Moder Physics WS 21/22 Sören Bieling

Problem Sheet 1

Wave equation

Ex. 1.1: a) general 3D wave-eq.: $\boxed{\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi = 0}$

with $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ in cart. coord. $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

in 1D: $\nabla^2 \to \frac{\partial^2}{\partial x^2}$: $\boxed{\frac{\partial^2}{\partial x^2}\psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi}$

$\Psi_1 = A \sin(kx - \omega t)$
$\hookrightarrow \cos($
$\hookrightarrow -\sin($

b) Ψ_1

L.E.: $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\Psi_1\right) = \frac{\partial}{\partial x}\left(k \cdot A\cos(kx-\omega t)\right) = -k^2\Psi_1$

R.E.: $\frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\Psi_1\right) = \frac{1}{v^2}\frac{\partial}{\partial t}\left((-\omega)\cdot A\cos(kx-\omega t)\right) = -\frac{1}{v^2}\omega^2\Psi_1$

$\hookrightarrow$ fulfilled for $v^2 = \frac{\omega^2}{k^2}$ ✓ (dispersion relation)

Ψ_2

Same for $\Psi_2 = A\sin(kx)\cos(\omega t)$

L.E.: $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\Psi_2\right) = \frac{\partial}{\partial x}\left(k\ A\cos(kx)\cos(\omega t)\right) = -k^2\Psi_2$

R.E.: $\frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\Psi_2\right) = \frac{1}{v^2}\frac{\partial}{\partial t}\left(-\omega\ A\sin(kx)\sin(\omega t)\right) = -\frac{1}{v^2}\omega^2\Psi_2$

fulfilled for $v^2 = \frac{\omega^2}{k^2}$ ✓

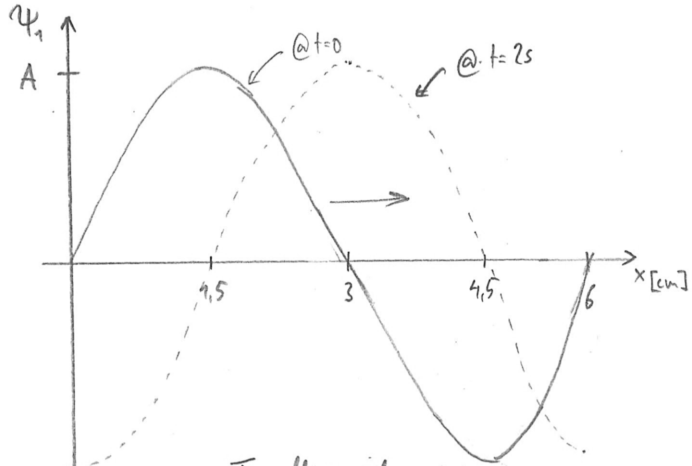
c) sketch Ψ_1 & Ψ_2 at $t_1 = 0s$ & $t_2 = 2s$:

$k = \frac{1}{3}\pi\frac{1}{cm}$ $\quad \omega = \frac{1}{4}\pi\frac{1}{s}$

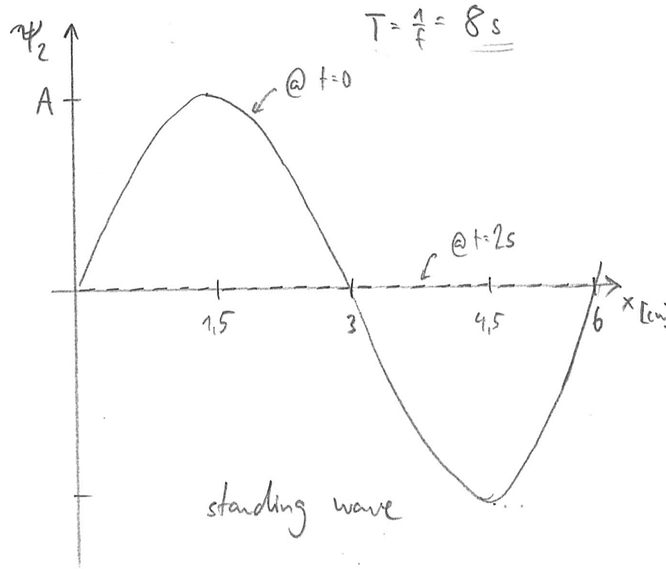
$\lambda, f, T = ?$ $\quad k = \frac{2\pi}{\lambda} \leftrightarrow \lambda = \frac{2\pi}{k} = \underline{\underline{6\,cm}}$

