme? Give reasons for your answer.

a) Quantum numbers in the ground state of the [$1s^2$] configuration
- total spin $S = 0$
- total orbital angular momentum $L = 0$
- total angular momentum $J = 0$

The term symbol for the ground state is

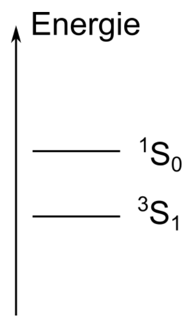
$$^{2S+1}L_J \quad \rightarrow \quad {}^1\mathrm{S}_0$$

b) Quantum numbers of the [1s2s] configuration
- total spin $S = 0$ or $S = 1$
- total orbital angular momentum $L = 0$
- total angular momentum $J = 0$ oder $J = 1$

The term symbols of the [1s2s] configuration are

$$^1\mathrm{S}_0 \quad \text{and} \quad {}^3\mathrm{S}_1$$

c) Energy level scheme of the [1s2s] configuration



d) The level $^1S_0$ has a larger energy than the $^3S_1$ levels due to the exchange interaction. The reason is the Pauli principle, which causes a larger Coulomb repulsion between the electrons in the $^1S_0$ state than in the $^3S_1$ states.

Problem 5 (4 Points)

a) Write down the energy and momentum of a phonon and explain the quantities used.
b) Why is the energy of the phonons a periodic function in the reciprocal lattice?
c) The momentum of a phonon is a crystal momentum. What is the difference between a crystal momentum and the momentum in Newtonian mechanics?
d) How many phonon modes are there in a crystal lattice with n atoms per primitive unit cell?

a) Energy of a phonon

$$E(\vec{q}) = \hbar\omega(\vec{q})$$

Momentum of a phonon

$$\vec{p} = \hbar\vec{q}$$

- ω is the angular frequency of the lattice vibrations
- $\vec{q}$ is a wave vector within the 1^{st} Brillouin zone

b) If one adds a vector of the reciprocal lattice to the wave vector $\vec{q}$, the motion of the atoms does not change, since the motion of the atoms is determined by the wave function $\psi(\vec{R}, t) = \psi_0 \exp(i(\vec{q}\vec{R} - \omega(\vec{q})t)) = \psi_0 \exp(i((\vec{q} + \vec{K})\vec{R} - \omega(\vec{q})t))$, and $\vec{K}\vec{R} = 2\pi n$ (this is the definition of the reciprocal lattice). The amplitude ψ_0 is a complex number determined by the atom and its position in the primitive unit cell of the crystal lattice. Since adding a reciprocal lattice vector does not change the motion of the atoms, the energy of the vibration does not change either, i.e. $E(\vec{q}) = \hbar\omega(\vec{q}) = \hbar\omega(\vec{q} + \vec{K})$.

c) Since the motion of the atoms cannot be changed by adding a reciprocal lattice vector, the momentum of a phonon cannot be changed by adding $\hbar\vec{K}$ either. In Newtonian mechanics, $\hbar\vec{q} + \hbar\vec{K}$ would be a different momentum. Since the momentum of a phonon behaves differently, it is called crystal momentum to distinguish it from Newtonian momentum.

d) The number of phonon modes is $3nN$. N denotes the number of primitive unit cells of the crystal lattice.

Problem 6 (4 Points)

The work function of metallic sodium is $W_A = 2.28\,\text{eV}$ and the Fermi energy is $E_F = 3.24\,\text{eV}$. Metallic Sodium is illuminated with light of wavelength $\lambda = 200\,\text{nm}$.

a) What is the largest value of the kinetic energy of the photoelectrons?
b) What is the smallest value of the kinetic energy of the photoelectrons?
c) Calculate the cut-off wavelength for which the photoelectric effect can be observed.
d) All photoelectrons of a certain energy are measured with a spectrometer. Sketch the spectrum of the photoelectrons as a function of the kinetic energy of the photoelectrons.

a) The energy of the photon is

$$E = \frac{hc}{\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^{8}\,\text{ms}^{-1}}{200 \cdot 10^{-9}\,\text{m}} = 6.21\,\text{eV}$$

The largest value of the kinetic energy of the photoelectrons is

$$E_{kin} = E - W_A = 6.21\,\text{eV} - 2.28\,\text{eV} = 3.93\,\text{eV}$$

b) The smallest value of the kinetic energy of the photoelectrons is

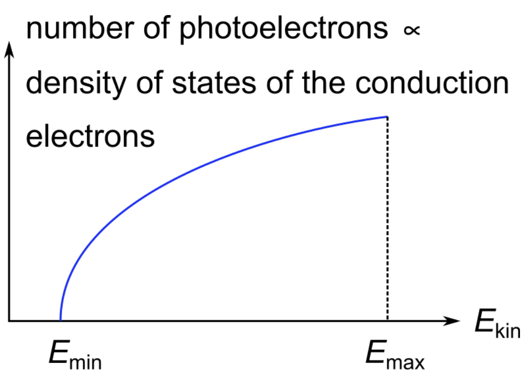
$$E_{kin} = E - W_A - E_F = 6.21\,\text{eV} - 2.28\,\text{eV} - 3.24\,\text{eV} = 0.69\,\text{eV}$$

c) The limit wavelength results when the most weakly bound electrons can just barely or just not be released by a photon

$$\lambda_G = \frac{hc}{W_A} = \frac{4.14 \cdot 10^{-15}\,\text{eV}\,3 \cdot 10^{8}\,\text{ms}^{-1}}{2.28\,\text{eV}} = 5.45 \cdot 10^{-7}\,\text{m}$$

Photoelectrons are no longer released for $\lambda > 545\,\text{nm}$.

d) Spectrum of the photoelectrons



The spectrum is proportional to the density of states of the conduction electrons, which in turn is proportional to the square root of the electron energy.

Problem 7 (4 Points)

a) What is the Fermi energy?
b) For sodium, the electron in the 3s orbital is the valence electron. These electrons are the conduction electrons in metallic sodium. Calculate the Fermi wavenumber of sodium.
c) Give the definition of the density of states of conduction electrons and explain the quantities used.
d) Sketch the density of states that results for a quasi-free electron gas.

Hint: The density of sodium is $\rho = 0.968 \mathrm{gcm}^{-3}$ and the atomic mass is $22.98\,u$

a) The Fermi energy is the highest energy that conduction electrons can have at the temperature $T = 0$.
b) The Fermi wave number k_F is the largest wave number that conduction electrons can have at the temperature $T = 0$. It is

$$N = 2\frac{\frac{4\pi k_F^3}{3}}{\frac{(2\pi)^3}{V}} \quad \rightarrow \quad k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$$

Here N denotes the number of conduction electrons and V the volume of the sodium crystal. With the density

$$\rho = \frac{m}{V} = \frac{22.98\,u\,N}{V}$$

results for the density of the conduction electrons

$$\frac{N}{V} = \frac{\rho}{22.98\,\mathrm{u}} = \frac{0.968\,\mathrm{gcm}^{-3}}{22.98 \cdot 1.66 \cdot 10^{-24}\,\mathrm{g}} = 0.0254 \cdot 10^{24}\,\mathrm{cm}^{-3}$$

and the Fermi wave number is

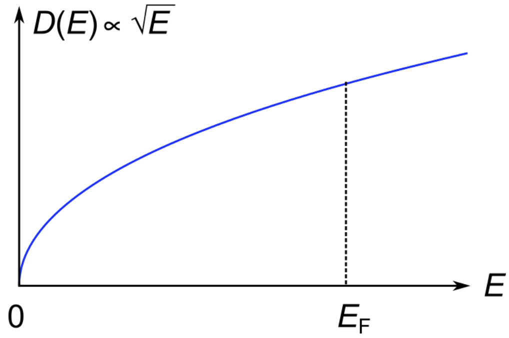
$$k_F = \left(3\pi^2\, 0.0254 \cdot 10^{24}\,\mathrm{cm}^{-3}\right)^{1/3} = 0.91 \cdot 10^{8}\,\mathrm{cm}^{-1} \quad \text{or} \quad 0.91 \cdot 10^{10}\,\mathrm{m}^{-1}$$

c) Definition of the density of states

$$D(E) = \frac{1}{V}\frac{\mathrm{d}N}{\mathrm{d}E}$$

Here
- $\mathrm{d}N$ is the number of states of the conduction electrons in the energy interval $\mathrm{d}E$
- and V the volume of the crystal

d) The density of states of the quasi-free electron gas is proportional to the square root of the energy of the electrons

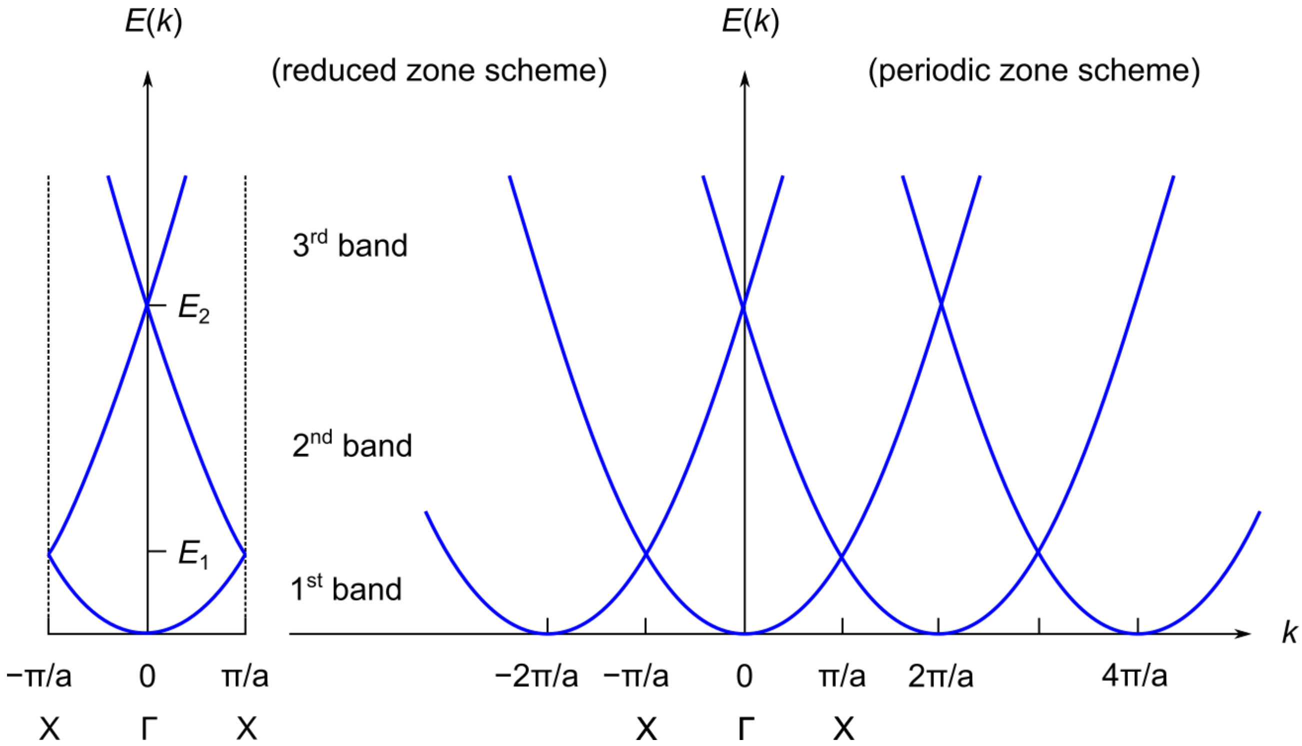

$D(E) \propto \sqrt{E}$
E
0
E_F

Problem 8 (4 Points)

Conduction electrons can move quasi freely in a simple cubic lattice with the lattice constant $a = 3.6 \cdot 10^{-10}$ m.

a) Sketch the band structure in the reduced and in the periodic zone scheme along the Γ-X direction in case the influence of the periodic crystal potential is negligibly small.
b) The energy zero point is in the minimum of the first band. Calculate the highest energy of the 1st and 2nd bands, when the mass of the charge carriers is $m_e = 500\,\text{keV}/c^2$.
c) Sketch the band structure in the reduced zone scheme along the Γ-X direction in case the influence of the periodic crystal potential is not negligibly small.
d) Mark the energy ranges in the band structure in which the charge carriers can be assigned a positive charge. Justify your sketch.

a) band structure of quasi free electrons in a simple cubic lattice along the Γ-X direction



