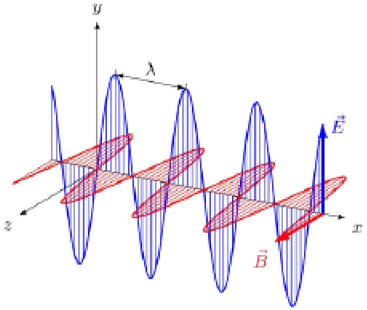
Light **a**mplification by **s**timulated **e**mission of **r**adiation

Amplification of a wave with a wave vector $\vec{k}$ and a small spectral width $\Delta\lambda$



spectral energy density

$$u(\lambda) = \frac{1}{V}\frac{\Delta E_\lambda}{\Delta\lambda} \quad \rightarrow \quad \boxed{u(k) = \frac{1}{V}\frac{\Delta E_k}{\Delta\lambda}} \propto n_k$$

adaptation of Einstein's equations for absorption and stimulated emission

$$\frac{dN_2^{abs}}{dt} = +\,u(k)B_{21}N_1 \quad \rightarrow \quad \frac{d{n_k}^{abs}}{dt} = -n_k W_{21} N_1$$

$$\frac{dN_2^{stim}}{dt} = -\,u(k)B_{21}N_2 \quad \rightarrow \quad \frac{d{n_k}^{stim}}{dt} = +n_k W_{21} N_2$$

$$\frac{dN_2^{spon}}{dt} = -\,A_{21}N_2$$

with $u(k) = \frac{\Delta E_\lambda}{V\Delta\lambda} = \frac{hc}{\lambda\Delta\lambda} n_k$