$\omega = 2\pi f = \frac{2\pi}{T} \to f = \frac{\omega}{2\pi} = \underline{\underline{\frac{1}{8}\,Hz}}$

$T = \frac{1}{f} = \underline{\underline{8\,s}}$



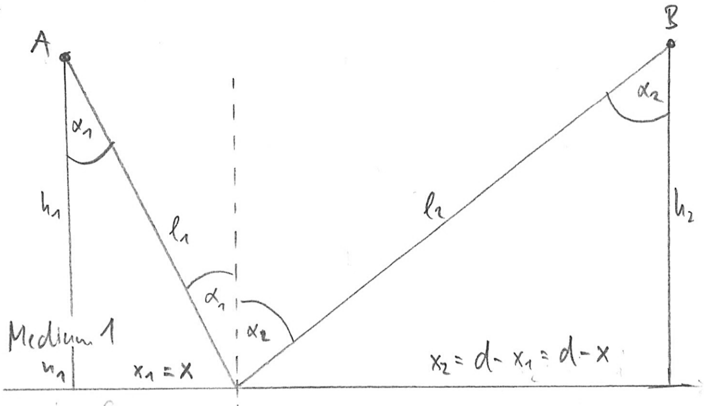
Travelling plane wave



standing wave

Ex. 1.2 Use Fermat's principle to derive a) law of reflection: $\alpha_1 = \alpha_2$

b) law of refraction: $n_1 \sin\alpha_1 = n_2 \sin\alpha_2$



$$\cos(\alpha_1) = \frac{h_1}{l_1} \qquad \cos(\alpha_2) = \frac{h_2}{l_2}$$

$$\sin(\alpha_1) = \frac{x}{l_1} \qquad \sin(\alpha_2) = \frac{d-x}{l_2}$$

$$\tan(\alpha_1) = \frac{x}{h_1} \qquad \tan(\alpha_2) = \frac{d-x}{h_2}$$

a)

$$t_{total} = t_1 + t_2 \stackrel{!}{=} \text{minimal} \qquad c_n = \frac{c}{n}$$

$$= \frac{1}{c_1}\left[l_1 + l_2\right]$$

$$x_1 + x_2 = d = \text{fix}$$

$$h_1 = \text{fix} \quad h_2 = \text{fix}$$

Ansatz 1.)

$$d = x + d - x = \text{fix}$$

$$= h_1 \tan(\alpha_1) + h_2 \tan(\alpha_2)$$

$$\text{fix} \Leftrightarrow \frac{\partial}{\partial\alpha_1}(d) = 0 = \frac{h_1}{\cos^2(\alpha_1)} + \frac{h_2}{\cos^2(\alpha_2)}\cdot\frac{\partial\alpha_2}{\partial\alpha_1} \quad (\mathrm{I})$$

2.)

$$\frac{\partial}{\partial\alpha_1}(t_{total}) = 0 = \frac{1}{c_1}\frac{\partial}{\partial\alpha_1}\left[l_1 + l_2\right]$$

$$= \frac{1}{c_1}\frac{\partial}{\partial\alpha_1}\left[\frac{h_1}{\cos(\alpha_1)} + \frac{h_2}{\cos(\alpha_2)}\right]$$

$$= \frac{1}{c_1}\left[h_1\frac{\sin(\alpha_1)}{\cos^2(\alpha_1)} + h_2\frac{\sin(\alpha_2)}{\cos^2(\alpha_2)}\frac{\partial\alpha_2}{\partial\alpha_1}\right] \quad (\mathrm{II})$$