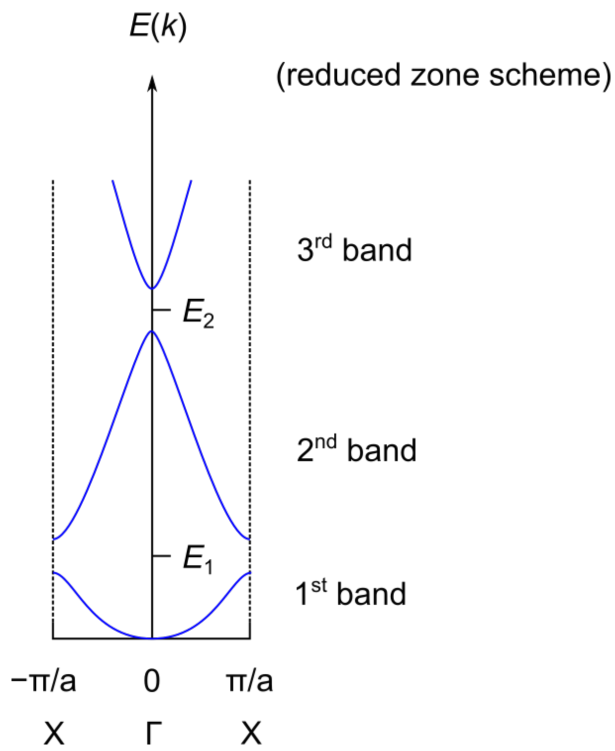
b) The highest energy of the 1st band is

$$E_1 = \frac{\hbar^2\pi^2}{2m_e a^2} = \frac{h^2}{8m_e a^2} = \frac{(4.14 \cdot 10^{-15}\,\text{eVs})^2(3 \cdot 10^8\,\text{ms}^{-1})^2}{8 \cdot 500 \cdot 10^3\,\text{eV}\,(3.6 \cdot 10^{-10}\,\text{m})^2} = 2.97\,\text{eV}$$

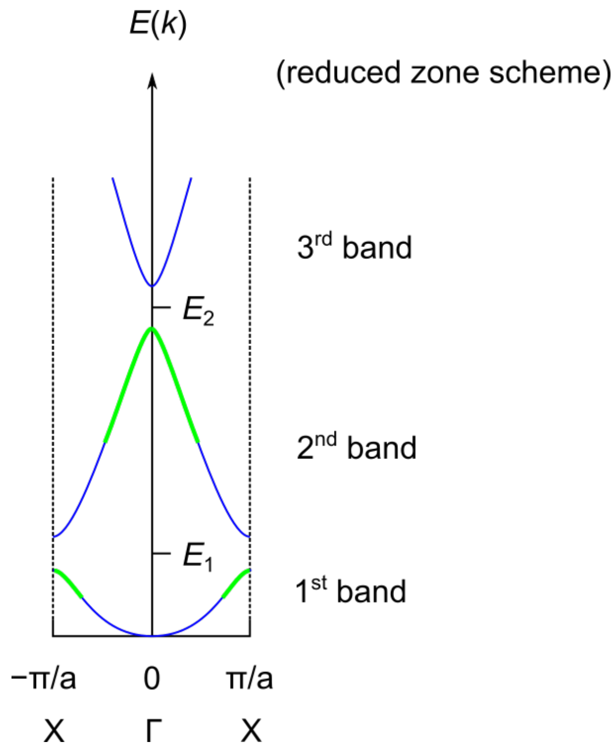
The highest energy of the 2nd band is

$$E_2 = 4 \cdot E_1 = 11.9\,\text{eV}$$

c) Energy gaps open up at the crossing points of the energy band if the periodic potential is not negligibly small



d) The reciprocal of the effective mass of the charge carriers is proportional to the second derivative of the band energy with respect to the components of the wave number vector, i.e. $m^{-1} = \frac{1}{\hbar^2}\frac{\partial^2 E(k)}{\partial k^2}$. In the sketch, the areas of negative effective masses, i.e. positive charge carriers, are marked in green.



Problem 1 (4 Points)

a) Write down the wave equation and the wave function of a plane wave in an isotropic medium.
b) How is the wave number defined and how is the wave number related to the wave number vector?
c) The speed of a light wave in a medium is determined by the refractive index n and $v = c/n$ applies. How does the wave number depend on the refractive index n?
d) When a light wave is refracted, the projection of the wave number vector onto the interface does not change. Show that this statement agrees with the law of refraction $n_1 \sin\alpha_1 = n_2 \sin\alpha_2$.

Problem 2 (Jönsson experiment 1961) (4 Points)

a) Electrons are accelerated with a voltage of 50 kV. What is the wavelength of the resulting electron beam?
b) The electron beam strikes a double slit perpendicularly. The distance between the slits is $d = 2\,\mu\text{m}$ from center to center. The single slits have a width of $b = 0.4\,\mu\text{m}$. At what angles to the central main maximum are the maxima of intensity observed?
c) The electrons are collected on a fluorescence screen set up perpendicular to the beam at a distance of $\ell = 35\,\text{cm}$ from the double slit. What is the distance between the interference maxima on the screen?
d) Sketch the intensity on the screen as a function of the path difference $\Delta s = d \sin\alpha$.

Hint: Use $\sin\alpha \approx \tan\alpha \approx \alpha$.

Problem 3 (4 Points)

a) With what speed must an electron move in relation to an observer so that 0.5 s elapse in the electron frame while 1 s elapses in the observer frame?
b) By how much does the distance between the electron and the observer change during this time, in the reference system of the observer and in the reference system of the electron, respectively?
c) What is the mass of the electron when it moves at this speed?
d) With what voltage is the electron accelerated to reach this speed?

Problem 4 (4 Points)

a) Sketch Planck's law of radiation, i.e. the intensity $I(\lambda)$ of an ideal thermal radiation source as a function of the wavelength λ.
b) What fundamental assumption does Max Planck have to make in order to derive the law of radiation?
c) To which fundamental processes does Albert Einstein attribute thermal radiation?
d) Which of these fundamental processes forms the basis for the function of a laser?
e) Which condition must be fulfilled so that Laser light can be generated?

Problem 5 (4 Points)

^{137}Cs transforms into ^{137}Ba by a β decay. A photon of energy $E_\gamma = 662\,\text{keV}$ is emitted with a probability of 92%.

a) What is the Compton effect?
b) A photon is scattered back from an electron exactly in its opposite direction. Calculate the energy of the backscattered photon.
c) What is the kinetic energy of the electron after scattering if its kinetic energy before scattering is negligibly small?
d) How is the Compton edge formed and what is the energy of the Compton edge?
e) Sketch the Compton spectrum of a ^{137}Cs sample. Name the essential characteristics of the spectrum.

Problem 6 (4 Points)

a) What is the binding energy of an electron in the 2s and the 2p orbital of the hydrogen atom, respectively?
b) Sketch the radial wave functions $R_{2s}(r)$ and $R_{2p}(r)$ for the hydrogen atom, as well as the radial probability densities $r^2R_{2s}^2(r)$ and $r^2R_{2p}^2(r)$.
c) Justify your sketches in part b).

Problem 7 (4 Points)

The electron configuration of sodium is $1s^2, 2s^2, 2p^6, 3s^1$.

a) Write down the eigenvalue equations for the total angular momentum $\vec{J}$.
b) What are the quantum numbers of the total angular momentum of sodium?
c) Calculate the magnetic moment of sodium.
d) Sodium atoms fly through an inhomogeneous magnetic field oriented in the z-direction. The field gradient is $\partial B_z/\partial z = 100\,\text{Tm}^{-1}$ and zero else. Calculate the force acting on the sodium atoms in this magnetic field. What determines the direction of the force? Hint: $\vec{F} = (\vec{\mu}\cdot\nabla)\vec{B}$.

Problem 8 (4 Points)

In the Sommerfeld model, the formula $C = V\frac{\pi^2}{3}D(E_F)k_B^2T$ results for the heat capacity of the electron gas.

a) How is the density of states of the electrons $D(E)$ defined?
b) Show that the density of states in the Sommerfeld model is proportional to the square root of the energy, i.e. it is $D(E) \propto \sqrt{E}$.
c) The density of the conduction electrons of copper is $N/V = 8.45\cdot 10^{22}\,\text{cm}^{-3}$. Calculate the Fermi temperature of copper, i.e. $T_F = E_F/k_B$.
d) Why is the density of states of the electrons at the Fermi energy E_F needed to calculate the heat capacity?

Required physical constants:

Speed of light:	$c = 3\cdot 10^8\ \text{m/s}$
Boltzmann's constant:	$k_B = 8.6\cdot 10^{-5}\text{eV/K} = 1.38\cdot 10^{-23}\text{J/K}$
Planck's constant:	$h = 4.14\cdot 10^{-15}\,\text{eVs} = 6.62\cdot 10^{-34}\,\text{Ws}^2$

Compton wave length:	$\lambda_C = 2.43 \cdot 10^{-12}$ m
Rydberg unit of energy:	$R = 13.6\,\text{eV}$
Bohr's magneton:	$\mu_B = 9.274 \cdot 10^{-24}\,\text{Am}^2$
Elementary charge:	$e = 1.6 \cdot 10^{-19}\,\text{As}$
Rest mass of the electron:	$m_e = 511\,\text{keV}/c^2 = 9.1 \cdot 10^{-31}\,\text{kg}$

Problem 1 (4 Points)

a) Write down the wave equation and the wave function of a plane wave in an isotropic medium.
b) How is the wave number defined and how is the wave number related to the wave number vector?
c) The speed of a light wave in a medium is determined by the refractive index n and $v = c/n$ applies. How does the wave number depend on the refractive index n?
d) When a light wave is refracted, the projection of the wave number vector onto the interface does not change. Show that this statement agrees with the law of refraction $n_1 \sin\alpha_1 = n_2 \sin\alpha_2$.

a) wave equation

$$\frac{\partial^2\psi}{\partial t^2} = v^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) \qquad \text{bzw.} \qquad \frac{\partial^2\psi}{\partial t^2} = v^2\nabla^2\psi$$

plane wave

$$\psi(\vec{r},t) = \psi_0 e^{i\left(\vec{k}\vec{r}-\omega t\right)}$$

b) wave number

$$k = \frac{2\pi}{\lambda}$$

The wave number is the absolute value of the wave number vector, i.e. $k = |\vec{k}|$.

c) With $\lambda = \frac{c}{n}\cdot T = \frac{c}{n\cdot\nu}$ one has

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\cdot\nu}{c}\cdot n = k_0\cdot n$$

k_0 denotes the wave number in vacuum.

d) Since the angles α_1 and α_2 are measured with respect to the surface normal in the case of the law of refraction, the following formulas apply to the projection of the wave number vector onto the interface

$$k_1 \sin\alpha_1 \qquad \text{and} \qquad k_2 \sin\alpha_2$$

With the condition

$$k_1 \sin\alpha_1 = k_2 \sin\alpha_2 \qquad \text{the law of refraction follows, i.e.} \qquad n_1 \sin\alpha_1 = n_2 \sin\alpha_2$$

Problem 2 (Jönsson experiment 1961) (4 Points)

a) Electrons are accelerated with a voltage of 50 kV. What is the wavelength of the resulting electron beam?
b) The electron beam strikes a double slit perpendicularly. The distance between the slits is $d = 2\,\mu\text{m}$ from center to center. The single slits have a width of $b = 0.4\,\mu\text{m}$. At what angles to the central main maximum are the maxima of intensity observed?
c) The electrons are collected on a fluorescence screen placed perpendicular to the beam at a distance of $\ell = 35\,\text{cm}$ from the double slit. What is the distance between the interference maxima on the screen?
d) Sketch the intensity on the screen as a function of the path difference $\Delta s = d \sin\alpha$.

Hint: Use $\sin\alpha \approx \tan\alpha \approx \alpha$.

a) de Broglie wavelength of the electron beam

$$\lambda = \frac{h}{p}$$

the momentum is

$$E_{\text{kin}} = \frac{p^2}{2m_e} \qquad \rightarrow \qquad p = \sqrt{2m_e E_{\text{kin}}} = \sqrt{2 \cdot 511\,\text{keV}/c^2\, 50\,\text{keV}} = 226\,\text{keV}/c$$

and the wave length

$$\lambda = \frac{h}{p} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\, 3 \cdot 10^8\,\text{ms}^{-1}}{226 \cdot 10^3\,\text{eV}} = 5.5 \cdot 10^{-12}\,\text{m}$$

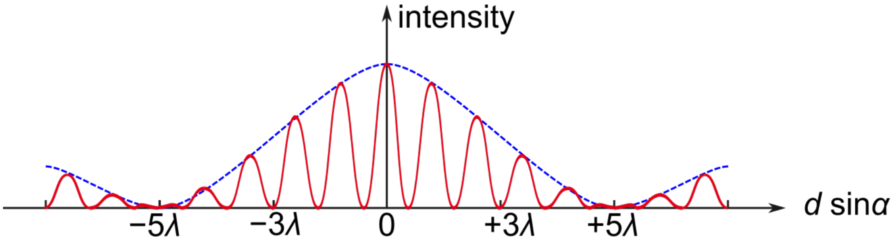
b) Condition for constructive interference at the double slit

$$n\lambda = d \cdot \sin\alpha_n \qquad \rightarrow \qquad \alpha_n = n\frac{\lambda}{d} = n \cdot \frac{5.5 \cdot 10^{-12}\,\text{m}}{2 \cdot 10^{-6}\,\text{m}} = n \cdot 2.75 \cdot 10^{-6}$$

c) δ denotes the distance of the interference maxima on the screen. Since the angles are very small, $\tan\alpha = \alpha$ can be used

$$\delta = \alpha \cdot \ell = 2.75 \cdot 10^{-6} \cdot 0.35\,\text{m} = 0.96\,\mu\text{m}$$

d) Since $d = 5 \cdot b$ applies, every 5th interference maximum is suppressed due to the single-slit diffraction



Problem 3 (4 Points)

a) With what speed must an electron move in relation to an observer so that 0.5 s elapse in the electron frame while 1 s elapses in the observer frame?
b) By how much does the distance between the electron and the observer change during this time, in the reference system of the observer and in the reference system of the electron, respectively?
c) What is the mass of the electron when it moves at this speed?
d) With what voltage is the electron accelerated to reach this speed?

