Modern Physics

Ilias (https://ilias.studium.kit.edu)

- Lecture (Bernd Pilawa)
- Tutorials (Hossam Tohamy) tomorrow (Room 229.4 in this building)
- Next exams:
  - Tuesday March 19th 2024 8 a.m. - 11 a.m.
  - Tuesday September 24th 2024 8 a.m. - 11 a.m.

Modern Physics

Contents:

Classical Wave Phenomena

- General Wave Phenomena
- Classical wave optics

General Wave Phenomena

Video: (Wasserwelle.mp4)

General Wave Phenomena: Comment

In classical physics, waves denote propagating excitations that transport energy, momentum or another physical quantity.

In contrast, waves have a somewhat different meaning in quantum physics. There, waves define the probability of finding particles.

The video shows a stone falling into water.

The impact stimulates surface waves.

As a result, part of the kinetic energy of the stone is transported on the surface away from the point of impact.

The section “General Wave Phenomena” deals with classical waves and is divided into six subsections.

General Wave Phenomena

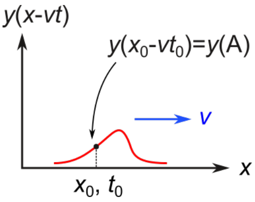
- Waves in one dimension
- Reflection and transmission
- Standing waves
- Waves in three dimensions
- Electromagnetic waves

1D-waves

General Wave Phenomena

- Waves in one dimension
- Reflection and transmission
- Standing waves
- Waves in three dimensions
- Electromagnetic waves

Waves in one dimension 1: Wave equation



(Ondeprogressive1.mp4)

The function of a wave propagating along the x-axis is

$$y(x,t) = y(x - vt)$$

The marked point with the coordinates t_0 and x_0 will have at time t the spacial coordinate x

$$x_0 - vt_0 = A = x - vt \quad \rightarrow \quad \boxed{x = A + vt = x_0 + v(t - t_0)}$$

Waves in one dimension 1 — Comment

Since the description of general waves is complicated, one-dimensional waves will be discussed first.

The figure and the animation show a wave propagating along the x-axis.

The underlined equation shows the general description of a wave.

The function f itself can be arbitrary. What matters is the argument of the function $x - vt$.

Thereby, v denotes the propagation speed of the wave.

The marked point in the diagram moves with this velocity to the right.

The shift of the marked point is given by the equation outlined in red.

Waves in one dimension 2: Wave equation

With the partial differentiations of $y(x,t) = y(x - vt)$

$$\frac{\partial^2 y}{\partial x^2} = y''$$
$$\frac{\partial^2 y}{\partial t^2} = v^2 y''$$

results the one dimensional wave equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

Waves in one dimension 2 Comment

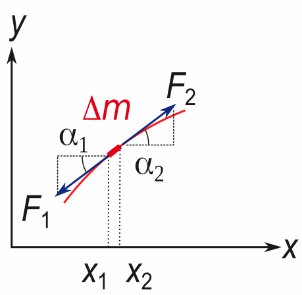
With the partial derivatives of the wave function $y(x,t) = y(x - vt)$, the differential equation outlined in red results.

This equation is the classical wave equation in one dimension.

Of course, there are many other differential equations whose solutions are also wave functions.

The differential equation outlined in red is important because Newton's mechanics and Maxwell's electrodynamics lead to this type of differential equation..

Waves in one dimension 3: Wave on a string



With

$$F_{1y} = -F_{1x} \tan \alpha_1 = -F_{1x} \left.\frac{\partial y}{\partial x}\right|_{x_1}$$

$$F_{2y} = F_{2x} \tan \alpha_2 = F_{2x} \left.\frac{\partial y}{\partial x}\right|_{x_2}$$

Waves in one dimension 3 — Comment

The illustration shows a section of a rope. The forces F_1 and F_2 act on the mass element Δm.

The mass Δm can move up and down along the y-axis, but not along the x-axis.

The forces that act on the mass Δm along the x-axis cancel each other out.

Only along the y-axis acts a force which accelerates the mass Δm.

The figure and the equations written in blue show that the forces along the y-axis are proportional to the derivative of the wave function on the left and right side of the mass Δm.

