Modern Physics WS 21/22

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Problem Sheet 3

Ex. 3.1 Ideal gas & Maxwell velocity distribution

1 l ^4He @ $T_1 = 293K$, $p_1 = 1 bar$ $\longmapsto$ $T_2 = 523K$, p_2 $\quad V = const.$

(~20°C) (~250°C)

a) Derive from $f(v) \to \langle v^2 \rangle, \langle E_{kin} \rangle$ at both T_1 & T_2:

$$f(v) = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_BT}\right)^{3/2} v^2 \frac{1}{e^{\left(\frac{1}{2}mv^2/k_BT\right)}}$$

$$\langle v^2 \rangle = \int_{v=0}^{\infty} v^2 f(v)\, dv = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_BT}\right)^{3/2} \int_0^{\infty} v^4 \exp\left(-\tfrac{1}{2}mv^2/k_BT\right) dv$$

with Hint: $\int_0^{\infty} x^4 \exp(-\alpha x^2)\, dx = \frac{3\sqrt{\pi}}{8\alpha^{5/2}}$, $\quad \alpha = \frac{1}{2} \cdot \frac{m}{k_BT}$

$$= \frac{\cancel{4}}{\cancel{\sqrt{\pi}}} \left(\frac{m}{2k_BT}\right)^{3/2} \frac{3}{\cancel{8}_2} \cancel{\sqrt{\pi}} \left(\frac{2k_BT}{m}\right)^{5/2 \, 1} = \frac{3}{2}\left(\frac{2k_BT}{m}\right)$$

$$\langle E_{kin}(T) \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{\cancel{2}} \cancel{m} \frac{3}{2} \left(\frac{\cancel{2}k_BT}{\cancel{m}}\right) = \frac{3}{2} k_BT \qquad \left(\tfrac{1}{2}k_BT \text{ per d.o.f}\right)$$

b) $\langle E_{kin}(293K) \rangle = 607 \cdot 10^{-23} J$ $\quad v_{rms,1} = 1352\ m/s$ $\quad m = 4u$ $\quad v_{rms} = \sqrt{\frac{3}{2} \frac{2k_BT}{m}}$

$\langle E_{kin}(523K) \rangle = 1083 \cdot 10^{-23} J$ $\quad v_{rms,2} = 1806\ m/s$

c) ideal gas eq. $pV = Nk_BT$ $\quad V, N = const$ $\quad \to \frac{p}{T} = \frac{Nk_B}{V} = const$

$$\frac{p_1}{T_1} = \frac{p_2}{T_2} \qquad p_2 = p_1 \frac{T_2}{T_1} = 1{,}78\ bar$$

Ex. 3.2 Special relativity - muons

1. mean lifetime $\tau_0 = 2{,}2\mu s$ (proper time), $\quad h_0 = 10\ km$ $\quad v = 0{,}998 \cdot c$
2. non-relativistic path $s = v \cdot \tau = 658\ m < 10 km$! Not enough to reach earth's surface from 10 km above.

a) Perspective of observer on earth:

1. muon moves with $v = 0{,}998 \cdot c$ relative to observer
2. due to time dilation the muon's lifetime increases to

$$\tau(v) = \gamma \tau_0 = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \tau_0 = 34{,}8\ \mu s \longrightarrow s = v \cdot \tau(v) = 10{,}4\ km \to$$

They can reach earth's surface!

b) Perspective of the muon:

1. Earth's surface moves towards the muon with $v = 0{,}998 \cdot c$
2. It perceives the distance to earth's surface contracted by

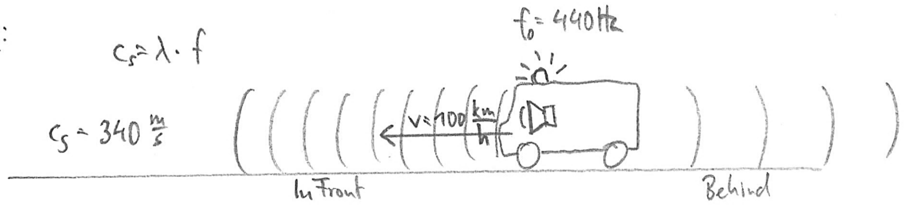
$$h(v) = \frac{h_0}{\gamma} = h_0 \sqrt{1-\left(\frac{v}{c}\right)^2} = 632\ m < 658 m \longrightarrow$$

Earth's surface can be reached!

Ex. 3.3:

$c_s = \lambda \cdot f$

$c_s = 340\ \frac{m}{s}$



$v = 100\ \frac{km}{h} \cdot \frac{1h}{3600s} \cdot \frac{1000m}{1km}$

a) $\lambda_0 = \frac{c_s}{f_0} = 77{,}3\ cm$ $\qquad T_0 = \frac{1}{f_0} = 2{,}\overline{27}\ ms$ $\qquad 1\,\text{halftone} = \left(\sqrt[12]{2} - 1\right) \cdot 440\,Hz = 26\,Hz$

