Modern Physics WS 21/22 Sören Bieling

Problem Sheet 6

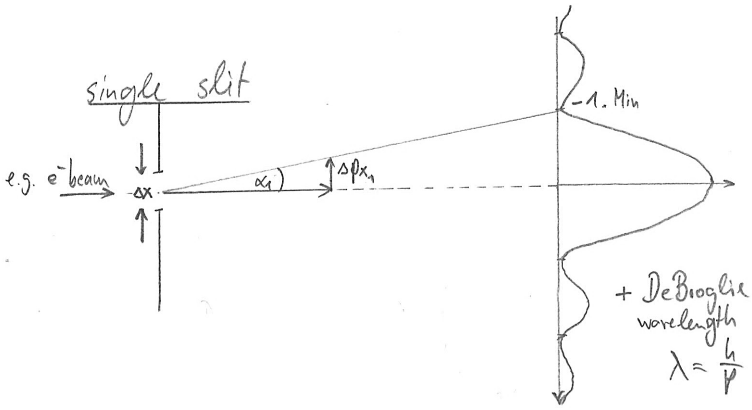
Ex. 6.1: Heisenberg's uncertainty principle

a) $\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$ $\Delta p_x = m_e \Delta v_x$ $\Rightarrow$ $\Delta v_x \geq \frac{\hbar}{2 m_e \Delta x}$ $m_e = 9{,}11 \cdot 10^{-31}\,kg$, $\Delta x = 2\,Å$

$\rightarrow$ uncertainty (spread) not affected by realistic effects! $\geq 2{,}89 \cdot 10^5 \frac{m}{s} \ll c$

$\rightarrow$ independant of absolute velocity: $v_n = \alpha c \frac{Z}{n} \rightarrow \frac{1}{137} c$ also not relativistic! $\approx 2{,}2 \cdot 10^6 \frac{m}{s}$

$\alpha = \frac{e^2}{4\pi \varepsilon_0 \hbar c} \approx \frac{1}{137}$

b) 

+ DeBroglie wavelength $\lambda = \frac{h}{p}$

$\Delta x \cdot \sin \alpha_n = n \cdot \lambda$ $n = \pm 1, \pm 2, \ldots$

$\tan \alpha_n = \frac{\Delta p_x}{p}$

for width of main diffraction peak $n=1$

$\lambda = \Delta x \sin \alpha_n \approx \Delta x \frac{\Delta p_x}{p}$

$\sin \alpha \approx \alpha \approx \tan \alpha$

$\frac{h}{\cancel{p}} = \Delta x \cdot \Delta p_x \cdot \frac{1}{\cancel{p}}$

$\boxed{\Delta x \cdot \Delta p_x = h > \frac{\hbar}{2}}$ ✓

c) grain of sand ($m = 1\,mg$)

$\Delta x \approx \lambda \approx 500\,nm$ $\Delta v_x \geq \frac{\hbar}{2 m \Delta x} = 1{,}055 \cdot 10^{-22} \frac{m}{s}$

$\boxed{\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}}$

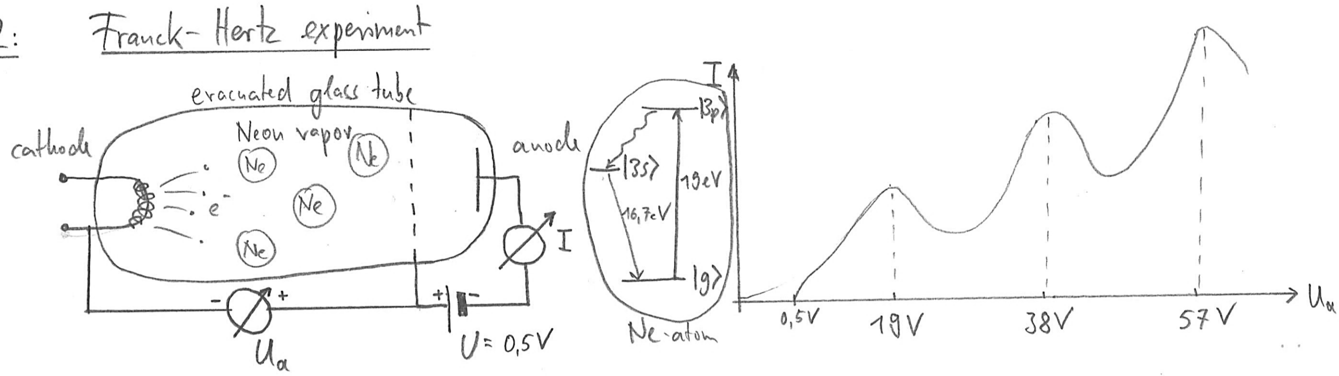
$\Delta x \cdot m \Delta v_x \geq \frac{\hbar}{2}$