a) The proper time in the reference frame of the electron is $T_0 = 0.5\,\text{s}$. The time $T(v) = 1\,\text{s}$ elapses while the observer observes the moving electron, i.e.

$$T(v) = T_0 \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

the velocity of the electron is thus

$$v = c\sqrt{1 - (T_0/T(v))^2} = c\sqrt{1 - (0.5)^2} = c\frac{\sqrt{3}}{2} = 2.6 \cdot 10^8\,\text{m/s}$$

b) While 1 s elapses in the observer's frame of reference, the electron covers the distance $2.6 \cdot 10^8$ m. During this time, only 0.5 s elapse in the electron's frame of reference and the electron can only cover the distance $2.6 \cdot 10^8\,\text{m} \cdot 0.5 = 1.3 \cdot 10^8\,\text{m}$. This is length contraction.

c) The relativistic mass of the electron is

$$m(v) = \frac{m_0}{\sqrt{1 - (v/c)^2}} = \frac{511\,\text{keV}/c^2}{\sqrt{1 - (\sqrt{3}/2)^2}} = 2 \cdot 511\,\text{keV}/c^2$$

d) The energy of the electron is

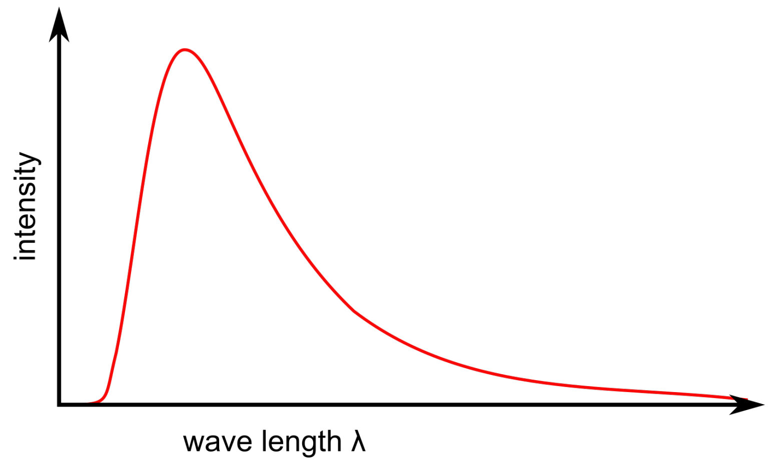
$$E = m(v)c^2 = 2 \cdot 511\,\text{keV}.$$

That is twice the rest energy of an electron. The kinetic energy equals therefore the rest energy and the acceleration voltage must therefore be 511 kV.

Problem 4 (4 Points)

a) Sketch Planck's law of radiation, i.e. the intensity $I(\lambda)$ of an ideal thermal radiation source as a function of the wavelength λ.
b) What fundamental assumption does Max Planck have to make in order to derive the law of radiation?
c) To which fundamental processes does Albert Einstein attribute thermal radiation?
d) Which of these fundamental processes forms the basis for the function of a laser?
e) Which condition must be fulfilled so that Laser light can be generated?

a) Sketch of Planck's radiation law



b) Max Planck makes the assumption that electromagnetic waves can only be absorbed and emitted in quanta of energy $E = h\nu$.
c) In order to derive Planck's law of radiation, Albert Einstein needed the fundamental processes of spontaneous emission, stimulated emission and the absorption of an energy quantum of electromagnetic radiation.
d) The stimulated emission amplifies an electromagnetic wave coherently and thus forms the basis for the laser (Light Amplification by Stimulated Emission of Radiation)
e) Since the stimulated emission must exceed the effect of absorption, the excited state of the laser is more populated than the ground state. This condition is called population inversion.

Problem 5 (4 Points)

^{137}Cs transforms into ^{137}Ba by a β decay. A photon of energy $E_\gamma = 662\,\text{keV}$ is emitted with a probability of 92%.

a) What is the Compton effect?
b) A photon is scattered back from an electron exactly in its opposite direction. Calculate the energy of the backscattered photon.
c) What is the kinetic energy of the electron after scattering if its kinetic energy before scattering is negligibly small?
d) How is the Compton edge formed and what is the energy of the Compton edge?
e) Sketch the Compton spectrum of a ^{137}Cs sample. Name the essential characteristics of the spectrum.

a) The Compton effect refers to the scattering of a photon by a quasi-resting electron, i.e. the kinetic energy of the electron before colliding with the photon is negligibly small compared to the energy of the photon.
b) Compton formula

$$\lambda' - \lambda = \lambda_C(1 - \cos\theta)$$

λ' denotes the wavelength after scattering. The scattering angle for backscattering is $\theta = 180°$. The wavelength of the photon before the collision is

$$\lambda = \frac{hc}{E_\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^{8}\,\text{ms}^{-1}}{662 \cdot 10^{3}\,\text{eV}} = 1.88 \cdot 10^{-12}\,\text{m}$$

and the wavelength of the backscattered photon is

$$\lambda' = \lambda + \lambda_C(1 - \cos\theta) = 1.88 \cdot 10^{-12}\,\text{m} + 2 \cdot 2.43 \cdot 10^{-12}\,\text{m} = 6.74 \cdot 10^{-12}\,\text{m}.$$

The energy of the backscattered photon is thus

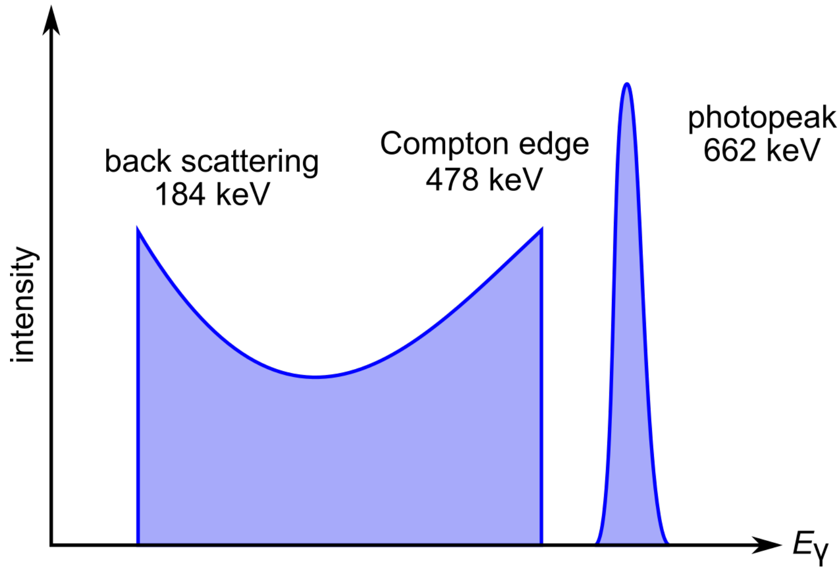
$$E_{\lambda'} = \frac{hc}{\lambda'} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\,3 \cdot 10^{8}\,\text{ms}^{-1}}{6.74 \cdot 10^{-12}\,\text{m}} = 1.84 \cdot 10^{5}\,\text{eV} = 184\,\text{keV}$$

c) The kinetic energy of the electron is

$$E_{\text{kin}} = E_\lambda - E_{\lambda'} = 662\,\text{keV} - 184\,\text{keV} = 478\,\text{keV}$$

d) The Compton edge is the largest energy of the continuous Compton spectrum. The largest energy of the continuous spectrum results when the backscattered electrons radiate all their kinetic energy with a single photon. The energy of the Compton edge thus corresponds to the maximum kinetic energy of the scattered electrons.

e) Schematic sketch of the Compton spectrum of ^{137}Cs



Just as the photopeak has a finite line width, the backscatter peak and the Compton edge also have a finite width (cf. e.g. the Compton spectrum of ^{60}Co in the lecture.)

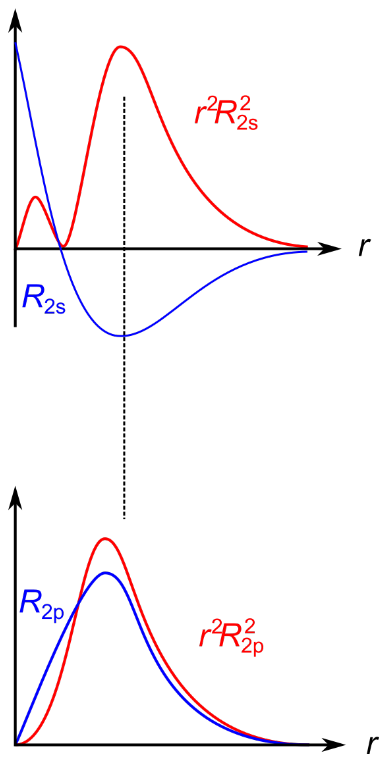
Problem 6 (4 Points)

a) What is the binding energy of an electron in the 2s and the 2p orbital of the hydrogen atom, respectively?
b) Sketch the radial wave functions $R_{2s}(r)$ and $R_{2p}(r)$ for the hydrogen atom, as well as the radial probability densities $r^2R_{2s}^2(r)$ and $r^2R_{2p}^2(r)$.
c) Justify your sketches in part b).

a) The binding energies of 2s and 2p electrons are the same in the hydrogen atom to a first approximation. With $E_n = -\frac{R}{n^2}$ and $n = 2$ is the binding energy

$$E_2 = -\frac{R}{2^2} = -\frac{13.6\,\text{eV}}{2^2} = -3.4\,\text{eV}$$

b) The radial wave functions and probability densities of the 2s- and 2p-orbitals of the hydrogen atom



c)
- The p-electron has an angular momentum with the quantum number $\ell = 1$. The centrifugal force pushes the p-electron away from the nucleus, so that the radial wave function increases proportionally with the distance to the nucleus r, and then decays exponentially with $e^{-r/2a_B}$ for larger r-values.
- The s-electron has no angular momentum. It penetrates the nucleus and reappears on the other side, completely unaffected by the strong forces acting within and between the nucleons. The wave function is maximum at the nucleus. Since the s-electron and the p-electron have the same energy, the additional rotational energy of the p-electron must be balanced by an increased radial energy of the s-electron. Since the range of the radial wave

function is determined by the same exponential function for the s- and p-orbitals, the slope of the radial wave function can be increased by adding a zero crossing in the radial wave function of the s-electron $R_{2s}(r)$. This increases the radial energy since it depends on the derivatives of the wave function.

- The comparison of the radial probability densities of s- and p-electron, respectively, shows that the p-electron is pushed away from the nucleus due to the rotational movement and the maximum of the radial probability density of the s-electron due to the larger radial energy is at a greater distance to the nucleus than in the case of the p-electron.

Problem 7 (4 Points)

The electron configuration of sodium is $1s^2, 2s^2, 2p^6, 3s^1$.

a) Write down the eigenvalue equations for the total angular momentum $\vec{J}$.
b) What are the quantum numbers of the total angular momentum of sodium?
c) Calculate the magnetic moment of sodium.
d) Sodium atoms fly through an inhomogeneous magnetic field oriented in the z-direction. The field gradient is $\partial B_z/\partial z = 100\,\mathrm{Tm}^{-1}$ and zero else. Calculate the force acting on the sodium atoms in this magnetic field. What determines the direction of the force? Hint: $\vec{F} = (\vec{\mu} \cdot \nabla)\vec{B}$.

a) The eigenvalue equations for total angular momentum are

$$\hat{\vec{J}}^2 \,|\, j, m_j\rangle = j(j+1)\hbar^2 \,|\, j, m_j\rangle$$
$$\hat{J}_z \,|\, j, m_j\rangle = m_j\hbar \,|\, j, m_j\rangle$$

b) The $1s^2$, $2s^2$ and $2p^6$ configurations correspond to closed shells and have no angular momentum. The $3s^1$ configuration has no orbital angular momentum and spin $s = 1/2$. The total angular momentum thus corresponds to the spin of the electron.

c) The spin magnetic moment is

$$\vec{\mu}_s = -g\mu_B \frac{\vec{s}}{\hbar}$$

The magnitude of the magnetic moment is with $g = 2$ and $s = 1/2$

$$|\vec{\mu}_s| = g\mu_B \sqrt{s(s+1)} = 2 \cdot 9.274 \cdot 10^{-24}\,\mathrm{Am}^2 \sqrt{\frac{1}{2}(1+\frac{1}{2})} = 16.1 \cdot 10^{-24}\,\mathrm{Am}^2$$

d) Only the projection of $\vec{\mu}_s$ onto the z-axis is effective. Depending on the orientation of the spin, the force acts parallel or antiparallel to the z-axis. The force is

$$F_z = \pm\mu_B \cdot \frac{\partial B_z}{\partial z} = \pm 9.274 \cdot 10^{-24}\,\mathrm{Am}^2 \cdot 100\,\mathrm{Tm}^{-1} = \pm 9.274 \cdot 10^{-22}\,\mathrm{N}$$

Problem 8 (4 Points)

In the Sommerfeld model, the formula $C = V\frac{\pi^2}{3}D(E_F)k_B^2 T$ results for the heat capacity of the electron gas.