$$W_{21} = \frac{hcB_{21}}{\lambda\Delta\lambda}$$

Laser

$\rightarrow$ population inversion

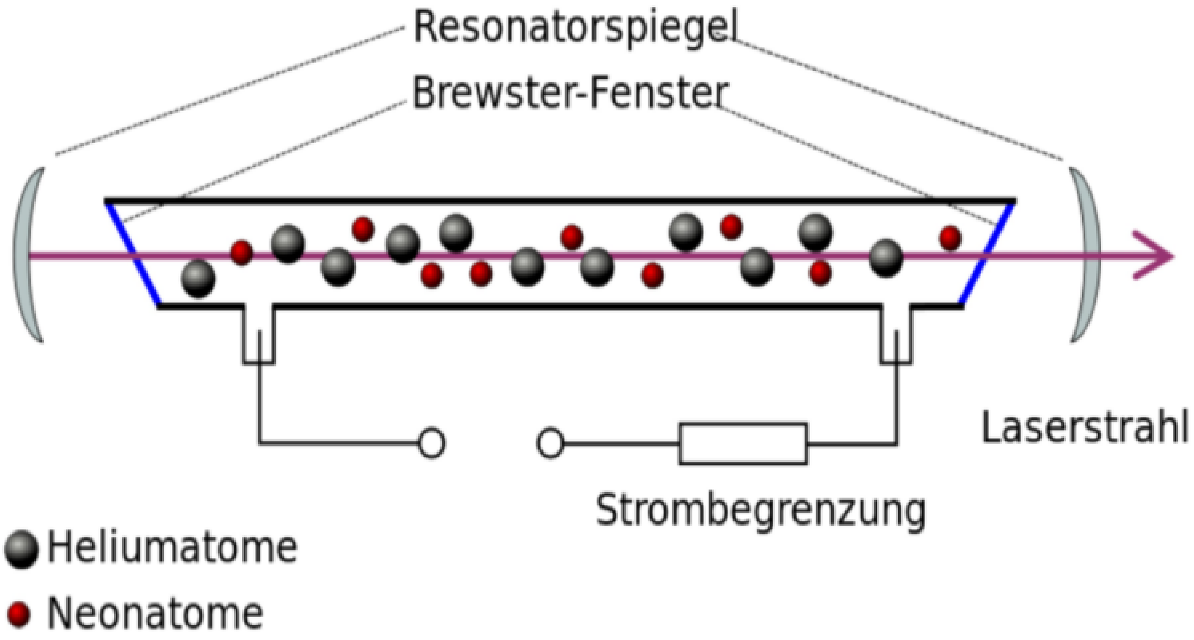
$$N_2 - N_1 > \frac{1}{W_{21}\tau}$$

with

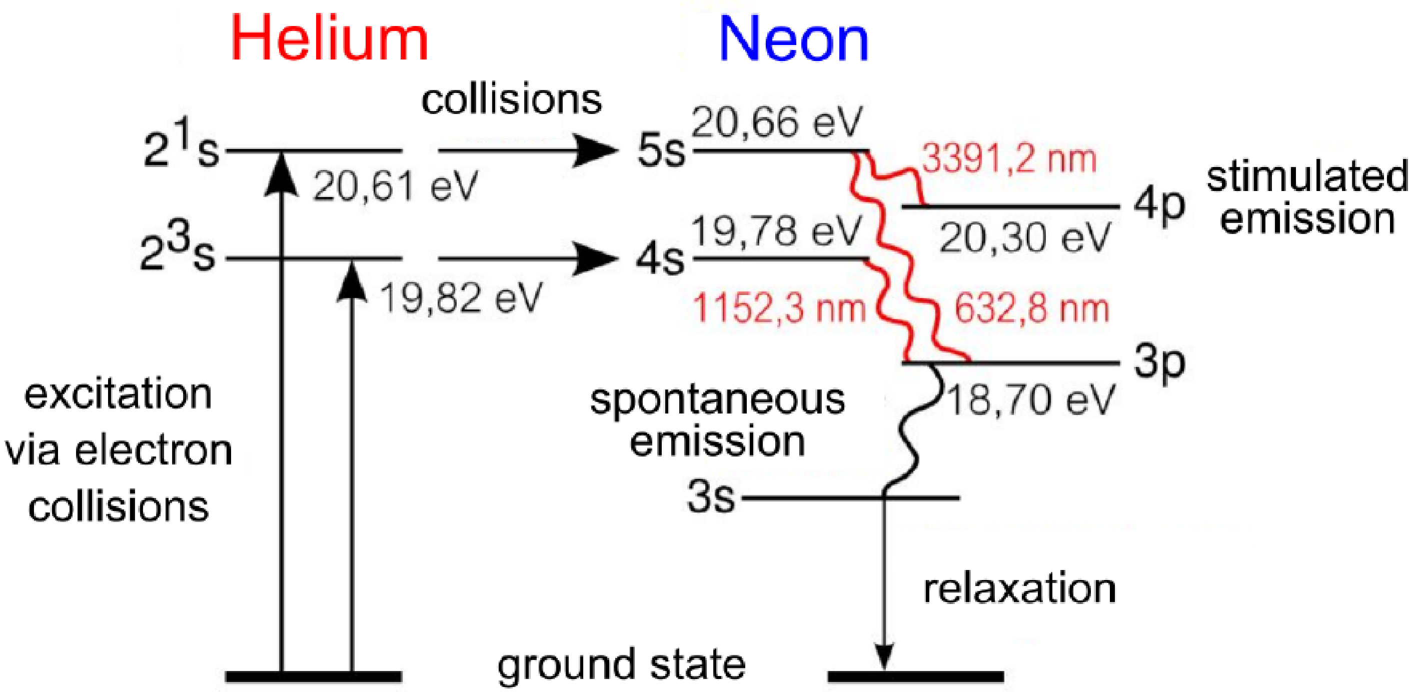
$$\frac{1}{W_{21}} = \frac{\lambda \Delta \lambda}{hcB_{21}}$$

- small spectral width $\Delta\lambda$
- the wavelength λ should not be to small due $B_{21} \propto \lambda^2$
- for a continuous laser $W_{21}(N_2 - N_1) = \tau^{-1}$