$S_x = \Delta v_x \cdot t$ $S_x = 1\,\mu m$

$t = \frac{S_x}{\Delta v_x} = 9{,}5 \cdot 10^{15}\,s = 3 \cdot 10^9$ years

for comparison: age of the universe $= 13{,}8 \cdot 10^9$ years

Ex. 6.2: Franck-Hertz experiment



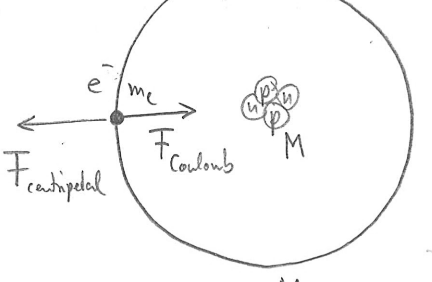
a) The energy of the emitted electrons increases with $E_{pot} = e \cdot U_a$
Every 19V in acceleration voltage they have accumulated enough potential $\rightarrow$ kinetic energy to excite a Neon atom a further time by collision. At that point they might not have enough energy to reach the anode $\rightarrow$ the anode current drops.

b) $E_{pot,e} = e \cdot U_0 = \Delta E_{Ne} = 19\,eV$ ($\hat{=}$ 65,25 nm)

c) $\lambda = 632{,}8\,nm$ $E = h \cdot \nu = \frac{hc}{\lambda} = 1{,}96\,eV$ ($\hat{=}$ $3p \rightarrow 3s$)

Ex. 6.3: Bohr's atomic model

He^+ ion: $Z=2$, $n \in \mathbb{N}$



$\mu = \frac{m_e M}{m_e + M} \approx m_e$

① $F_{centripetal} \overset{!}{=} F_{Coulomb}$

② circumference $\overset{!}{=}$ $n \cdot$ DeBroglie-λ

$$2\pi \cdot r = n \cdot \frac{h}{\mu v}$$

$$L = r\mu v = n \cdot \hbar \quad (\mathrm{I})$$

$$\frac{\mu v^2}{r} = \frac{Ze^2}{4\pi\varepsilon_0 r^2} \quad (\mathrm{II})$$

$\overset{(\mathrm{I})}{\rightarrow}$

$$\frac{L^2}{\mu r^{\cancel{3}}} = \frac{Ze^2}{4\pi\varepsilon_0 \cancel{r^2}}$$

$$r = \frac{4\pi\varepsilon_0}{Ze^2\mu} L^2 = \frac{\cancel{4\pi}\varepsilon_0}{Ze^2\mu} n^2 \frac{h^2}{\cancel{(2\pi)^2}}$$

$$r_n = \underbrace{\frac{\varepsilon_0 h^2}{\pi e^2 \mu}}_{=: a_0 = 0{,}529\,\text{Å}} \frac{n^2}{Z}$$

Bohr's radius

$$\boxed{\vec{F} = -q\vec{\nabla}\phi}$$

$$\boxed{E_{pot} = -\frac{e^2}{4\pi\varepsilon_0}\frac{Z}{r}}$$

$$\boxed{E_{kin} = \frac{1}{2} m_e v^2}$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$

$$(\mathrm{II}) \rightarrow -E_{pot} = 2E_{kin} \quad (\mathrm{III})$$

Total Energy: $E = E_{kin} + E_{pot}$

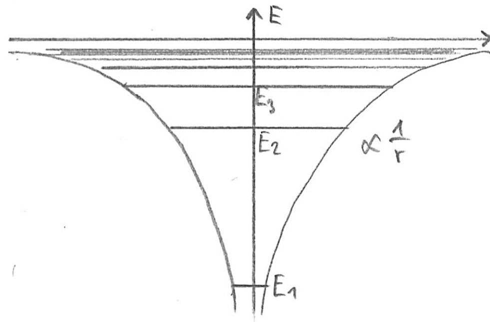
$$\overset{(\mathrm{III})}{=} -\frac{1}{2}E_{pot} + E_{pot} = \frac{1}{2}E_{pot} = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{\pi e^2 m_e}{\varepsilon_0 h^2}\frac{Z^2}{n^2}$$

$$= -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 \hbar c}\frac{e^2}{4\pi\varepsilon_0 \hbar c} m_e c^2 \frac{Z^2}{n} = -\frac{1}{2}\alpha^2 m_e c^2 \frac{Z^2}{n^2}$$

$$\boxed{E_n = -\underbrace{\frac{1}{2}\alpha^2 m_e c^2}_{\text{Rydberg const.} \; R_\infty = 13{,}6\,\text{eV} = 1\,\text{Ry}} \frac{Z^2}{n^2}}$$