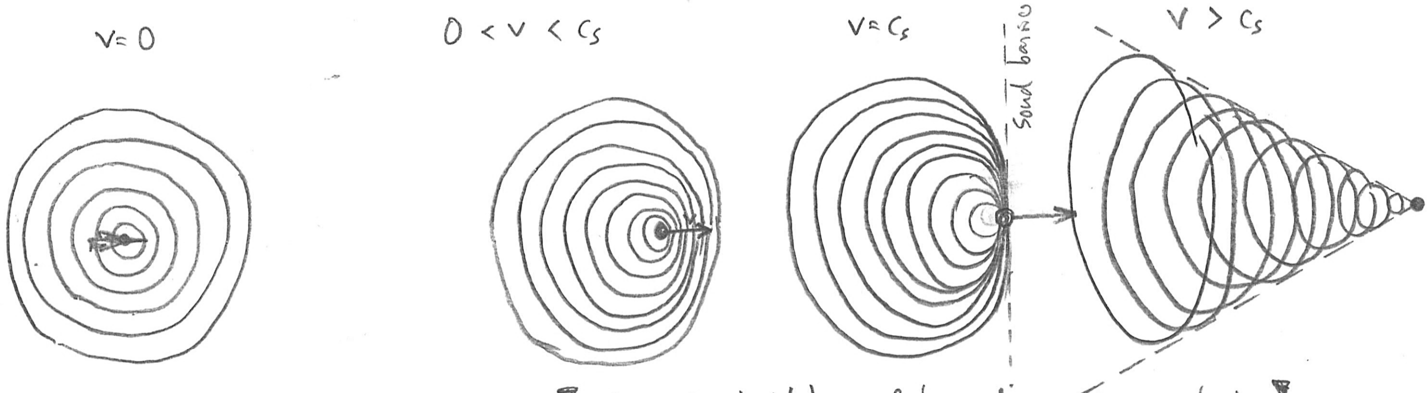
$\lambda_{In\,Front} = \frac{c_s - v}{f} = (c_s - v)\,T_0 = 70{,}\overline{95}\ cm \longrightarrow f_{In\,Front} = \frac{c_s}{\lambda_{In\,Front}} = \frac{c_s}{c_s - v} f_0 = \underline{\underline{479\ Hz}}$ $\quad \Delta = 39\,Hz$

$440\,Hz$

$\Delta = -33\,Hz$

$\lambda_{Behind} = \frac{c_s + v}{f} = (c_s + v)\,T_0 = 83{,}\overline{58}\ cm \longrightarrow f_{Behind} = \frac{c_s}{\lambda_{Behind}} = \frac{c_s}{c_s + v} f_0 = \underline{\underline{407\ Hz}}$

b) $v = 0$ $\qquad 0 < v < c_s$ $\qquad v = c_s$ $\qquad v > c_s$



! all circles should be in fact round = 1:1 aspect radio !

c) from a)

In Front: −

$$f_{In\,Front} = \frac{c_s}{c_s - v} \frac{1}{T_0} = \frac{1}{1 - \frac{v}{c_s}} \cdot \frac{\sqrt{1 - \left(\frac{v}{c}\right)^2}}{T_0} = \underbrace{\frac{1}{T_0}}_{f_0} \frac{\sqrt{1 - \left(\frac{v}{c}\right)^2}}{1 - \frac{v}{c}} = f_0 \frac{\sqrt{1 + \frac{v}{c}}\,\cancel{\sqrt{1 - \frac{v}{c}}}}{\sqrt{1 - \frac{v}{c}}\,\cancel{\sqrt{1 - \frac{v}{c}}}}$$

The time dilation has to be taken into account!

$$\boxed{T(v) = \gamma\, T_0 = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} T_0}$$

$$\boxed{f_{In\,Front}(v) = f_0 \sqrt{\frac{1 + (v/c)}{1 - (v/c)}}}$$

approaching

Behind: +

$$\boxed{f_{Behind}} = \frac{c_s}{c_s + v} \frac{1}{T} = \frac{1}{1 + \frac{v}{c_s}} \frac{\sqrt{1 - \left(\frac{v}{c}\right)^2}}{T_0} = f_0 \frac{\sqrt{1 - \frac{v}{c}}\,\cancel{\sqrt{1 + \frac{v}{c}}}}{\sqrt{1 + \frac{v}{c}}\,\cancel{\sqrt{1 + \frac{v}{c}}}} = \boxed{f_0 \sqrt{\frac{1 - (v/c)}{1 + (v/c)}}}$$

leaving

d) sign of formulae from c) changes for λ $\qquad (f \nearrow \,=\, \lambda \searrow)$

$$\lambda_{In\,Front}(v) = \lambda_0 \sqrt{\frac{1 - (v/c)}{1 + (v/c)}} \doteq \lambda_v$$

$$\left(\frac{\lambda_v}{\lambda_0}\right)^2 = \frac{1 - v/c}{1 + v/c} = \frac{c - v}{c + v}$$

$$(c + v)\left(\frac{\lambda_v}{\lambda_0}\right)^2 = c - v$$

$\lambda_0 = 633\,nm$ $\qquad \lambda_v \doteq 540\,nm$ $\qquad v = ?$

$$v\left[\left(\frac{\lambda_v}{\lambda_0}\right)^2 + 1\right] = \left[1 - \left(\frac{\lambda_v}{\lambda_0}\right)^2\right] c$$

$$v = c\, \frac{1 - \left(\frac{\lambda_v}{\lambda_0}\right)^2}{\left(\frac{\lambda_v}{\lambda_0}\right)^2 + 1} = 47\,240\ km/s$$

$$\hat{=}\ 170 \cdot 10^6\ km/h$$