a) How is the density of states of the electrons $D(E)$ defined?
b) Show that the density of states in the Sommerfeld model is proportional to the square root of the energy, i.e. it is $D(E) \propto \sqrt{E}$.
c) The density of the conduction electrons of copper is $N/V = 8.45 \cdot 10^{22}\,\text{cm}^{-3}$. Calculate the Fermi temperature of copper, i.e. $T_F = E_F/k_B$.
d) Why is the density of states of the electrons at the Fermi energy E_F needed to calculate the heat capacity?

a) Definition of the density of states

$$D(E) = \frac{1}{V}\frac{dN}{dE}$$

Here, dN denotes the number of $\vec{k}$ states in the energy interval dE at the energy E.

b) In the Sommerfeld model, the energy of the conduction electrons is simply the kinetic energy $E = \frac{\hbar^2\vec{k}^2}{2m_e}$. The energy is thus determined by the wave number k and is independent of the direction in which the vector $\vec{k}$ points. The number of $\vec{k}$-states in the wavenumber interval dk is thus

$$dN = 4\pi k^2 dk.$$

With $E = \hbar^2k^2/2m_e$ is $dE \propto k dk$ and $dN \propto k dE \propto \sqrt{E}\,dE$ results $D(E) \propto \sqrt{E}$.

c) The Fermi energy is

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} \qquad \text{with} \qquad k_F = \left((3\pi^2)\frac{N}{V}\right)^{1/3}$$

With $N/V = 8.45 \cdot 10^{22}\,\text{cm}^{-3} = 8.45 \cdot 10^{28}\,\text{m}^{-3}$ is

$$k_F = \left((3\pi^2)\frac{N}{V}\right)^{1/3} = \left((3\pi^2)\cdot 8.45\cdot 10^{28}\,\text{m}^{-3}\right)^{1/3} = 1.36\cdot 10^{10}\,\text{m}^{-1}$$

and the Fermi temperature

$$T_F = \frac{\hbar^2 k_F^2}{2m_e k_B} = \frac{(4.14\cdot 10^{-15}\,\text{eVs}\cdot 1.36\cdot 10^{10}\,\text{m}^{-1}\,3\cdot 10^8\,\text{ms}^{-1})^2}{2\cdot 2^2\pi^2\,511\cdot 10^3\,\text{eV}\,8.6\cdot 10^{-5}\,\text{eVK}^{-1}} = 82222\,\text{K}$$

d) Only electrons occupying $\vec{k}$ states near the Fermi energy can be excited by thermal energies in the $\approx k_B T$ range. All other electrons cannot leave their k-states. Since $T << T_F$ applies to all temperatures below the vaporization temperature of copper (2868 K), only electrons in the range of the Fermi energy can contribute to the heat capacity.

Problem 1 (4 Points)

a) What energy is required to accelerated an electron to 90 % of the speed of light?
b) What is the momentum of the electron after it has been accelerated to 90 % of the speed of light?
c) An observer moves at 50 % of the speed of light perpendicular to the motion of the electron.
   i) What is the momentum of the electron perpendicular to the velocity of the observer?
   ii) What is the energy of the electron in the observer's frame of reference?

Problem 2 (4 Points)

a) Ideal thermal radiation is often referred to as blackbody radiation. What characterizes an ideal thermal radiation source?
b) Sketch the radiation spectrum of a blackbody as a function of wavelength.
c) How does the temperature of the radiation source affect the radiation spectrum?
d) ^{210}Po decays into lead by an α-decay and represents a heat source with the thermal power per kg of $P_{\text{decay}}/m = 141\,\text{W/kg}$. What is the equilibrium temperature on the surface of a polonium sphere (radius 1 cm) in an environment at room temperature $T_0 = 300$ K if the sphere is only cooled by thermal radiation? The density of polonium is $\rho = 9{,}2 \cdot 10^3\,\text{kgm}^{-3}$.

Problem 3 (4 Points)

a) How is the orbital angular momentum defined in Newtonian mechanics and how does the orbital angular momentum of classical physics differ from the orbital angular momentum in quantum physics?
b) Write down the orbital angular momentum operator of quantum physics and give the eigenvalue equations for the orbital angular momentum.
c) Which values can the eigenvalues of the orbital angular momentum have?
d) A mass m can rotate at a distance r around a fixed point in space. Write down the Schrödinger equation of the mass m and give the energy eigenvalues of this mass.

Problem 4 (4 Points)

The electron has an intrinsic angular momentum: the spin.

a) Give the spin operator of the electron.
b) Since the eigenfunctions of the spin operator are not wave functions, Dirac introduced a generalized notation of the quantum states. Explain the Dirac notation and use it to write down the eigenvalue equations of the spin.
c) Which values can the eigenvalues of the electron spin have?
d) The spin of the electron is linked to a magnetic moment. Calculate the change in energy of the electron when a magnetic field of strength $B_0 = 1$ T is applied.

Please note the reverse side

Problem 5 (4 Points)

The valence electron of sodium is a 3s electron.

a) The yellow sodium line corresponds to the 3s $\leftrightarrow$ 3p transition. Explain why the yellow sodium line splits into two components and write down the transitions that correspond to the two components.
b) Which component has the smaller energy?
c) Explain why in a magnetic field one of the two components splits into four and the other into six spectral lines. Give the transitions corresponding to these spectral lines.

Problem 6 (4 Points)

a) What is the difference between Rayleigh and Brillouin scattering?
b) Sketch a Brillouin spectrum, give the typical frequency range and label the Stokes and anti-Stokes components.
c) State the energy and momentum conservation laws relevant to phonon-phonon scattering.
d) What is umklapp scattering and what is the experimental evidence that umklapp scattering occurs?

Problem 7 (4 Points)

a) Explain the Sommerfeld theory of metals.
b) Calculate the Fermi wave number k_F in the Sommerfeld model.
c) Give the definition of the Fermi energy E_F.
d) Calculate the density of states $D(E)$ of the electrons in the Sommerfeld model.

Problem 8 (4 Points)

a) Sketch the 1st Brillouin zone of a simple cubic lattice with the lattice constant a and mark the symmetry points.
b) Calculate the Fermi wave number for the case that there are 3 conduction electrons in the primitive cell of the simple cubic lattice.
c) Calculate the Fermi energy in units $E_0 = \frac{\hbar^2}{2m_e}(\frac{\pi}{a})^2$ and sketch the band structure of the quasi-free electron gas up to the Fermi energy along the ΓX and the ΓM direction in the reduced zone scheme.
d) Sketch the intersection of the Fermi surface of the 2nd energy band with the ΓXM plane in the reduced zone scheme.

Required physical constants:

Speed of light:	$c = 3 \cdot 10^8$ m/s
Planck's constant:	$h = 4.14 \cdot 10^{-15}$ eVs $= 6.62 \cdot 10^{-34}$ Ws^2
Stefan-Boltzmann constant:	$\sigma = 5.67 \cdot 10^{-8}$ Wm^{-2}K^{-4}
Bohr's magneton:	$\mu_B = 9.274 \cdot 10^{-24}$ Am^2
Elementary charge:	$e = 1.6 \cdot 10^{-19}$ As
Rest mass of the electron:	$m_e = 511$ keV$/c^2 = 9.1 \cdot 10^{-31}$ kg

Problem 1 (4 Points)

a) What energy is required to accelerated an electron to 90 % of the speed of light?
b) What is the momentum of the electron after it has been accelerated to 90 % of the speed of light?
c) An observer moves at 50 % of the speed of light perpendicular to the motion of the electron.
   i) What is the momentum of the electron perpendicular to the velocity of the observer?
   ii) What is the energy of the electron in the observer's frame of reference?

a)

$$E = mc^2 = \frac{m_e c^2}{\sqrt{1-0.9^2}} = 2.29 m_e c^2$$

The energy to accelerate is

$$E_{\text{beschl.}} = E - m_e c^2 = 1.29 m_e c^2 = 1.29 \cdot 500\,\text{keV} = 645\,\text{keV}$$

(1 Point)

b) momentum of the electron

$$\vec{p} = m\vec{v} = \frac{m_e \vec{v}}{\sqrt{1-0.9^2}} \quad \rightarrow \quad p = 2.29 \cdot 9.1 \cdot 10^{-31}\,\text{kg}\,0.9 \cdot 3 \cdot 10^8\,\text{ms}^{-1}$$
$$= 56.3 \cdot 10^{-23}\,\text{kgms}^{-1} = 1053\,\text{keV}/c$$

(1 Point)

c) Lorentz transformation for the coordinates (Invariant: $(ct)^2 + \vec{r}^2 = s^2$)

$$x' = \gamma(\text{x} - \frac{\text{v}_{\text{obs.}}}{\text{c}}ct), \quad y' = \text{y}, \quad z' = \text{z}, \quad ct' = \gamma(ct - \frac{\text{v}_{\text{obs.}}}{\text{c}}\text{x})$$

Lorentz transformation for the components of the momentum
(Invariant: $(\frac{E}{c})^2 + \vec{p}^{\,2} = m_e^2$)

$$p'_x = \gamma(p_x - \frac{\text{v}_{\text{obs.}}}{\text{c}^2}E), \quad p'_y = p_y, \quad p'_z = p_z, \quad \frac{E'}{c} = \gamma(\frac{E}{c} - \frac{\text{v}_{\text{obs.}}}{\text{c}}p_x)$$

and $\gamma = \frac{1}{\sqrt{1-(\frac{v_{\text{obs.}}}{c})^2}}$.

   i) The momentum perpendicular to the velocity of the observer does not change

$$p'_y = p_y \quad \text{and} \quad p'_z = p_z \quad \rightarrow \quad p_\perp = 56{,}3 \cdot 10^{-23}\,\text{kgms}^{-1}$$

(1 Point)

ii) The energy of the electron in the frame of reference of the observer is with $p_x = 0$

$$E' = \gamma E = \frac{E}{\sqrt{1 - v_{obs.}^2/c^2}} = \frac{E}{\sqrt{1 - 0.5^2}} = 1.15 \cdot E = 1.15 \cdot 2.29 m_e c^2$$

$$E' = 2.63 \cdot m_e c^2 = 2.63 \cdot 500\,\text{keV} = 1.3\,\text{MeV}$$

(1 Point)

Alternatively, the velocity of the electron in the observer's reference system can be calculated:
With the Lorentz transformation

$$x' = \gamma(x - v_{obs.} \cdot t), \quad t' = \gamma(t - v_{obs.} \cdot x/c^2), \quad y' = y, \quad z' = z$$

with $\gamma = 1/\sqrt{1 - v_{obs.}^2/c^2}$ and the speed of the observer $v_{obs.} = 0{,}5\,c$.
With the velocity of the electron in the rest frame $v_x = 0$ $v_y = 0.9c$ is the velocity in the frame of the observer

$$v'_x = \frac{x'}{t'} = \frac{x - v_{obs.} \cdot t}{t - v_{obs.} \cdot x/c^2} = \frac{v_x - v_{obs.}}{1 - v_{obs.} \cdot v_x/c^2} = -v_{obs.}$$

and

$$v'_y = \frac{y'}{t'} = \frac{y}{\gamma(t - v_{obs.} \cdot x/c^2)} = \frac{v_y}{\gamma(1 - v_{obs.} \cdot v_x/c^2)} = \frac{v_y}{\gamma}$$

The energy of the electron is

$$E = \frac{m_e c^2}{\sqrt{1 - v_e^2}} \quad \text{und} \quad v_e^2 = (v'_x)^2 + (v'_y)^2 = v_{obs.}^2 + (v_y)^2(1 - v_{obs.}^2/c^2)$$

$$E = m_e c^2 \frac{1}{\sqrt{1 - v_{obs.}^2 - (v_y)^2(1 - v_{obs.}^2/c^2)}} = m_e c^2 \frac{1}{\sqrt{(1 - v_{obs.}^2)(1 - (v_y)^2)}}$$

and

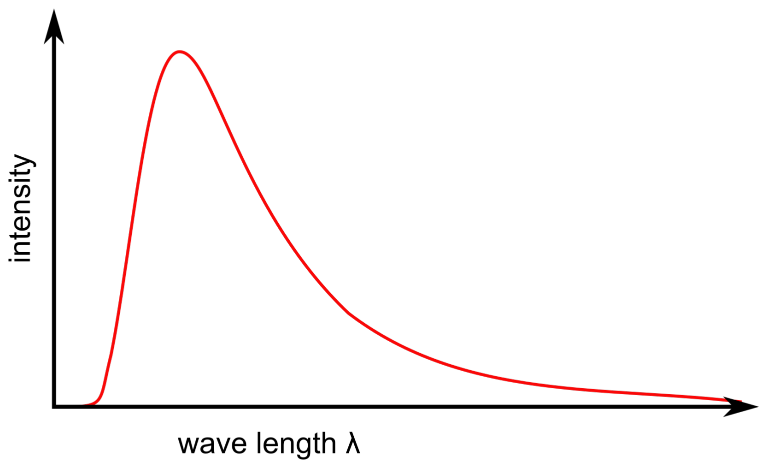
$$E = m_e c^2\, 1.15 \cdot 2.29$$

Problem 2 (4 Points)

a) Ideal thermal radiation is often referred to as blackbody radiation. What characterizes an ideal thermal radiation source?
b) Sketch the radiation spectrum of a blackbody as a function of wavelength.
c) How does the temperature of the radiation source affect the radiation spectrum?
d) ^{210}Po decays into lead by an α-decay and represents a heat source with the thermal power per kg of $P_{\text{decay}}/m = 141\,\text{W/kg}$. What is the equilibrium temperature on the surface of a polonium sphere (radius 1 cm) in an environment at room temperature $T_0 = 300$ K if the sphere is only cooled by thermal radiation? The density of polonium is $\rho = 9{,}2 \cdot 10^3\,\text{kgm}^{-3}$.