$Z=2$

$n=1$	$E_1 = -54{,}4$ eV
$n=2$	$E_2 = -13{,}6$ eV
$n=3$	$E_3 = -6{,}04$ eV
$n=4$	$E_4 = -3{,}4$ eV
$n=5$	$E_5 = -2{,}18$ eV



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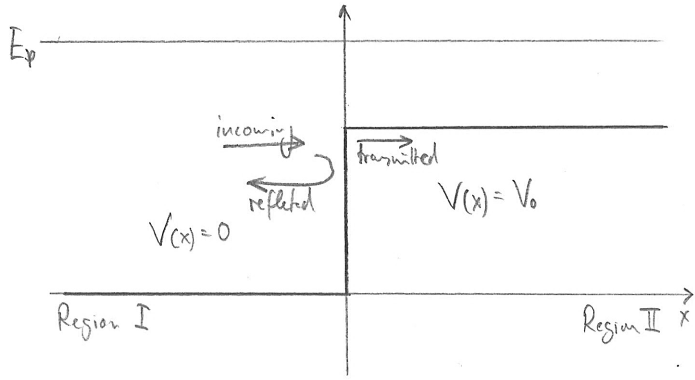
Ex. 6.4: Schrödinger eq.: potential step

$$\boxed{\begin{array}{lll} \text{SGL:} & i\hbar \frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle & \\ \text{with} & \hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V(x) & \hat{\vec{p}} = -i\hbar\vec{\nabla} \end{array}}$$



a) Region I: $V(x) = 0$

$$i\hbar \frac{\partial}{\partial t}\Psi_1 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_1$$

Region II: $V(x) = V_0$

$$i\hbar \frac{\partial}{\partial t}\Psi_2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_2 + V_0\Psi_0$$

$$E_p = \frac{p^2}{2m} + V(x) = \frac{\hbar^2 k^2}{2m} + V(x)$$

$\hookrightarrow \hbar k_1 = \sqrt{2mE_p}$

$\hookrightarrow \hbar k_2 = \sqrt{2m(E_p - V_0)}$

$\hbar\omega = E_p$

b)

1: $\Psi_1(x=0,t) \overset{!}{=} \Psi_2(x=0,t)$

2: $\frac{\partial}{\partial x}\Psi_1(x=0,t) \overset{!}{=} \frac{\partial}{\partial x}\Psi_2(x=0,t)$

I: incoming + reflected wave:

$$\Psi_1 = \Psi_0 e^{i(k_1 x - \omega t)} + r\Psi_0 e^{i(-k_1 x - \omega t)}$$

II: only transmitted wave:

$$\Psi_2 = t\Psi_0 e^{i(k_2 x - \omega t)}$$

$$\left|\begin{array}{l} 1: \quad 1 + r = t \\ 2: \quad k_1 - r k_1 = t k_2 \end{array}\right|$$

1 in 2: $k_1 - r k_1 = (1+r)k_2$

$$\boxed{r = \frac{k_1 - k_2}{k_1 + k_2} \qquad R = r^2}$$

$$t = 1 + r = \frac{2k_1}{k_1 + k_2}$$

c)

$$R \overset{!}{=} 0{,}5 = r^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

$$\frac{k_1 - k_2}{k_1 + k_2} = \frac{1}{\sqrt{2}} \iff \frac{k_2}{k_1} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\sqrt{2m(E_p - V_0)}}{\sqrt{2m E_p}}$$

$$\frac{k_2}{k_1} = \sqrt{1 - \frac{V_0}{E_p}}$$

$$\frac{k_2^2}{k_1^2} = 1 - \frac{V_0}{E_p}$$

$$\frac{V_0}{E_p} = 1 - \left(\frac{k_2}{k_1}\right)^2$$

$$\frac{E_p}{V_0} = \frac{1}{1 - \left(\frac{k_2}{k_1}\right)^2} = \frac{1}{1 - \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2} = \underline{\underline{1{,}03\ldots}}$$