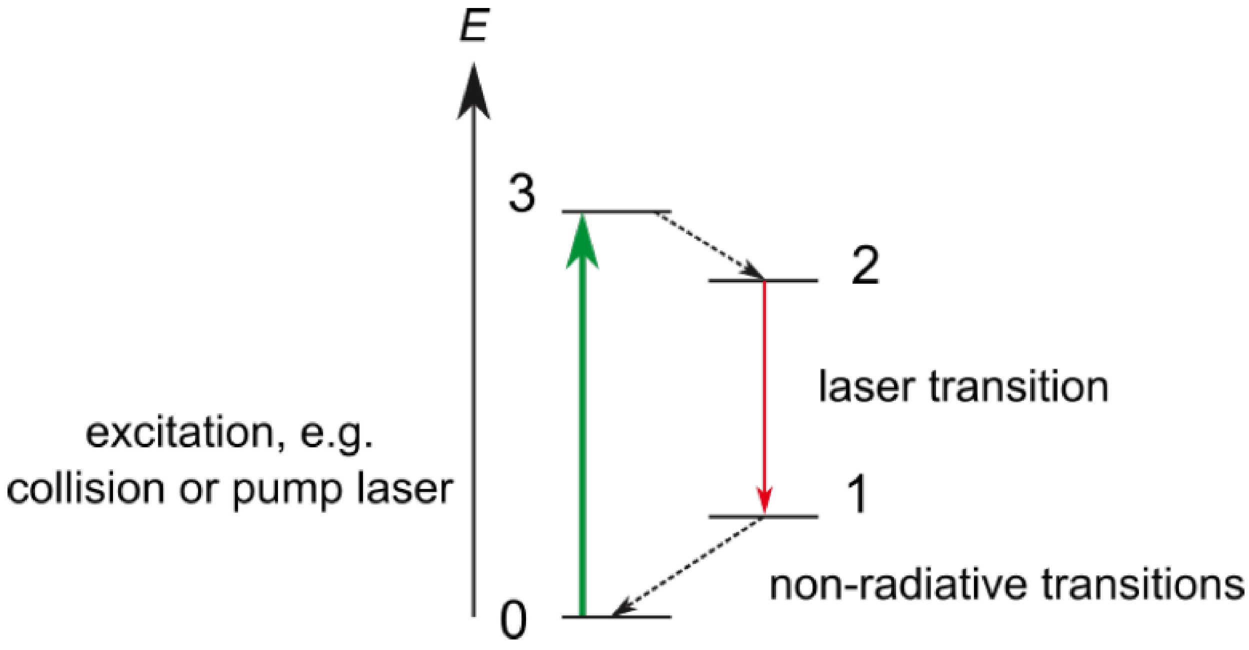
the basic set-up of a He-Ne laser



relevant energy levels of the He and Ne atoms



4 level scheme



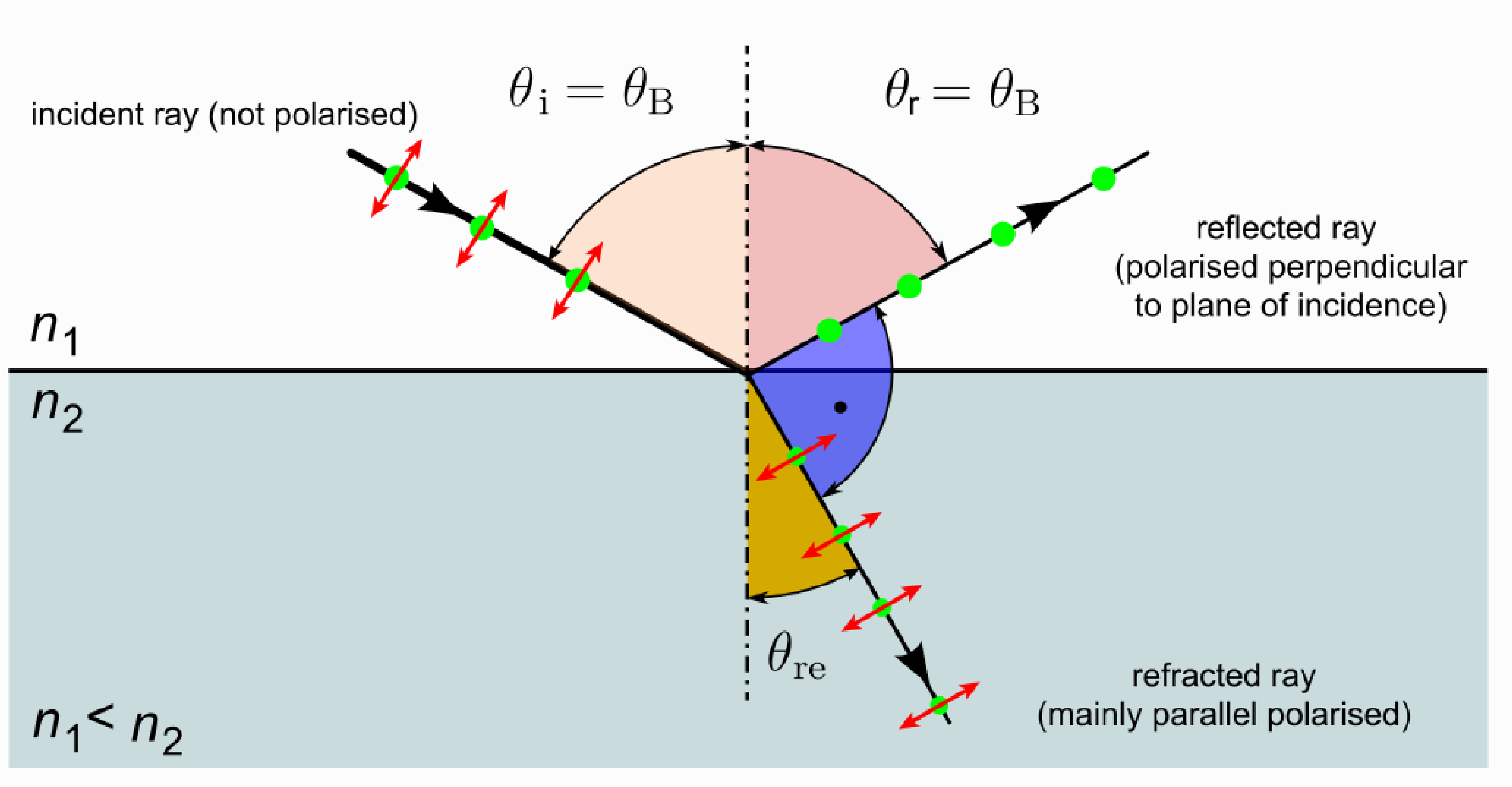
The video demonstrates the Brewster effect.

Only light which is polarised perpendicular to the direction of the beam and the normal of the glass plate is reflected, when the Brewster condition (Brewster angle) is fulfilled.

Only light which is polarised within the plane which is formed by the laser beam and the normal of the window can pass without loss.

The perpendicular component is strongly damped and not amplified by the laser process.

The suppression of one polarisation can be avoided when the mirrors form directly the end caps of the discharge tube.


incident ray (not polarised)
$\theta_i = \theta_B$
$\theta_r = \theta_B$
reflected ray
(polarised perpendicular
to plane of incidence)
n_1
n_2
θ_{re}
refracted ray
(mainly parallel polarised)
$n_1 < n_2$

The sketch illustrates the Brewster effect.

When the angle of incidence equals Brewster's angle only the polarisation perpendicular to the plane of incidence is reflected.

The heuristic explanation is based on Hertz's dipole.

The emitted radiation is zero along the dipole axis and maximal in the perpendicular direction.

