Modern Physics WS 21/22

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Problem Sheet 9

general nomenclature:

$$\boxed{n^{2S+1} L_J}$$

$L = 0, 1, 2, \dots N-1$
$\quad S, P, D \dots$

$m_L = -L, \dots, 0, \dots +L$

$\mu_B = \frac{e\hbar}{2m_e} = 9{,}274 \cdot 10^{-24} \frac{J}{T}$

E1 Selection Rules:

$\Delta L = \pm 1$

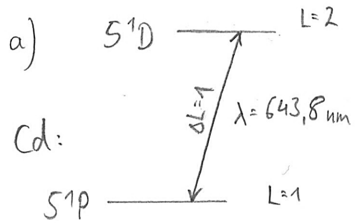
$\Delta m_e = 0 \quad \pi \quad$ lin. pol.

$= +1 \quad \sigma^+ \quad$ circ.$^+$ pol.

$= -1 \quad \sigma^- \quad$ circ.$^-$ pol.

Ex. 9.1:

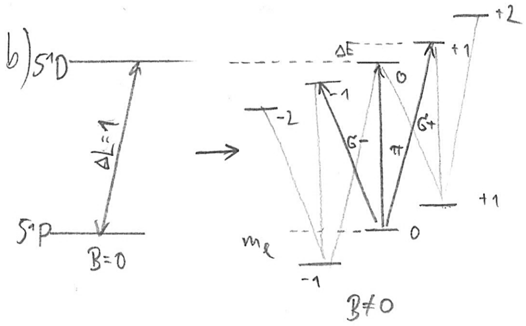
a)



S=0 → Normal Zeeman effect with Cd

$E_\lambda = \frac{hc}{\lambda} = 1{,}926\,eV$

b)



$\Delta E = \Delta m_e \cdot \mu_B \cdot B_z \qquad \Delta m_e = \pm 1$

$E_\lambda = \frac{hc}{\lambda} \rightarrow \frac{\Delta E}{\Delta \lambda} = -\frac{hc}{\lambda^2}$

$\Delta E = -\frac{hc}{\lambda^2} \Delta\lambda = \pm \mu_B \cdot B_z$

$\Delta\lambda = \mp \lambda \cdot \frac{\mu_B \cdot B_z}{E_\lambda}$

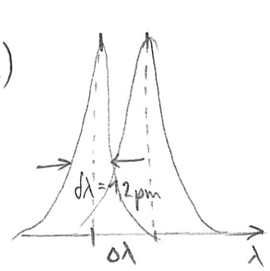
c)

$B = 0$ single transition $(\pi, \sigma^+, \sigma^-)$

$B \neq 0$ Splitting into 3 transition frequencies: σ^-, π, σ^+
due to E1 Selection rules $(\Delta L = \pm 1, \ \Delta m_e = \pm 1, 0)$

$\delta\lambda = 1{,}2\,pm$
$\lambda = 643{,}8\,nm$

d)



B_{min} so that Zeeman splitting is observable?

$|\Delta\lambda| > \delta\lambda = 1{,}2\,pm \qquad \lambda \cdot \frac{\mu_B \cdot B_z}{E_\lambda} > \delta\lambda$

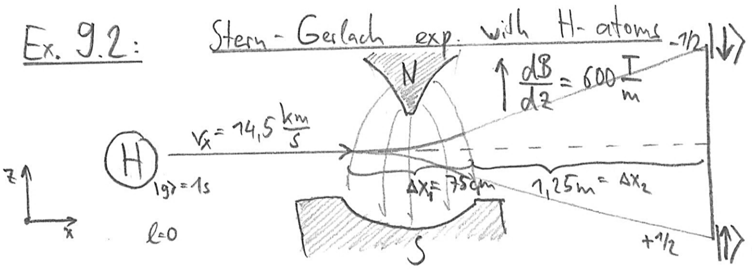
$B_z > \frac{\delta\lambda}{\lambda} \cdot \frac{hc}{\lambda} \cdot \frac{1}{\mu_B}$

$> 62\,mT$

btw: 100 year anniversary of Stern-Gerlach exp. on 8.2.2022

Ex. 9.2:

Stern-Gerlach exp. with H-atoms



$F_z = \mu_z \cdot \frac{dB_z}{dz}$

$\vec{\mu}_e = -g\, \mu_B \frac{\vec{S}}{\hbar} \qquad g = 2$

$S_z = m_s \hbar \qquad m_s = \pm 1/2$

$\mu_{e,z} = -g \mu_B \frac{S_z}{\hbar}$

a) max. acceleration?