Waves in one dimension 4: Wave on a string

With

$$F_{1x} = F_{2x} = F_0$$

and Newton's equation of motion

$$\Delta m \vec{a} = \vec{F}$$

becomes

$$\Delta m \frac{\partial^2 y}{\partial t^2} = F_{1y} + F_{2y} = F_0 \left(\left.\frac{\partial y}{\partial x}\right|_{x_2} - \left.\frac{\partial y}{\partial x}\right|_{x_1} \right)$$

Waves in one dimension 4 — Comment

It is now assumed that the displacement is so small that the forces $F_{1x} = F_{2x} = F_0$ along the x-axis are not changed.

The force F_0 is usually called the tension of the rope.

Newton's equation of motion gives then the underlined equation.

Waves in one dimension 5: Wave on a string

with the mass per unit length

$$\mu = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} \quad \text{and} \quad \Delta x = x_2 - x_1$$

and the second partial differentiation

$$\frac{\partial^2 y}{\partial x^2} = \lim_{\Delta x \to 0} \frac{\left(\left.\frac{\partial y}{\partial x}\right|_{x_2} - \left.\frac{\partial y}{\partial x}\right|_{x_1} \right)}{\Delta x}$$

becomes Newton's equation of motion

$$\lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} \frac{\partial^2 y}{\partial t^2} = \mu \frac{\partial^2 y}{\partial t^2} = F_0 \lim_{\Delta x \to 0} \frac{\left(\left.\frac{\partial y}{\partial x}\right|_{x_2} - \left.\frac{\partial y}{\partial x}\right|_{x_1} \right)}{\Delta x} = F_0 \frac{\partial^2 y}{\partial x^2}$$

Waves in one dimension 5 Comment

The equation outlined in red gives the mass per unit length.

The following equation defines the second partial derivative of the displacement.

With the definition of the mass per unit length and the second partial derivative of the displacement, the wave equation of a rope wave follows from Newton's equation of motion.

Waves in one dimension 6: Wave on a string

the comparison between Newton's equation of motion

$$\mu \frac{\partial^2 y}{\partial t^2} = F_0 \frac{\partial^2 y}{\partial x^2}$$

and the one dimensional wave equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

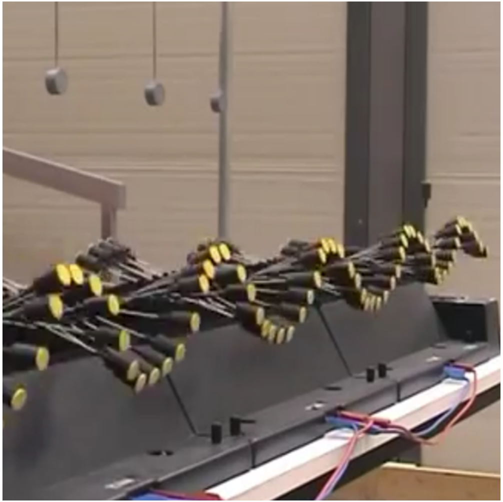
with the speed of the wave

$$\boxed{v = \sqrt{\frac{F_0}{\mu}}}$$

Waves in one dimension 6 Comment

The comparison between the one-dimensional wave equation and Newton's equation of motion gives the speed of a wave on a rope when the displacement does not change the tension of the robe.

Waves in one dimension 7: Harmonic waves



(HarmonischeWelle.mp4)

Waves in one dimension 7 Comment

The shape of a wave, i.e. the function f, depends on the excitation of the wave.

The video shows the excitation of a harmonic wave.

The excitation follows a sine or cosine function.

The wave propagates on a chain of coupled pendulums and each pendulum performs a harmonic oscillation.

Harmonic waves are characterised by the wavelength λ and the period T of the oscillation.

The wavelength denotes the distance between two succeeding maxima or minima of the displacement.

Waves in one dimension 8: Harmonic waves

The wave function of a harmonic wave is e.g.

$$y(x,t) = y(x - vt) = y_0 \cos(kx - \omega t + \varphi_0)$$

k denotes the wave number

$$\boxed{k = \frac{2\pi}{\lambda}}$$

and ω die angular frequency

$$\boxed{\omega = \frac{2\pi}{T}}$$

Waves in one dimension 8 — Comment

The wave function of a harmonic wave is a sine or cosine function.

The argument of a sine or cosine function is called a phase.

The sine and cosine functions are periodic functions.

The function values repeat when 2π is added to the phase.