The exact explanation is given by Fresnel's equation for the reflection and transmission of electromagnetic waves on interfaces.

Problem 1 (4 Points)

An antenna on board of a satellite emits one short pulse per second. The satellite is moving the velocity of 90 % relative to a receiver.

a) Calculate the frequency of pulses arriving at the receiver.
b) A second antenna is placed 2 m behind the first antenna in the direction of the motion of the satellite. Calculate the time interval between two pulses recorded by the receiver, when the pulses are emitted simultaneously by the two antennae on board of the satellite.
c) The mass of the satellite at rest is 1 kg. Calculate the energy which is a least necessary to accelerate the satellite from zero to 90 % of the velocity of light?
d) Calculate the time which is necessary for a power station with an output power of 1 GW to provide this amount of energy.

a) During the time $\Delta t = 1\,\mathrm{s}$ covers the satellite the distance

$$\Delta \ell = \Delta t \cdot 0.9\,\mathrm{c}$$

and the additional time for the next pulse in the frame of the satellite to reach the receiver is

$$\delta t = \frac{\Delta \ell}{c}.$$

The time between two pulses measured in the frame of the satellite is

$$\Delta t_0 = \Delta t + \delta t = \Delta t(1 + 0.9).$$

The time between two pulses measured by the receiver at rest is due to time dilation

$$\Delta t_R = \Delta t_0 \frac{1}{\sqrt{1 - 0.9^2}}$$

or

$$\Delta t_R = \Delta t \frac{(1 + 0.9)}{\sqrt{1 - 0.9^2}} = \Delta t \sqrt{\frac{1 + 0.9}{1 - 0.9}}.$$

The frequency $\nu_R = 1/\Delta t_R$ is

$$\nu_R = 1\,\mathrm{Hz} \sqrt{\frac{1 - 0.9}{1 + 0.9}} = 0.23\,\mathrm{Hz}.$$

b) Due to length contraction in the frame of the receiver covers the pulse of the second antenna the additional distance

$$\Delta \ell = 2\,\mathrm{m} \sqrt{1 - (v/c)^2}$$

and the additional time for the pulse of the second antenna is

$$\Delta t = \frac{\Delta \ell}{c} = 2\,\mathrm{m} \sqrt{1 - (v/c)^2}/c = 2\,\mathrm{m}\sqrt{1 - 0.9^2}/3 \cdot 10^8\,\mathrm{m/s} = 0.29 \cdot 10^{-8}\,\mathrm{s}.$$

c) The total energy of the satellite is

$$E = mc^2.$$

The kinetic energy is

$$E = (m - m_0)c^2 = \frac{m_0 c^2}{\sqrt{1 - 0,9^2}} - m_0 c^2$$
$$= 1\,\mathrm{kg} \cdot (3 \cdot 10^8\,\mathrm{m/s})^2 \cdot 1,294 = 11,65 \cdot 10^{16}\,\mathrm{Ws}$$

d) To produce this amount of energy a power station needs the time

$$t = \frac{11,65 \cdot 10^{16}\,\mathrm{Ws}}{10^9\,\mathrm{W}} = 11,65 \cdot 10^7\,\mathrm{s} = 1348\,\mathrm{d} = 3,7\,\mathrm{a}.$$

Problem 3 (4 points)

a) Albert Einstein explained Planck's law by three fundamental processes. Explain these processes.

b) Make a sketch of Planck's law, i.e. plot the spectral radiance as a function of the wavelength λ.