a) blackbody radiation
   i) The electromagnetic radiation is in thermal equilibrium with the rest of matter. (1/2 Point)
   ii) The surface does not affect the radiation, i.e. electromagnetic radiation can leave or enter the thermal radiation source unhindered. (1/2 Point)

b) Spectrum of ideal thermal radiation



(1 Point)

c) The wavelength of the maximum λ_{max} is inversely proportional to the temperature of the radiation source, i.e. $\lambda_{\text{max}} \propto 1/T$ (Wien's displacement law). (1/2 Point)

d) In thermal equilibrium,

$$P_{\text{heating}} + A\sigma T_0^4 = A\sigma T^4 \quad \text{with} \quad P_{\text{heating}} = (P_{\text{decay}}/m)\,\rho\,\frac{4\pi r^3}{3} \quad \text{and} \quad A = 4\pi r^2$$

(1/2 Point)

surface temperature

$$T = (\frac{P_{\text{heating}}}{\sigma A} + {T_0}^4)^{(1/4)} = (\frac{141\,\text{Wkg}^{-1}\,9.2 \cdot 10^3\,\text{kgm}^{-3}\,4\pi\,10^{-2}\,\text{m}}{5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}\,3 \cdot 4\pi} + 300^4\,\text{K}^4)^{1/4}$$

(1/2 Point)

$$T = (763 \cdot 10^8\,\text{K}^4 + 300^4\,\text{K}^4)^{1/4} = (763 + 81)^{1/4} \cdot 100\,\text{K} = 539\,\text{K}$$

(1/2 Point)

Problem 3 (4 Points)

a) How is the orbital angular momentum defined in Newtonian mechanics and how does the orbital angular momentum of classical physics differ from the orbital angular momentum in quantum physics?
b) Write down the orbital angular momentum operator of quantum physics and give the eigenvalue equations for the orbital angular momentum.
c) Which values can the eigenvalues of the orbital angular momentum have?
d) A mass m can rotate at a distance r around a fixed point in space. Write down the Schrödinger equation of the mass m and give the energy eigenvalues of this mass.

a) Newton's definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p}.$$

Here, $\vec{r}$ denotes the position vector and $\vec{p}$ the momentum of a particle.

(1/2 Point)

With Newton's momentum, all three components of the vector can be assigned numerical values. In the case of angular momentum in quantum physics, only one component, commonly referred to as the z-component, can be assigned a fixed value. In addition, in quantum physics, the length of the angular momentum vector can also be assigned a fixed value. (1/2 Point)

b) With the momentum operator $\hat{\vec{p}} = -\mathrm{i}\hbar\nabla$ one can write for the angular momentum operator

$$\hat{\vec{L}} = -\mathrm{i}\,\hbar\vec{r} \times \nabla$$

(1/2 Point)

The eigenvalue equations are

$$\begin{aligned} \hat{L}_z Y_{\ell,m} &= m\hbar Y_{\ell,m} \\ \hat{\vec{L}}^2 Y_{\ell,m} &= \ell(\ell+1)\hbar^2 Y_{\ell,m} \end{aligned}$$

The eigenfunctions $Y_{\ell,m} = Y_{\ell,m}(\theta, \varphi)$ are spherical harmonics. (1/2 Point)

c) The angular momentum quantum number ℓ can assume the values 0, 1, 2 and so on. The angular momentum quantum number m varies in steps of 1 in the interval $-\ell \leq m \leq +\ell$.

(1 Point)

d) The Schrödinger equation of a rotating mass is

$$E_\ell Y_{\ell,m}(\theta,\varphi) = \frac{\hat{\vec{L}}^2}{2mr^2} Y_{\ell,m}(\theta,\varphi)$$

(1/2 Point)

The energy eigenvalues are

$$E_\ell = \frac{\ell(\ell+1)\hbar^2}{2mr^2}$$

(1/2 Point)

Problem 4 (4 Points)

The electron has an intrinsic angular momentum: the spin.

a) Give the spin operator of the electron.
b) Since the eigenfunctions of the spin operator are not wave functions, Dirac introduced a generalized notation of the quantum states. Explain the Dirac notation and use it to write down the eigenvalue equations of the spin.
c) Which values can the eigenvalues of the electron spin have?
d) The spin of the electron is linked to a magnetic moment. Calculate the change in energy of the electron when a magnetic field of strength $B_0 = 1\,\mathrm{T}$ is applied.

a) Spin operator of the electron

$$\hat{\vec{s}} = \frac{\hbar}{2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

Here $\sigma_{x,y,z}$ denote the Pauli matrices. (1 Point)

b) Dirac introduced a special bracket symbol $|\,...\,\rangle$ to designate quantum states, in which the quantum numbers are written that determine a quantum state, e.g.

$$|\text{quantum number}_1,\ \text{quantum number}_2,\ \text{quantum number}_3,\ ...\rangle$$

(1/2 Point)

Eigenvalue equations of the spin

$$\hat{s}_z\,|s,\ m_s\rangle = m_s\hbar\,|s,\ m_s\rangle$$
$$\hat{\vec{s}}^{\,2}\,|s,\ m_s\rangle = s(s+1)\hbar^2\,|s,\ m_s\rangle$$

(1/2 Point)

c) The eigenvalues of the electron spin are $s = 1/2$ and $m_s = \pm 1/2$. (1 Point)

d) Zeeman splitting

$$\Delta E_{m_s} = \pm g\mu_B B_0 m_s$$

with $g = 2$ (in a very good approximation) results

$$\Delta E_{m_s=\pm 1/2} = \pm\mu_B B_0 = \pm 9.274 \cdot 10^{-24}\,\mathrm{Am^2} \cdot 1\,\mathrm{Vsm^{-2}}$$
$$= \pm 9.274 \cdot 10^{-24}\,\mathrm{Ws}$$
$$= \pm 5.8 \cdot 10^{-5}\,\mathrm{eV}.$$

(1 Point)

Problem 5 (4 Points)

The valence electron of sodium is a 3s electron.

a) The yellow sodium line corresponds to the 3s $\leftrightarrow$ 3p transition. Explain why the yellow sodium line splits into two components and write down the transitions that correspond to the two components.
b) Which component has the smaller energy?
c) Explain why in a magnetic field one of the two components splits into four and the other into six spectral lines. Give the transitions corresponding to these spectral lines.

a) Spin-orbit coupling: in the 3p state, the magnetic moment of the electron aligns in the magnetic field caused by the orbital motion of the electron. Two settings are possible in which the orbital angular momentum with the quantum number $\ell = 1$ couple with the spin of the electron to become the total angular momentum with the quantum numbers $j = 1/2$ or $j = 3/2$.

(1 Point)

The transitions are

$$3s_{1/2} \leftrightarrow 3p_{1/2}$$
$$3s_{1/2} \leftrightarrow 3p_{3/2}$$

(1 Point)

b) The transition $3s_{1/2} \leftrightarrow 3p_{1/2}$ has the smaller energy, since the magnetic moments of orbit and spin are oriented antiparallel rather than parallel to each other (However, note that strictly parallel or antiparallel alignment is not possible for quantized angular momentum.).

(1/2 Point)

c) The ground state splits into the two quantum states $|3s_{1/2}, m_\mathrm{j} = \pm 1/2\rangle$ in the magnetic field.
Likewise, the first excited state $3p_{1/2}$ splits into the two quantum states $|3p_{1/2}, m_\mathrm{j} = \pm 1/2\rangle$,
while the second excited state $3p_{3/2}$ splits into the four quantum states $|3s_{3/2}, m_\mathrm{j} = \pm 1/2, \pm 3/2\rangle$.
With the selection rule $\Delta m_\mathrm{j} = 0, \pm 1$ and the fact that the splitting energy of the quantum states $3s_{1/2}$, $3p_{1/2}$ and $3s_{3/2}$ in the magnetic field is different,

(1/2 Point)

result the four transitions

$$|3s_{1/2}, \pm 1/2\rangle \leftrightarrow |3p_{1/2}, \pm 1/2\rangle$$
$$|3s_{1/2}, \pm 1/2\rangle \leftrightarrow |3p_{1/2}, \mp 1/2\rangle$$

(1/2 Point)

and the six transitions

$$|3s_{1/2}, +1/2\rangle \leftrightarrow |3p_{3/2}, +3/2\rangle \quad \text{and} \quad |3s_{1/2}, -1/2\rangle \leftrightarrow |3p_{3/2}, -3/2\rangle$$
$$|3s_{1/2}, +1/2\rangle \leftrightarrow |3p_{3/2}, \pm 1/2\rangle \quad \text{and} \quad |3s_{1/2}, -1/2\rangle \leftrightarrow |3p_{3/2}, \pm 1/2\rangle \,.$$

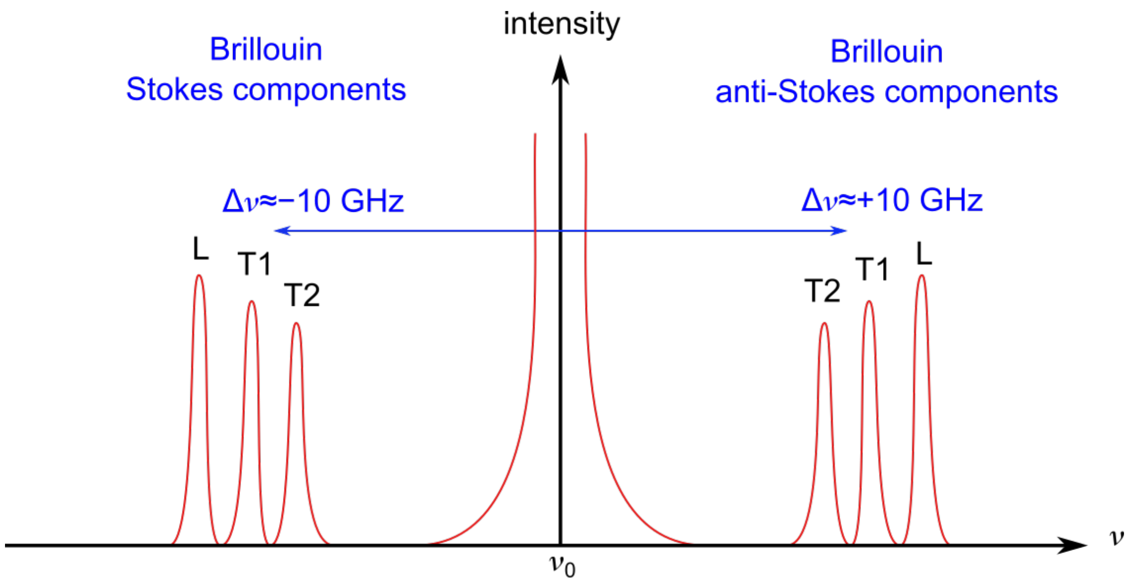
(1/2 Point)

Problem 6 (4 Points)

a) What is the difference between Rayleigh and Brillouin scattering?
b) Sketch a Brillouin spectrum, give the typical frequency range and label the Stokes and anti-Stokes components.
c) State the energy and momentum conservation laws relevant to phonon-phonon scattering.
d) What is umklapp scattering and what is the experimental evidence that umklapp scattering occurs?

a) Rayleigh scattering describes the elastic scattering of light due to density fluctuations of a material. (1/2 Point)
Brillouin scattering describes the absorption or emission of an acoustic phonon by a photon in a material. (1/2 Point)

b) Depending on the emission or absorption of a longitudinally or transversely polarized phonon, there are up to three lines. A phonon is emitted in the Stoke components, while a phonon is absorbed by a photon in the anti-Stokes components. The frequency shift of the Brillouin lines compared to the elastic scattering of the Rayleigh line is in the range of several GHz.