The coordinate x is mapped to the phase by the wave number k.

The time t is mapped to the phase by the angular frequency ω.

Waves in one dimension 9: Harmonic waves

The phase velocity is the propagation speed of a harmonic wave

$$\varphi = kx - \omega t = k(x - \frac{\omega}{k}t)$$

$$\boxed{v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda\nu}$$

frequency:

$$\boxed{\nu = T^{-1}}$$

Unit of the frequency: Hz (Hertz)

Waves in one dimension 9 Comment

The first equation outlined in red gives the phase velocity.

The phase velocity is the propagation speed of a harmonic wave.

The reciprocal of the period T is the frequency of the wave.

In contrast to the angular frequency ω, the unit of frequency is named after Heinrich Hertz (1857-94).

Heinrich Hertz first demonstrated the existence of electromagnetic waves in 1886.

Waves in one dimension 10: Harmonic waves

Usually the complex exponential function is used

$$y(x,t) = y_0 \exp[i(kx - \omega t)] = y_0 \{\cos(kx - \omega t) + i \sin(kx - \omega t)\}$$

At the end of a calculation, only the real part has to be taken into account, since physical quantities are real

The amplitude y_0 can also be a complex number which contains the additional constant phase φ_0, i.e. $y_0 = |y_0| \cdot \exp(i\varphi_0)$

Waves in one dimension 10 Comment

Since the calculation with sine and cosine functions it is usually a nasty business complex exponential functions are used.

The equation outlined in red shows the relationship between the sine and cosine functions and the complex exponential function, i.e. $\exp\varphi = \cos\varphi + i\sin\varphi$.

Physical quantities are described by the real part of the complex exponential function.

A constant phase φ_0 can be included in the amplitude of the wave.

Waves in one dimension 11: Harmonic waves

Dispersion:

(DispersionWasserwellen.mp4

$$v = v(\lambda) \quad \text{or} \quad v = v(k)$$

Waves in one dimension 11 — Comment 1

The video shows again the excitation of a water surface by a falling stone.

The water wave is now observed over a longer period of time.

Further away from where the stone fell into the water, a harmonic wave can be observed. The wavelength becomes shorter over time.

Surface waves with a long wavelength propagate faster than waves whose wavelength is short.

The phase velocity depends on the wavelength.

The angular frequency ω is no longer proportional to the wave number k, since the phase velocity v is a function of the wavelength λ.

Waves in one dimension 11 — Comment 2

The falling stone causes a pulse-like stimulation of the water surface.

The Fourier transformation of a pulse results in a broad spectrum of harmonic waves.

The pulse-like excitation breaks down quickly because the harmonic components spread at different speeds.

The water waves therefore quickly disintegrate into their harmonic components.

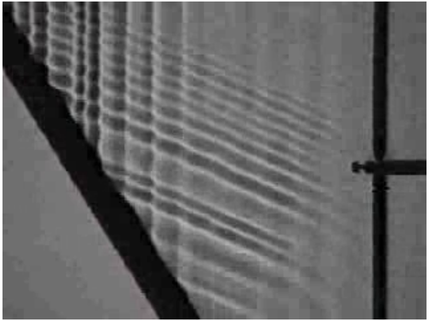
Reflexion/Transmission

General Wave Phenomena

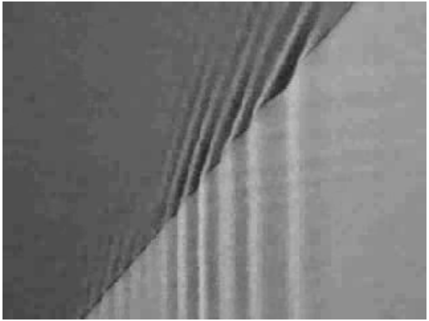
- Waves in one dimension
- Reflection and transmission
- Standing waves
- Waves in three dimensions
- Electromagnetic waves

Reflection and Transmission 1

Reflection and transmission are general wave phenomena



ReflexionvonWasserwellen.mp4



BrechungvonWasserwellen.mp4

Reflection and Transmission 1 Comment

Reflection and transmission are general wave phenomena.

The first video shows the reflection of water waves on a barrier.

Water waves are excited by a paddle which oscillates periodically.

The second video shows the transmission of water waves.