$\sum F = m_H\, a \qquad a_z = \frac{F_z}{m_H} = \frac{-g \mu_B m_s}{m_H} \cdot \frac{dB_z}{dz} = \mp 3{,}325 \cdot 10^6 \frac{m}{s^2} \qquad \mp$ for $m_s = \pm 1/2$

b) $\Delta x = v_x \cdot \Delta t_1 \qquad \Delta t_1 = \frac{\Delta x_1}{v_x} = 51{,}72\,\mu s$

time in magnet

time to detector

$\Delta t_2 = \frac{\Delta x_2}{v_x} = 86{,}21\,\mu s$

$\rightarrow \Delta S_{z,1} = \frac{1}{2} a \Delta t_1^2 = 4{,}45\,mm$

$\rightarrow v_z = a \cdot \Delta t_1 = 172 \frac{m}{s}$

$\rightarrow \Delta S_{z,2} = v_z \cdot \Delta t_2 = 14{,}83\,mm$

max. distance between $|\uparrow\rangle$ & $|\downarrow\rangle$ is

$2(\Delta S_{z,1} + \Delta S_{z,2})$

$\Delta S_H = 38{,}6\,mm$

$\times \frac{58^2}{107} = 31{,}44$

c) H-atom → Ag $\quad m_H \rightarrow m_{Ag} = 107 \cdot m_H$

F_z identical, $a_z \rightarrow \frac{1}{107} a_z$, $v_x \rightarrow \frac{1}{58} v_x$,
$(v_x = 14{,}5 \frac{km}{s} \rightarrow 250 \frac{m}{s})$

$\Delta t_2 \rightarrow 58 \cdot \Delta t_2$

$\Delta t_1 \rightarrow 58 \cdot \Delta t_1, \quad \Delta S_{z,1} \rightarrow \frac{1}{107} (58)^2 \cdot \Delta S_{z,1}$

$v_z \rightarrow \frac{1}{107} \cdot 58 \cdot v_z, \quad \Delta S_{z,2} \rightarrow \frac{58}{107} \cdot 58\, \Delta S_{z,1}$

$\Delta S_{Ag} = 1{,}214\,m$

Ex. 9.3: Eigenvalue problem

$\hat{A}|\phi\rangle = a|\phi\rangle, \ a \in \mathbb{C}$

conjugated operator $\hat{A}^\dagger$: $\langle\psi|\hat{A}^\dagger|\xi\rangle = \langle\xi|\hat{A}|\psi\rangle^*$

hermitean operator: $\hat{A} = \hat{A}^\dagger$

$z_1, z_2 \in \mathbb{C}$

unity operator $\hat{1} = \sum_{n=1}^{N} |n\rangle\langle n|$

$(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$

a) Show $(\hat{A}\hat{B})^\dagger = B^\dagger A^\dagger$:

let's start with the definition of a conjugated operator:

$$\langle c|\hat{A}\hat{B}|d\rangle = \underbrace{\langle d|(\hat{A}\hat{B})^\dagger|c\rangle^*}_{\mathrm{I}}$$

$$\langle c|\hat{A}\hat{B}|d\rangle = \sum_{n=1}^{N} \langle c|\hat{A}|n\rangle\langle n|\hat{B}|d\rangle = \sum_{n=1}^{N} \langle n|\hat{A}^\dagger|c\rangle^* \langle d|\hat{B}^\dagger|n\rangle^* = \sum_{n=1}^{N} \langle d|\hat{B}^\dagger|n\rangle^* \langle n|\hat{A}^\dagger|c\rangle^*$$

1. insert $\hat{1}$

2. conjugate each

$$= \sum_{n=1}^{N} \left(\langle d|\hat{B}^\dagger \underbrace{|n\rangle\langle n|}_{3.\text{ extract } \hat{1}} \hat{A}^\dagger|c\rangle\right)^* = \underbrace{\langle d|\hat{B}^\dagger\hat{A}^\dagger|c\rangle^*}_{\mathrm{II}}$$

$\Rightarrow$ $\mathrm{I} = \mathrm{II}$: $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$

b)

let's start with the definition of a hermitean operator: $\hat{A} = \hat{A}^\dagger$

$\Downarrow$

III $\langle c|\hat{A}|d\rangle = \langle c|\hat{A}^\dagger|d\rangle = \langle d|\hat{A}|c\rangle^*$

$$\langle\phi|\hat{A}|\phi\rangle = a\langle\phi|\phi\rangle$$

$$\mathrm{III} \longrightarrow \ \|$$

$$\langle\phi|\hat{A}|\phi\rangle^* = a^* \underbrace{\langle\phi|\phi\rangle^*}_{\in\mathbb{R}} = a^*\langle\phi|\phi\rangle$$

$\Rightarrow$ $a = a^* \Rightarrow a \in \mathbb{R}$