(1 Point)

c) energy and momentum conservation laws

$$\hbar\omega(\vec{q}_1) + \hbar\omega(\vec{q}_2) = \hbar\omega(\vec{q}_3) \qquad \text{and} \qquad \vec{q}_1 + \vec{q}_2 + \vec{K} = \vec{q}_3$$

(1 Point)

d) The momentum of a phonon is a crystal or quasi-momentum, i.e. a vector of the reciprocal lattice can be added. Only the shortest wave vector of a phonon determines the direction of propagation of the lattice wave. If the sum $\vec{q}_1 + \vec{q}_2$ can be reduced to a shorter q-vector by adding a reciprocal lattice vector, the resulting wave can propagate into the opposite direction of the original two waves. The corresponding scattering is referred to as umklapp scattering, since the energy of the two original waves can be transported in the opposite direction. (1/2 Point)

Umklapp scattering reduces the thermal conductivity of crystals at high temperatures. (1/2 Point)

Problem 7 (4 Points)

a) Explain the Sommerfeld theory of metals.
b) Calculate the Fermi wave number k_F in the Sommerfeld model.
c) Give the definition of the Fermi energy E_F.
d) Calculate the density of states $D(E)$ of the electrons in the Sommerfeld model.

a) In the Sommerfeld theory of metals, the electrons are described by plane waves that can propagate in all spatial directions. The wave number vectors are quantized according to the formula

$$\vec{k}_{n_1, n_2, n_3} = \frac{2\pi}{L} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

(1/2 Point)

Here L denotes the edge length of a cube-shaped metal sample.
According to the Pauli principle, each quantum state that is determined by a wave number vector $\vec{k}_{n_1, n_2, n_3}$ can be occupied by two electrons. (1/2 Point)

b) The Fermi wave number is determined by the condition

$$N = 2 \cdot \frac{4\pi k_F^3/3}{(2\pi)^3/L^3}$$

Since all spatial directions are equal, all k-states within a sphere can be occupied by electrons up to the Fermi wave number. $4\pi k_F^3/3$ is the volume of a sphere, $(2\pi)^3/L^3$ is the volume of a k-state. The factor 2 takes into account that each k-state can be occupied by two electrons and N denotes the number of electrons. The Fermi wave number is only determined by the electron density

$$k_F = (3\pi^2 \frac{N}{L^3})^{1/3} = (3\pi^2 \frac{N}{V})^{1/3}.$$

(1 Point)

c) The Fermi energy is given by the kinetic energy of the electrons with the Fermi wave number

$$E_F = \frac{\hbar^2 k_F^2}{2m_e}$$

(1/2 Point)

d) The density of states of the electrons is

$$D(E) = \frac{1}{V}\frac{dN}{dE}.$$

(1/2 Point)

Since all spatial directions are equal, the number of states dN at a certain energy is given by the number of k-states in a spherical shell with radius k and thickness dk

$$\mathrm{d}N = 2 \cdot \frac{4\pi k^2 \mathrm{dk}}{(2\pi)^3/V}.$$

With $E = \hbar^2 k^2/2m_\mathrm{e} \rightarrow k^2 = 2m_\mathrm{e}E/\hbar^2$ and $\mathrm{d}E = \hbar^2 k \mathrm{d}k/m_\mathrm{e} \rightarrow \mathrm{d}k = m_\mathrm{e}\mathrm{d}E/(\hbar^2 k)$

$$\mathrm{d}N = \frac{V}{\pi^2} \cdot \frac{2m_\mathrm{e}E}{\hbar^2} \cdot \frac{m_\mathrm{e}\mathrm{d}E}{\hbar^2 k} = \frac{V}{\pi^2} \cdot \frac{2m_\mathrm{e}E}{\hbar^2} \cdot \frac{m_\mathrm{e}\mathrm{d}E}{\hbar\sqrt{2m_\mathrm{e}E}}$$

and

$$D(E) = \frac{1}{V}\frac{\mathrm{d}N}{\mathrm{d}E} = \frac{1}{\pi^2}\frac{\sqrt{2m_\mathrm{e}^3}}{\hbar^3}\sqrt{E}.$$

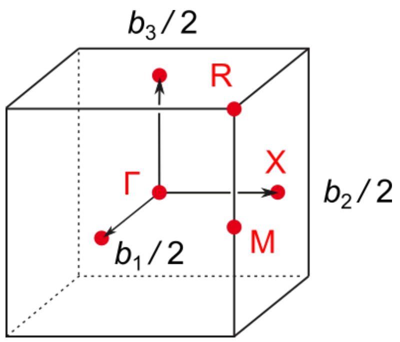
(1 Point)

Problem 8 (4 Points)

a) Sketch the 1st Brillouin zone of a simple cubic lattice with the lattice constant a and mark the symmetry points.
b) Calculate the Fermi wave number for the case that there are 3 conduction electrons in the primitive cell of the simple cubic lattice.
c) Calculate the Fermi energy in units $E_0 = \frac{\hbar^2}{2m_e}(\frac{\pi}{a})^2$ and sketch the band structure of the quasi-free electron gas up to the Fermi energy along the ΓX and the ΓM direction in the reduced zone scheme.
d) Sketch the intersection of the Fermi surface of the 2nd energy band with the ΓXM plane in the reduced zone scheme.

a) With the basis vectors of the simple cubic lattice $\vec{a}_1 = a\vec{e}_x$, $\vec{a}_2 = a\vec{e}_y$ and $\vec{a}_3 = a\vec{e}_z$ are the basis vectors of the reciprocal lattice $\vec{b}_1 = (2\pi/a)\vec{e}_x$, $\vec{b}_2 = (2\pi/a)\vec{e}_y$ and $\vec{b}_3 = (2\pi/a)\vec{e}_z$.

Sketch of the 1st Brillouin zone



(1 Point)

b) Fermi wave number

$$k_F = (3\pi^2 \frac{N}{V})^{1/3} = (3\pi^2 \cdot 3/a^3)^{1/3} = \frac{4.46}{a}$$

(1 Point)

c) Fermi energy

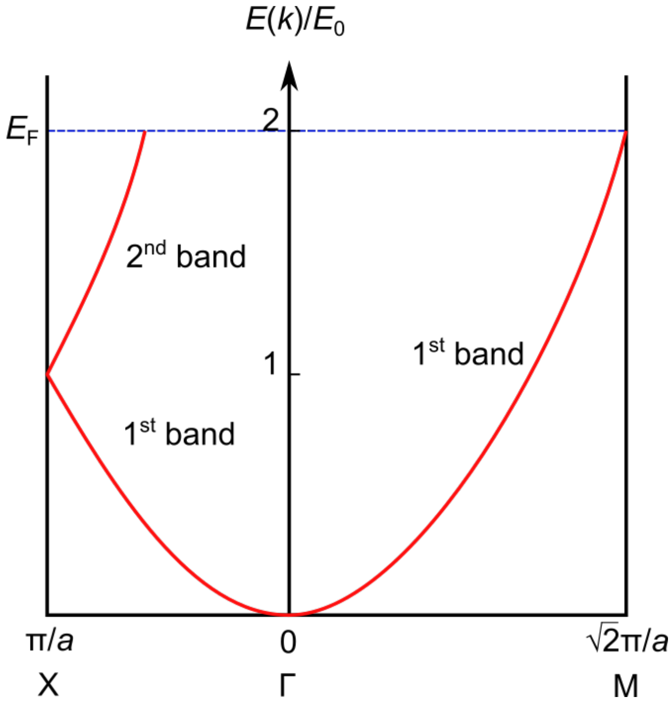
$$E_F = \frac{\hbar^2}{2m_e} k_F^2$$

and

$$\frac{E_F}{E_0} = \left(\frac{k_F\, a}{\pi}\right)^2 = \left(\frac{4.46}{\pi}\right)^2 = 2.02$$

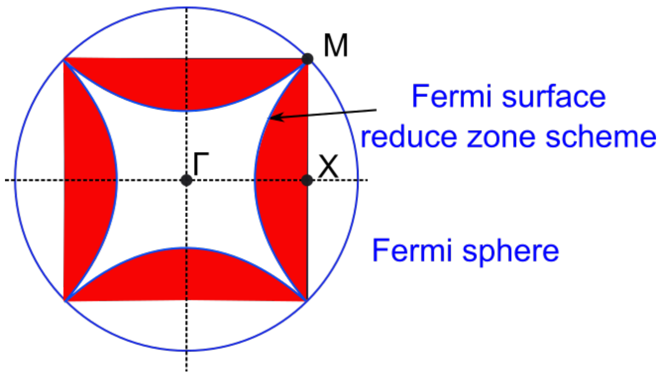
(1/2 Point)

Band structure



(1/2 Point)

d) Intersection of the Fermi surface of the 2nd energy band with the ΓXM plane in the reduced zone scheme



(1 Point)

Problem 1 (4 Points)

a) Write down the wave function of a harmonic wave and explain the quantities involved.
b) Two harmonic electromagnetic waves with amplitude $\vec{E}_0$ are propagating in the same direction along the x-axis with frequency $\nu_1 = 101$ MHz and $\nu_2 = 99$ MHz. At what times does the electric field strength $\vec{E}$ at location $x = 0$ become zero?
c) The two waves now propagate in opposite directions along the x-axis in vacuum with a frequency of 100 MHz. How does the electric field strength $\vec{E}$ vary at location $x = 0$ as a function of time?
d) At which locations along the x-axis does the amplitude of the electric field strength $\vec{E}(x)$ reach its maximum in subtask c)?

Problem 2 (4 Points)

a) An X-ray has the wavelength $\lambda = 5.5 \cdot 10^{-12}$ m. What is the energy of the photons of the X-ray beam?
b) The X-ray is perpendicular to a double slit. The distance between the slits is $d = 2$ µm from center to center. At what angles to the central main maximum are the maxima of intensity observed?
c) The diffracted photons are collected on a fluorescence screen set up perpendicular to the beam at a distance of $\ell = 50$ cm from the double slit. What is the distance between the interference maxima on the screen?
d) Sketch the intensity on the screen as a function of the path difference $\Delta s = d \sin \alpha$ if each slit has a width of $b = 0.5$ µm.

Hint: Use the approximation of small angles, i.e. $\sin \alpha \approx \tan \alpha \approx \alpha$.

Problem 3 (4 Points)

a) Write down the formula of the de Broglie wavelength.
b) Electrons are accelerated from rest by a voltage U. The resulting electron beam has a wavelength of $\lambda = 10^{-11}$ m. Calculate the acceleratiom voltage U. Hint: To answer this question, calculate the momentum of the electrons and give reasons whether or not the acceleration voltage can be calculated using the formulas of classical mechanics.
c) Write down the relativistic energy-momentum relationship for a particle that moves freely, and give the equation for the kinetic energy of a relativistic particle.
d) Calculate the acceleration voltage if the wavelength of the electron beam is $\lambda = 10^{-17}$ m.

Problem 4 (4 Points)

a) State the three Bohr postulates.
b) Derive the radius r_n of the nth Bohr orbit and the velocity v_n of an electron on this orbit. Put numbers in the formulas and calculate the numerical values of r_n and v_n as well.
c) Calculate the total energy of an electron on the nth Bohr orbit.

Problem 5 (4 Points)

a) Sketch the energy level scheme of the hydrogen atom for the principal quantum numbers $n = 1, 2,$ and 3 resulting from the Schrödinger equation when relativistic effects are not considered. For each energy level give the energy, the principal quantum number n, the angular momentum quantum numbers ℓ and m, and the spectroscopic notation.

b) The atomic number of sodium is $Z = 11$. Give the electron configuration in the ground state for the neutral sodium atom.

c) The sodium D line has a wavelength of about 589 nm and can be assigned to the transition from the 1st excited state to the ground state. Explain why the sodium D line is split into two components and write down the transitions using the spectroscopic notation of atomic states.

Problem 6 (4 Points)

a) Write down the equation for the angular momentum in Newtonian mechanics and the corresponding angular momentum operator in quantum physics.

b) What are the eigenvalue equations for the orbital angular momentum? Which values can the quantum numbers of the orbital angular momentum have?

c) Explain why in quantum physics there is only an eigenvalue equation for one component of the angular momentum operator.

d) Write down the spin operator of the electron and give the quantum numbers that describe the spin of the electron.

Problem 7 (4 Points)

a) Both orbital angular momentum and spin are associated with a magnetic moment. Write down the equations for the magnetic moments of orbital angular momentum $\vec{\mu}_L$ and spin $\vec{\mu}_S$.

b) Spin and orbital angular momentum add up to the total angular momentum $\vec{J} = \vec{S} + \vec{L}$. The magnetic moments add up to the total magnetic moment $\vec{\mu} = \vec{\mu}_L + \vec{\mu}_S$. The effective magnetic moment results from the projection of $\vec{\mu}$ onto the direction of $\vec{J}$, i.e. $\mu_{\text{eff}} = \vec{\mu} \cdot \vec{J}/|\vec{J}|$. Calculate μ_{eff}.

c) Express the effective magnetic moment $\vec{\mu}_{\text{eff}} = \mu_{\text{eff}}\vec{J}/|\vec{J}|$ in terms of the eigenvalues of $\vec{J}^{\,2}$, $\vec{L}^{\,2}$, and $\vec{S}^{\,2}$.

Problem 8 (4 Points)

Electrons are trapped in an infinitely deep cube-shaped potential well. The potential energy of the electrons inside the cube is zero and infinite outside. Standing electron waves form within the cube. The standing electron waves are described by the wave functions $\psi(\vec{r}, t) = \psi_0 \sin(k_x x) \sin(k_y y) \sin(k_z z) \cdot \exp(-iEt/\hbar)$.

a) Give the numerical values which the wave numbers k_x, k_y, and k_z can assume if the edge length of the potential well is $L = 1$ mm?
b) Calculate the energy of the electron waves when only the kinetic energy of the electrons needs to be considered in the Schrödinger equation.
c) What does the Pauli principle say?
d) What is the highest energy the electrons can have in the ground state if there are $N = 10^{20}$ electrons in the potential well?