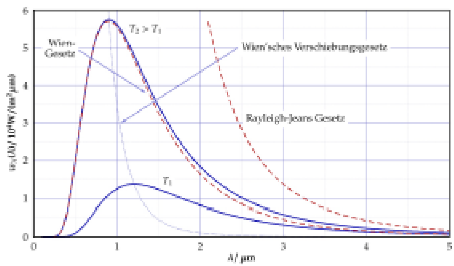
c) Write up and explain Stefan-Boltzmann's law.

d) What is a black body?

e) A sphere (radius 10 cm) approximating a black body is exposed to the sun ($150 W/m^2$) and absorbs the whole arriving power of the sun. Calculate the temperature of the sphere in thermal equilibrium when the temperature of the surrounding is 300 K.

a) The processes are the absorption, the stimulated and the spontaneous emission of photons.

b) Planck's law



c) Stefan-Boltzmann's law calculates to total power emitted by electromagnetic radiation by a body with the temperature T.

$$P = \varepsilon \cdot A \cdot \sigma \cdot T^4.$$

ε is a coefficient which describes the fraction of radiation that can pass the surface. A is the surface and σ Stefan-Boltzmann's constant.

d) A black body absorbs all radiation falling on it and emits from its interior all radiation arriving at the surface.

e) The equilibrium condition is $P_{Absorption} = P_{Emission}$. For the absorption of the sun light is the cross section of the sphere $A_Q = \pi \cdot r^2$ relevant. For the absorption and emission of thermal radiation the whole surface of the sphere has to be considered $A = 4\pi \cdot r^2$. With these considerations one gets

$$A_Q \cdot 150 W/m^2 = A \cdot \sigma \cdot (T^4 - (300K)^4)$$

$$T^4 = (300\,K)^4 + 150\,W/m^2 \frac{1}{4\sigma} = 8,1 \cdot 10^9\,K^4 + 6,6 \cdot 10^8\,K^4$$

$$\rightarrow T = 306\,K.$$

Problem 4 (4 Points)

a) Write up the Schrödinger equation for an electron moving within a region with constant potential energy V. The total energy of the electron is $E > V$.

b) Calculate the wave function and momentum of an electron moving along the x-axis.

c) The electron moves with the total energy E within a region with $V = V_1 < E$ in the direction of increasing x values and hits a barrier where the potential energy changes from V_1 to $V_2 < E$. How large is the probability that the electron is reflected?
Hint: The wave function at the interface of the barrier is both continuous and continuously differentiable.

a) Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2\nabla^2}{2m_e} + \mathrm{V}\right)\psi$$

b) The wave function is

$$\psi = \psi_0 e^{i(Px-Et)/\hbar}.$$

The momentum becomes with

$$E = \frac{P^2}{2m_e} + \mathrm{V} \rightarrow P = \sqrt{2m_e(E-V)}.$$

c) The condition on the interface $V_1 \rightarrow V_2$ is

$$\psi_i + \psi_r = \psi_t.$$

Thereby denotes

$$\psi_i = \psi_0 e^{i(P_1x-Et)/\hbar}$$
$$\psi_r = R\psi_0 e^{i(-P_1x-Et)/\hbar}$$
$$\psi_t = T\psi_0 e^{i(P_2x-Et)/\hbar}$$

the incident, reflected and transmitted wave. Due to the condition that the wave function is continuous and continuously differentiable on the interface one gets the usual formula for the reflection coefficient

$$R = \frac{P_1 - P_2}{P_1 + P_2}$$

and the probability for reflection is R^2 with $P_1 = \sqrt{2m_e(E-V_1)}$ and $P_2 = \sqrt{2m_e(E-V_2)}$.

Problem 5 (4 Points)

Spherical coordinates r, ϑ, φ are used to solve the Schrödinger equation of an electron in the electric field of a nucleus.

a) Sketch and write up the equation of the effective potential energy $\phi_{\text{eff}}(r)$ of the electron.