3.) Insert eq. (I) into (II):

$$(\mathrm{I}) \Leftrightarrow \frac{\partial\alpha_2}{\partial\alpha_1} = -\frac{h_1}{h_2}\cdot\frac{\cos^2(\alpha_2)}{\cos^2(\alpha_1)}$$

into (II)

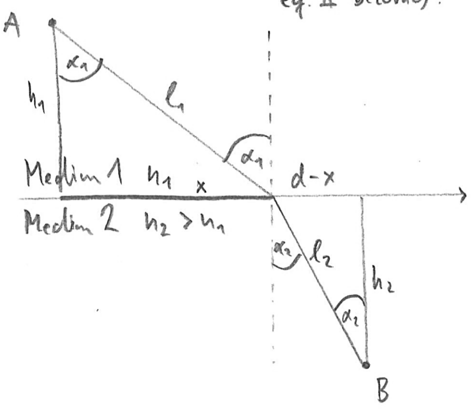
$$0 = \frac{1}{c_1}h_1\frac{\sin(\alpha_1)}{\cos^2(\alpha_1)} + \frac{1}{c_1}\cancel{h_2}\frac{\sin(\alpha_2)}{\cancel{\cos^2(\alpha_2)}}\cdot\left(-\frac{h_1}{\cancel{h_2}}\cdot\frac{\cancel{\cos^2(\alpha_2)}}{\cos^2(\alpha_1)}\right)$$

$$0 = \frac{h_1}{c_1\cos^2(\alpha_1)}\left[\sin(\alpha_1) - \sin(\alpha_2)\right] \Rightarrow \sin(\alpha_1) = \sin(\alpha_2) \quad \text{for } \alpha_{1,2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \boxed{\alpha_1 = \alpha_2}$$

b) identical to a) except eq. II becomes:

$$t_{total} = t_1 + t_2 = \frac{l_1}{c_1} + \frac{l_2}{c_2} = \frac{1}{c}\left[n_1 l_1 + n_2 l_2\right]$$

$$= \frac{1}{c}\left[n_1\frac{h_1}{\cos\alpha_1} + n_2\frac{h_2}{\cos\alpha_2}\right]$$



$$\frac{\partial}{\partial\alpha_1}(t_{total}) \stackrel{!}{=} 0 = \frac{1}{c}\left[n_1 h_1\frac{\sin(\alpha_1)}{\cos^2(\alpha_1)} + n_2\cancel{h_2}\frac{\sin\alpha_2}{\cancel{\cos^2(\alpha_2)}}\left(-\frac{h_1}{\cancel{h_2}}\cdot\frac{\cancel{\cos^2(\alpha_2)}}{\cos^2(\alpha_1)}\right)\right]$$

$$0 = \frac{h_1}{c\cdot\cos^2(\alpha_1)}\left[n_1\cdot\sin(\alpha_1) - n_2\cdot\sin(\alpha_2)\right]$$

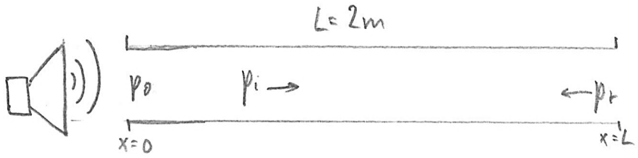
$$\Rightarrow \boxed{n_1\sin(\alpha_1) = n_2\sin(\alpha_2)}$$

Modern Physics WS 21/22 Sören Bieling

$$p(\vec{r},t) = P_0 + p_s(\vec{r},t)$$

wave equation: $\boxed{\nabla^2 p_s - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} p_s = 0}$

Ex. 1.3:



a) pressure wave fct.:

$$\left.\begin{aligned} p_i(x,t) &= p_0\, e^{i(kx-\omega t)} \\ p_r(x,t) &= r \cdot p_0 \cdot e^{i(-kx-\omega t)} \end{aligned}\right\} \quad p(x,t) = p_i(x,t) + p_r(x,t)$$