Required physical constants:

speed of light in vacuum:	$c = 3 \cdot 10^{8}$ m/s
Planck's constant:	$h = 4.14 \cdot 10^{-15}$ eVs $= 6.62 \cdot 10^{-34}$ Ws^2
reduced Planck's constant:	$\hbar = h/2\pi = 0.66 \cdot 10^{-15}$ eVs $= 1.05 \cdot 10^{-34}$ Ws^2
elementary charge:	$e = 1.6 \cdot 10^{-19}$ As
rest mass of the electron:	$m_e = 511$ keV$/c^2 = 9.1 \cdot 10^{-31}$ kg
electric field constant:	$\varepsilon_0 = 8.86 \cdot 10^{-12}$As/Vm

Problem 1 (4 Points)

a) Write down the wave function of a harmonic wave and explain the quantities involved.
b) Two harmonic electromagnetic waves with amplitude $\vec{E}_0$ are propagating in the same direction along the x-axis with frequency $\nu_1 = 101$ MHz and $\nu_2 = 99$ MHz. At what times does the electric field strength $\vec{E}$ at location $x = 0$ become zero?
c) The two waves now propagate in opposite directions along the x-axis in vacuum with a frequency of 100 MHz. How does the electric field strength $\vec{E}$ vary at location $x = 0$ as a function of time?
d) At which locations along the x-axis does the amplitude of the electric field strength $\vec{E}(x)$ reach its maximum in subtask c)?

a) harmonic wave

$$\psi(x, t) = \psi_0 \exp\{i(kx - \omega t)\}$$

wave number

$$k = \frac{2\pi}{\lambda}$$

λ denotes the wavelength and ω the circular frequency

$$\omega = \frac{2\pi}{T}$$

T denotes the period of the oscillation. The frequency is $\nu = 1/T$.

(1 point)

b) superposition of the waves

$$\vec{E}(x, t) = \vec{E}_0 \exp\{i(k_1 x - \omega_1 t)\} + \vec{E}_0 \exp\{i(k_2 x - \omega_2 t)\}$$

at $x = 0$

$$\begin{aligned}\vec{E}(x = 0, t) =& \vec{E}_0 \exp\{-i\omega_1 t\} + \vec{E}_0 \exp\{-i\omega_2 t\}\\ =& \vec{E}_0 \exp\{-i(\omega_1 + \omega_2)t/2\}(\exp\{-i(\omega_1 - \omega_2)t/2\} + \exp\{+i(\omega_1 - \omega_2)t/2\})\\ =& 2\vec{E}_0 \exp\{-i(\omega_1 + \omega_2)t/2\} \cos(\omega_1 - \omega_2)t/2.\end{aligned}$$

The electric field strength vanishes if

$$\frac{(\omega_1 - \omega_2)t_n}{2} = \frac{\pi}{2}(2n-1) = \frac{2\pi(\nu_1 - \nu_2)t_n}{2} \quad \rightarrow \quad t_n = \frac{2n - 1}{2(\nu_1 - \nu_2)} = \frac{2n - 1}{4} \cdot 10^{-6}\,\text{s},$$

and $n = 1, 2$ etc., i.e. $t_1 = 0.25\,\mu\text{s}$, $t_2 = 0.75\,\mu\text{s}$, $t_3 = 1.25\,\mu\text{s}$ ...

(1 point)

c) now $k = k_1 = -k_2$ and $\omega = \omega_1 = \omega_2$ and the superposition of the waves is

$$\vec{E}(x, t) = \vec{E}_0 \exp\{i(kx - \omega t)\} + \vec{E}_0 \exp\{i(-kx - \omega t)\}$$

at $x = 0$ is $\vec{E}(x = 0, t) = 2\vec{E}_0 \exp\{-i\omega t\}$ (1 point)

d) with

$$\begin{aligned}\vec{E}(x,t) &= \vec{E}_0 \exp\{i(kx-\omega t)\} + \vec{E}_0 \exp\{i(-kx-\omega t)\} \\ &= \vec{E}_0 \exp\{-i\omega t\}(\exp\{ikx\} + \exp\{-ikx\}) \\ &= 2\vec{E}_0 \exp\{-i\omega t\}\cos(kx)\end{aligned}$$

is the amplitude of the electric field strength for

$$kx_n = n\pi \quad \text{und} \quad n = 0, 1, \ 2 \text{ etc.}$$

maximal, i.e.

$$x_n = \frac{n\lambda}{2} = n\frac{c}{2\nu} = n\frac{3\cdot 10^8\,\mathrm{ms^{-1}}}{2\cdot 100\cdot 10^6\,\mathrm{s^{-1}}} = n\cdot 1.5\,\mathrm{m}$$

(1 point)

Problem 2 (4 Points)

a) An X-ray has the wavelength $\lambda = 5.5 \cdot 10^{-12}$ m. What is the energy of the photons of the X-ray beam?

b) The X-ray is perpendicular to a double slit. The distance between the slits is $d = 2$ µm from center to center. At what angles to the central main maximum are the maxima of intensity observed?

c) The diffracted photons are collected on a fluorescence screen set up perpendicular to the beam at a distance of $\ell = 50$ cm from the double slit. What is the distance between the interference maxima on the screen?

d) Sketch the intensity on the screen as a function of the path difference $\Delta s = d \sin\alpha$ if each slit has a width of $b = 0.5$ µm.

Hint: Use the approximation of small angles, i.e. $\sin\alpha \approx \tan\alpha \approx \alpha$.

a) energy of the photons

$$E = h\nu = \frac{hc}{\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}\, 3 \cdot 10^{8}\,\text{ms}^{-1}}{5.5 \cdot 10^{-12}\,\text{m}} = 2.26 \cdot 10^{5}\,\text{eV} = 226\,\text{keV}$$

(1 point)

b) Condition for the maxima of the intensity at the double slit

$$n\lambda = d\sin\alpha_n \quad \rightarrow \quad \sin\alpha_n \approx \alpha_n = n \cdot \frac{5.5 \cdot 10^{-12}\,\text{m}}{2 \cdot 10^{-6}\,\text{m}} = 2.75 \cdot 10^{-6}$$

(1 point)

c) With the position of the maxima Δ_n on the screen

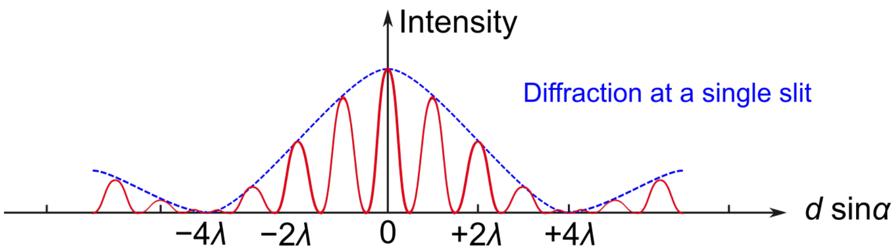
$$\tan\alpha_n \approx \alpha_n = \frac{\Delta_n}{\ell}$$

Distance of the maxima on the screen $\delta = \Delta_{n+1} - \Delta_n$

$$\delta = \Delta_{n+1} - \Delta_n = \ell(\alpha_{n+1} - \alpha_n) = 0{,}5\,\text{m} \cdot 2{,}75 \cdot 10^{-6} = 1{,}375\,\mu\text{m}.$$

(1 point)

d) With $d = 4 \cdot b$ every fourth maximum of the double slit meets a minimum of the single slit



(1 point)

Problem 3 (4 Points)

a) Write down the formula of the de Broglie wavelength.
b) Electrons are accelerated from rest by a voltage U. The resulting electron beam has a wavelength of $\lambda = 10^{-11}$ m. Calculate the acceleratiom voltage U. Hint: To answer this question, calculate the momentum of the electrons and give reasons whether or not the acceleration voltage can be calculated using the formulas of classical mechanics.
c) Write down the relativistic energy-momentum relationship for a particle that moves freely, and give the equation for the kinetic energy of a relativistic particle.
d) Calculate the acceleration voltage if the wavelength of the electron beam is $\lambda = 10^{-17}$ m.

a) The de Broglie-wavelength is

$$\lambda = \frac{h}{p}$$

thereby p denotes the momentum of the particles

(1 point)

b) with a wavelength of $\lambda = 10^{-11}$ m is the momentum

$$p = \frac{h}{\lambda} = \frac{4.14 \cdot 10^{-15}\,\mathrm{eVs}}{10^{-11}\,\mathrm{m}} = 4.14 \cdot 10^{-4}\,\mathrm{eVsm}^{-1}$$

according to Newtonian mechanics is the kinetic energy

$$E_{\mathrm{kin}} = \frac{p^2}{2m_0} = \frac{p^2c^2}{2m_0c^2}$$

The rest mass or energy is in the denominator and one gets

$$E_{\mathrm{kin}} = \frac{(4.14 \cdot 10^{-4}\,\mathrm{eVsm}^{-1}\,3 \cdot 10^{8}\,\mathrm{ms}^{-1})^2}{2 \cdot 511 \cdot 10^{3}\,\mathrm{eV}} = 15.1\,\mathrm{keV} << m_0c^2$$

Since the kinetic energy is much smaller than the rest energy of the electron, it can be calculated classically and the acceleration voltage is

$$eU = E_{\mathrm{kin}} \quad \rightarrow \quad U = \frac{E_{\mathrm{kin}}}{e} = 15.1\,\mathrm{kV}$$

(1 point)

c) The relativistic energy-momentum relation of a free particle is

$$E^2 - c^2p^2 = m_0^2c^4$$

and its kinetic energy is

$$E_{\mathrm{kin}} = E - m_0c^2$$

(1 point)

d) The momentum is with the wavelength of $\lambda = 10^{-17}$ m

$$p = \frac{h}{\lambda} = \frac{4.14 \cdot 10^{-15}\,\text{eVs}}{10^{-17}\,\text{m}} = 414\,\text{eVsm}^{-1}$$

and

$$cp = 414\,\text{eVsm}^{-1}\,3 \cdot 10^{8}\,\text{ms}^{-1} = 1242 \cdot 10^{8}\,\text{eV} >> 511\,\text{keV}$$

i.e. the rest energy of the electron can be safely ignored so that

$$E_{\text{kin}} = E = cp$$

The acceleration voltage is then

$$U = \frac{E}{e} = \frac{cp}{e} = 124\,\text{GV}$$

(1 point)

Problem 4 (4 Points)

a) State the three Bohr postulates.
b) Derive the radius r_n of the nth Bohr orbit and the velocity v_n of an electron on this orbit. Put numbers in the formulas and calculate the numerical values of r_n and v_n as well.
c) Calculate the total energy of an electron on the nth Bohr orbit.

a) Bohr's postulates are
   1. The electrons move in circular orbits around the nucleus
   2. The orbital angular momentum of the electrons is quantized according to the formula $L = \hbar n$
   3. A photon of energy $|E_n - E_m|$ is only emitted or absorbed at the transition from the n-th to the m-th Bohr orbit

(1 point)

b) The radius of Bohr's orbit follows from the force balance of Coulomb force and centrifugal force with the 2nd postulate (i.e. $m_0 v r = \hbar n$).

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = \frac{m_0 v^2}{r} \quad \rightarrow \quad r = 4\pi\varepsilon_0 \frac{m_0 v^2 r^2}{e^2} = 4\pi\varepsilon_0 \frac{m_0^2 v^2 r^2}{m_0 e^2}$$

and

$$\begin{aligned} r_n =& 4\pi\,\varepsilon_0 \frac{\hbar^2}{m_0 e^2} n^2 \\ =& 4\pi\, 8.86 \cdot 10^{-12} \mathrm{As/Vm} \frac{(0.66 \cdot 10^{-15}\,\mathrm{eVs})^2}{9.1 \cdot 10^{-31}\,\mathrm{kg}\, e^2} n^2 \\ =& 5.33 \cdot 10^{-11}\,\mathrm{m}\, n^2 \end{aligned}$$

(1.5 point)

The speed is

$$v_n = \frac{\hbar n}{m_0 r_n} = \frac{\hbar c^2}{m_0 c^2 r_n} n = \frac{0.66 \cdot 10^{-15}\,\mathrm{eVs}\,(3 \cdot 10^8\,\mathrm{ms}^{-1})^2}{511 \cdot 10^3\,\mathrm{eV}\, 5.33 \cdot 10^{-11}\,\mathrm{m}} \frac{1}{n} = 2.1 \cdot 10^6\,\mathrm{ms}^{-1} \frac{1}{n}$$

(0.5 point)

c) The total energy is

$$E = E_{\mathrm{kin}} + E_{\mathrm{pot}}$$

The potential energy of an electron in the field of the elementary charge of the proton is

$$E_{\mathrm{pot}} = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

From the force balance of Coulomb force and centrifugal force follows

$$2E_{\mathrm{kin}} = -E_{\mathrm{pot}}$$

i.e.

$$E = \frac{1}{2} E_{\text{pot}}$$

and

$$\begin{aligned} E_n &= -\frac{1}{8\pi\varepsilon_0}\frac{e^2}{r_n} \\ &= -\frac{1}{8\pi\, 8.86\cdot 10^{-12}\,\text{AsV}^{-1}\text{m}^{-1}}\frac{1.6\cdot 10^{-19}\,\text{As}\cdot e}{5.33\cdot 10^{-11}\,\text{m}}\frac{1}{n^2} \\ &= -13.5\,\text{eV}\frac{1}{n^2} \end{aligned}$$

(1 point)