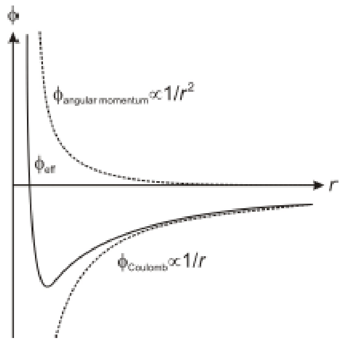
b) Explain the quantum numbers characterizing the radial part of the wave function.

c) Sketch the radial wave functions of the first excited state of the electron.

d) Which additional quantum numbers are necessary to characterize the quantum state of the electron?

a) The potential energy of the electron is due to the Coulomb potential in the electric field of the nucleus and there is an effective potential energy due to the motion of the electron around the nucleus. This contribution is proportional to the square of the angular momentum L.

$$\phi_{\text{eff}} = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r} + \frac{L^2}{2m_e r^2}.$$



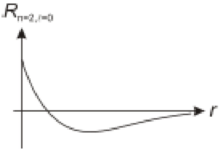
b) The radial part of the wave function is characterized by the main quantum number n

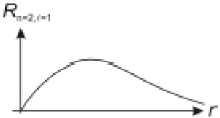
$$E_n = -13.6\,\text{eV}\left(\frac{Z}{n}\right)^2$$

and the angular momentum quantum number ℓ

$$L^2 Y_{\ell,m} = \hbar^2 \ell(\ell+1) Y_{\ell,m}.$$

c) For n = 2 there are two radial function $R_{n=2,\ell=0}$ and $R_{n=2,\ell=1}$.
$R_{n=2,\ell=0}$ starts at $r = 0$ with a finite value, changes the sign and approaches zero for $r \to \infty$.
Due to the centri-fugal potential starts $R_{n=2,\ell=1}$ for $r = 0$ at zero and approaches after a maximum zero again for $r \to \infty$.





d) For the orbital part one needs in addition to n and ℓ

the magnetic quantum number $|m| \leq \ell$ of the angular momentum.

To characterize the spin of the electron one needs the quantum numbers $s = 1/2$ and $m_s = \pm 1/2$.

s and ℓ add up to the total angular momentum which is characterized by the quantum numbers j and m_j.

Problem 6 (4 Points)

The orbital angular momentum $\vec{\ell}$ of the electron and its spin $\vec{s}$ couple to the total angular momentum $\vec{j}$ according to $\xi\vec{\ell}\cdot\vec{s}$.

a) Explain the origin for the coupling of $\vec{\ell}$ and $\vec{s}$.
b) Give the sign of the coupling constant ξ.
c) Calculate the energy difference between the energy of the total angular momentum state with the highest and the smallest j-value, respectively.
d) Calculate the potential energy of an electron in an homogeneous magnetic field $\vec{B}$ in terms of the quantum numbers of $\vec{j}$.

a) The magnetic moment due to the spin of the electron orients in the magnetic field caused by the nucleus due to the orbital motion of the nucleus around the electron.

b) The sign of the coupling constant ξ is negative, i.e. spin and orbital angular momentum orient in opposite directions for minimal energy.

c) With $\vec{\ell}\cdot\vec{s} = \frac{1}{2}(\vec{j}^2 - \vec{\ell}^2 - \vec{s}^2)$ one gets for the energy $E_{j,\ell,s}$

$$E_{j,\ell,s} = \frac{\xi}{2}\hbar^2(j(j+1) - \ell(\ell+1) - s(s+1))$$

and

$$E_{j=\ell+1/2,\ell,s} - E_{j=\ell-1/2,\ell,s} = \frac{\xi}{2}\hbar^2\left((\ell+\frac{1}{2})(\ell+\frac{3}{2}) - (\ell-\frac{1}{2})(\ell+\frac{1}{2})\right)$$
$$= \frac{\xi}{2}\hbar^2(\ell+\frac{1}{2})(\ell+2).$$

d) The potential energy in an homogeneous magnetic field of an electron characterized by j and ℓ is

$$E_{\ell,j,m_j} = g_j\mu_B B m_j$$

with the g_j-factor of the electron.

($g_j = \frac{3j(j+1)-\ell(\ell+1)+3/4}{2j(j+1)}$, and $g_e = 2$)