b) $c_s = 340\,\frac{m}{s}$ $\qquad \omega = 2\pi f$

$c_s = \lambda \cdot f$

boundary conditions: $p(x=0,t) = 0 = p(x=L,t)$

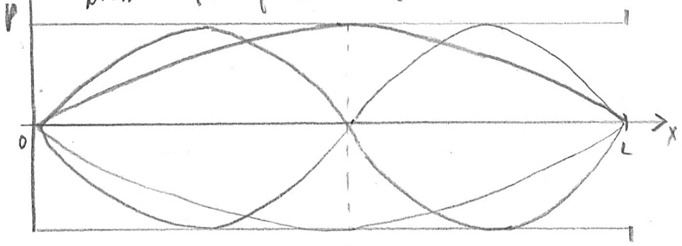
$$Re\{p(x,t)\} = p_0\left(\sin(kx-\omega t) + r\sin(-kx-\omega t)\right)$$

resonance condition: $L = n \cdot \frac{\lambda_n}{2} \quad n \in \mathbb{N}$

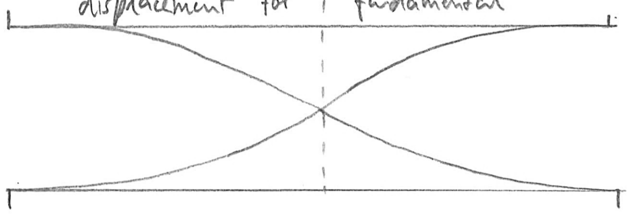
$$\Leftrightarrow \quad \lambda_n = \frac{2L}{n} \quad \rightarrow \quad f_n = \frac{c_s}{\lambda_n} = n \cdot \frac{c_s}{2L}$$

$$= n \cdot \underbrace{85\,Hz}_{\text{fundamental}}$$

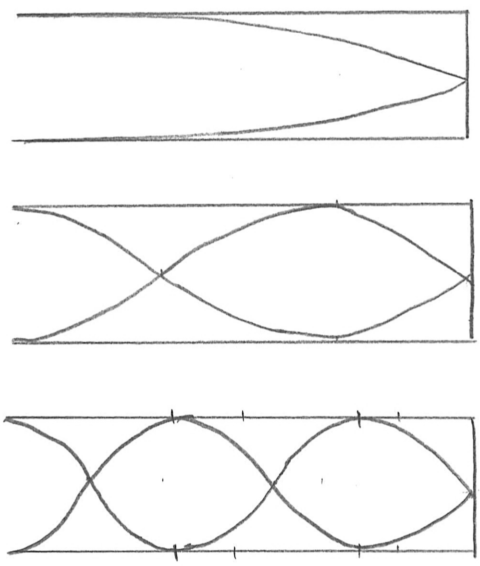
c) pressure for fundamental + 1st harmonic:



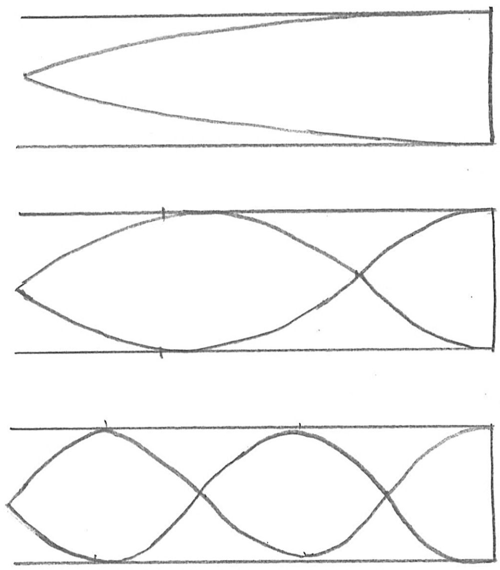
d) displacement for fundamental



e) displacement:



pressure:



$L = \frac{1}{4}\lambda \qquad f_1 = \frac{c_s}{4L}$

$= \underline{\underline{42{,}5\,Hz}}$

$L = \frac{3}{4}\lambda \qquad f_3 = \frac{3}{4}\frac{c_s}{L}$

$= \underline{\underline{127{,}5\,Hz}}$

$L = \frac{5}{4}\lambda \qquad f_5 = \frac{5}{4}\frac{c_s}{L}$

$= \underline{\underline{212{,}5\,Hz}}$

L = 2m $\qquad$ L = 2m