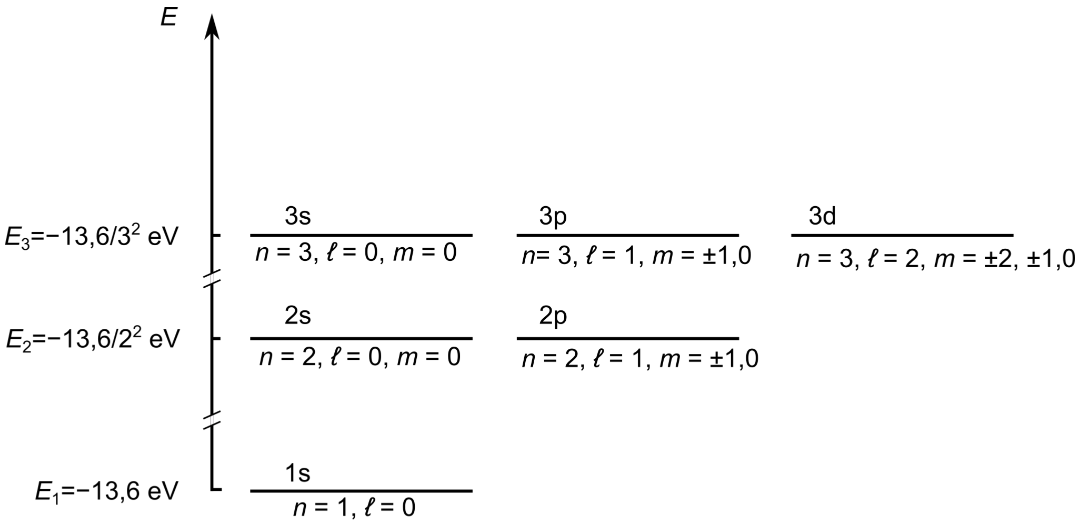
Problem 5 (4 Points)

a) Sketch the energy level scheme of the hydrogen atom for the principal quantum numbers $n = 1, 2,$ and 3 resulting from the Schrödinger equation when relativistic effects are not considered. For each energy level give the energy, the principal quantum number n, the angular momentum quantum numbers ℓ and m, and the spectroscopic notation.

b) The atomic number of sodium is $Z = 11$. Give the electron configuration in the ground state for the neutral sodium atom.

c) The sodium D line has a wavelength of about 589 nm and can be assigned to the transition from the 1st excited state to the ground state. Explain why the sodium D line is split into two components and write down the transitions using the spectroscopic notation of atomic states.

a) Energy level scheme of the hydrogen atom without relativistic effects



(2 points)

(1/2 wenn alle Energieniveaus eingezeichnet sind, 1/2 wenn die Energie angeben ist, 1/2 wenn die Quantenzahlen angegeben sind und 1/2 für die spektroskopische Notation)

b) The electron configuration in the ground state is $[1s^2, 2s^2, 2p^6, 3s^1]$.

(0.5 point)

c) The excited energy level is 3p. Due to spin-orbit coupling, it splits into a level with total angular momentum $j = 1/2$ and a level with total angular momentum $j = 3/2$. (Die Erwähnung der Spin-Bahn-Kopplung genügt)

(0.5 point)

The transitions are

$$3p_{3/2} \to 3s_{1/2}$$
$$3p_{1/2} \to 3s_{1/2}$$

(1 point)

Problem 6 (4 Points)

a) Write down the equation for the angular momentum in Newtonian mechanics and the corresponding angular momentum operator in quantum physics.
b) What are the eigenvalue equations for the orbital angular momentum? Which values can the quantum numbers of the orbital angular momentum have?
c) Explain why in quantum physics there is only an eigenvalue equation for one component of the angular momentum operator.
d) Write down the spin operator of the electron and give the quantum numbers that describe the spin of the electron.

a) The angular momentum $\vec{L}$ in Newtonian mechanics is

$$\vec{L} = \vec{r} \times \vec{p}$$

Here, $\vec{r}$ denotes the position vector of the particle and $\vec{p}$ its momentum.

(1/2 point)

The angular momentum operator results when the momentum is replaced by the momentum operator, i.e. $\vec{p} \to -\mathrm{i}\hbar\nabla$.

$$\hat{\vec{L}} = -\mathrm{i}\hbar\vec{r} \times \nabla$$

(1/2 point)

b) The eigenvalue equations for orbital angular momentum are

$$\begin{aligned} \hat{\vec{L}}^2 y_{\ell,m} &= \ell(\ell + 1)\hbar^2 y_{\ell,m} \\ \hat{\vec{L}}_z y_{\ell,m} &= m\hbar y_{\ell,m} \end{aligned}$$

(1/2 point)

The ranges for the quantum numbers are $\ell = 0,\ 1,\ 2, ...$ and $m = 0,\ \pm 1,\ \pm 2, ...$ with $|m| \leq \ell$.

(1/2 point)

c) Due to the uncertainty relation $\Delta L_z \Delta\varphi \geq \hbar/2$ the angle φ is undetermined if the z-component of L is assigned a fixed value. φ determines the components L_x and L_y, i.e. the projection of $\vec{L}$ onto the x- or y-axis. If φ cannot be determined, then the L_x and L_y components can no longer be determined either.

(1 point)

d) The spin operator of the electron is

$$\hat{\vec{S}} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

$\sigma_{x,\,y,\,z}$ denote the Paul matrices

(1/2 point)

zur Vollständigkeit aber nicht gefordert

$$\sigma_\mathrm{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\mathrm{y} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_\mathrm{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The quantum numbers of the electron spin are $s = 1/2$ and $m_\mathrm{s} = \pm 1/2$.
(1/2 point)

Problem 7 (4 Points)

a) Both orbital angular momentum and spin are associated with a magnetic moment. Write down the equations for the magnetic moments of orbital angular momentum $\vec{\mu}_L$ and spin $\vec{\mu}_S$.

b) Spin and orbital angular momentum add up to the total angular momentum $\vec{J} = \vec{S} + \vec{L}$. The magnetic moments add up to the total magnetic moment $\vec{\mu} = \vec{\mu}_L + \vec{\mu}_S$. The effective magnetic moment results from the projection of $\vec{\mu}$ onto the direction of $\vec{J}$, i.e. $\mu_{\text{eff}} = \vec{\mu} \cdot \vec{J}/|\vec{J}|$. Calculate μ_{eff}.

c) Express the effective magnetic moment $\vec{\mu}_{\text{eff}} = \mu_{\text{eff}}\vec{J}/|\vec{J}|$ in terms of the eigenvalues of $\vec{J}^2$, $\vec{L}^2$, and $\vec{S}^2$.

a) The magnetic moment of the orbital angular momentum

$$\vec{\mu}_L = -\mu_{\text{B}}\frac{\vec{L}}{\hbar}$$

(1/2 point)

The magnetic moment of the electron spin

$$\vec{\mu}_S = -g\mu_{\text{B}}\frac{\vec{S}}{\hbar}$$

with $g = 2$ in a very good approximation

(1/2 point)

b) The projection of the magnetic moment onto the direction of $\vec{J}$ is

$$\mu_{\text{eff}} = \vec{\mu} \cdot \vec{J}/|\vec{J}| = -\frac{\mu_{\text{B}}}{\hbar}\frac{(\vec{L}+2\vec{S})(\vec{L}+\vec{S})}{|\vec{J}|} = -\frac{\mu_{\text{B}}}{\hbar}\frac{(\vec{L}^2 + 3\vec{S}\vec{L} + 2\vec{S}^2)}{|\vec{J}|}$$

(1 point)

c) The effective magnetic moment is given by

$$\vec{\mu}_{\text{eff}} = \mu_{\text{eff}}\frac{\vec{J}}{|\vec{J}|} = -\frac{\mu_{\text{B}}}{\hbar}\frac{(\vec{L}^2 + 3\vec{S}\vec{L} + 2\vec{S}^2)}{|\vec{J}|}\frac{\vec{J}}{|\vec{J}|} = -\mu_{\text{B}}\frac{(\vec{L}^2 + 3\vec{S}\vec{L} + 2\vec{S}^2)}{\vec{J}^2}\frac{\vec{J}}{\hbar}$$

with

$$\vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + 2\vec{S}\vec{L} + \vec{S}^2 \quad \rightarrow \quad \vec{S}\vec{L} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$$

(1/2 point)

one gets

$$\vec{\mu}_{\text{eff}} = -\mu_{\text{B}} \frac{(\vec{L}^2 + 3\vec{S}\vec{L} + 2\vec{S}^2)}{\vec{J}^2} \frac{\vec{J}}{\hbar} = -\mu_{\text{B}} \frac{(\frac{3}{2}\vec{J}^2 - \frac{1}{2}\vec{L}^2 + \frac{1}{2}\vec{S}^2)}{\vec{J}^2} \frac{\vec{J}}{\hbar}$$

(1/2 point)

an with the eigenvalues of $\vec{J}^2$: $\hbar^2 J(J+1)$, $\vec{L}^2$: $\hbar^2 L(L+1)$ and $\vec{S}^2$: $\hbar^2 S(S+1)$ results

$$\vec{\mu}_{\text{eff}} = -\mu_{\text{B}} \frac{(3J(J+1) - L(L+1) + S(S+1))}{2J(J+1)} \frac{\vec{J}}{\hbar}$$

(1 point)

Problem 8 (4 Points)

Electrons are trapped in an infinitely deep cube-shaped potential well. The potential energy of the electrons inside the cube is zero and infinite outside. Standing electron waves form within the cube. The standing electron waves are described by the wave functions $\psi(\vec{r},t) = \psi_0 \sin(k_x x)\sin(k_y y)\sin(k_z z)\cdot\exp(-\mathrm{i}Et/\hbar)$.

a) Give the numerical values which the wave numbers k_x, k_y, and k_z can assume if the edge length of the potential well is $L = 1$ mm?
b) Calculate the energy of the electron waves when only the kinetic energy of the electrons needs to be considered in the Schrödinger equation.
c) What does the Pauli principle say?
d) What is the highest energy the electrons can have in the ground state if there are $N = 10^{20}$ electrons in the potential well?

a) Since the wave functions on the faces of the cube have the value zero, the wave numbers must apply

$$k_x L = n_x \pi, \quad k_y L = n_y \pi, \quad \text{and} \quad k_z L = n_z \pi$$

n_x, n_y and n_z are integers, i.e. 1, 2, 3, etc.

(0.5 point)

$$\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \frac{\pi}{L}\begin{pmatrix} n_x \\ n_y \\ n_x \end{pmatrix} = 1000\,\pi \begin{pmatrix} n_x \\ n_y \\ n_x \end{pmatrix} \text{ m}^{-1}$$

(0.5 point)

b) The energy of the electron waves results from the Schrödinger equation

$$\begin{aligned} E_{n_x,n_y,n_z}\psi_{n_x,n_y,n_z}(\vec{r},t) &= -\frac{\hbar^2\nabla^2}{2m_0}\psi_{n_x,n_y,n_z}(\vec{r},t) \\ &= -\frac{\hbar^2\nabla^2}{2m_0}\psi_0 \sin(k_x x)\sin(k_y y)\sin(k_z z)\exp(-\mathrm{i}Et/\hbar) \\ &= \frac{\hbar^2(k_x^2+k_y^2+k_z^2)}{2m_0}\psi_{n_x,n_y,n_z}(\vec{r},t) \end{aligned}$$

(0.5 point)

and

$$\begin{aligned} E_{n_x,n_y,n_z} &= \frac{\hbar^2\,\pi^2\,10^6\,\text{m}^{-2}}{2m_0}(n_x^2+n_y^2+n_z^2) = \frac{h^2\,10^6\,\text{m}^{-2}\,c^2}{8m_0c^2}(n_x^2+n_y^2+n_z^2) \\ &= \frac{(4.14\cdot 10^{-15}\,\text{eVs})^2\,10^6\,\text{m}^{-2}(3\cdot 10^8\,\text{ms}^{-1})^2}{8\cdot 511\cdot 10^3\,\text{eV}}(n_x^2+n_y^2+n_z^2) \\ &= 3.8\cdot 10^{-13}\,\text{eV}(n_x^2+n_y^2+n_z^2) \end{aligned}$$

(0.5 point)

c) The Pauli principle states that two electrons in a quantum system cannot have the same quantum number.

(1 point)

d) In the wave number space, in the 1st octant (i.e. positive k-values), k-states can be occupied up to a highest energy in the ground state of the quantum system. According to the Pauli principle, each k-state can be occupied by 2 electrons. The absolute value of the largest occupied k-states is called the Fermi wave number k_F. The volume of a k-state is $(\pi/L)^3$ for standing waves. With the volume of a sphere in wave number space $4\pi k^3/3$ results

$$N = 2\frac{4\pi k_F^3/3}{8(\pi/L)^3} \quad \rightarrow \quad k_F^3 = 3\pi^2 \frac{N}{L^3} = 3\pi^2\, 10^{20} \cdot 10^9\,\mathrm{m}^{-3}$$

and

$$k_F = 1.44 \cdot 10^{10}\,\mathrm{m}^{-1}$$

(0.5 point)

The highest energy in the ground state at $T = 0$, i.e. Fermi energy is

$$E_F = \frac{\hbar^2 k_F^2 c^2}{2m_0 c^2} = \frac{(0.66 \cdot 10^{-15}\,\mathrm{eVs})^2\,(1.44 \cdot 10^{10}\,\mathrm{m}^{-1})^2 \cdot (3 \cdot 10^8\,\mathrm{ms}^{-1})^2}{2 \cdot 511 \cdot 10^3\,\mathrm{eV}}$$
$$= 7.95\,\mathrm{eV}$$

(0.5 point)