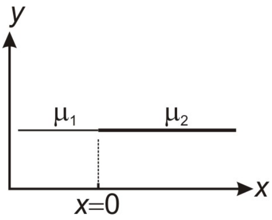
The water waves propagate from an area with the phase velocity v_1 into an area with the phase velocity v_2.

The phase velocity of a water waves depend on the depth of the water.

The depth of water is changed by a piece of acrylic glass in the water.

Reflection and Transmission 2: Strings

Two strings are connected at $x = 0$ with the mass per unit length μ_1 and μ_2, respectively



wave functions of the incoming, the reflected, and the transmitted wave

$$y_{in} = y_0 e^{i(k_1 x - \omega t)}$$

$$y_r = r\, y_0 e^{i(-k_1 x - \omega t)}$$

$$y_t = t\, y_0 e^{i(k_2 x - \omega t)}$$

Reflection and Transmission 2 Comment 1

The general description of reflection and transmission is complicated.

Therefore, the simple case is considered in which two ropes of different mass per unit length are connected to one another.

It turns out that the result is of general validity.

The figure shows the connection of two strings with the mass per unit length μ_1 and μ_2, respectively.

The strings are connected at $x = 0$.

This choice of the connection point determines the phases of the waves, which are then particularly simple.

Reflection and Transmission 2 Comment 2

The first equation describes the incoming wave.

The wave propagates from left to right.

The wave number is k_1.

The incoming wave hits the node at $x = 0$ and part of the wave is reflected.

The second equation describes the reflected wave.

The amplitude of the reflected wave is reduced by the factor r compared to the incoming wave.

r is called the reflection coefficient of the amplitude.

Reflection and Transmission 2 — Comment 3

The reflected wave propagates from right to left.

The wave number of the reflected wave is also k_1 since the wave propagates on string 1.

The negative sign of the wave number determines the propagation in the opposite direction.

It would be more intuitive to change the sign in front of the angular frequency ω, since the speed of the wave corresponds to the time.

It turns out, however, that in three-dimensional space the wavenumber is generalized to the wavenumber vector, which then determines the direction of propagation of the wave.

Reflection and Transmission 2 Comment 4

It therefore makes sense to describe the direction of propagation of the wave in one dimension using the sign of the wave number.

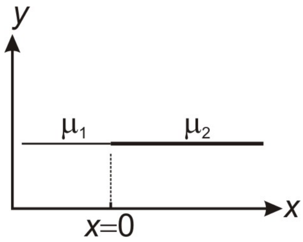
The last equation describes the transmitted wave.

The amplitude of the transmitted wave is changed by the factor t compared to the incoming wave.

The wave number is now k_2 as the wave propagates on the second rope.

t is called the transmission coefficient of the amplitude.

Reflection and Transmission 3: Strings



Boundary conditions at the connection of the strings

$$y_{in}(x=0,t)+y_r(x=0,t)=y_t(x=0,t)\rightarrow 1+r=t$$

$$\left.\frac{\partial y_{in}(x,t)}{\partial x}\right|_{x=0}+\left.\frac{\partial y_r(x,t)}{\partial x}\right|_{x=0}=\left.\frac{\partial y_t(x,t)}{\partial x}\right|_{x=0}\rightarrow k_1-k_1 r=k_2 t$$

Reflection and Transmission 3 Comment

The two equations describe the boundary conditions at the connection point of the two ropes.

The first equation says that the amplitudes to the left and right of the connection point of the two ropes must be the same.

The second equation says that the slopes of the wave functions to the left and right of the node must be the same.

Only then can the forces to the left and right of the connection point cancel each other out exactly.

Ideally, the connection point has no mass and the acceleration would be infinite if the forces did not cancel each other out.

With these two equations one finds the equations written in red for determining the reflection and transmission coefficient.

Reflection and Transmission 4: Strings

Reflection coefficient of the amplitude

$$r = \frac{k_1 - k_2}{k_1 + k_2}$$

Transmission coefficient of the amplitude

$$t = 1 + r = \frac{2k_1}{k_1 + k_2}$$

Remark.: These formulas are very general. E.g., if a light wave in a medium with the refractive index n_1 perpendicularly hits a boundary surface with a medium with the refractive index n_2, then the reflection and transmission coefficient is $r = \frac{n_1 - n_2}{n_1 + n_2}$ and $t = \frac{2n_1}{n_1 + n_2}$, since $v = \frac{\omega}{k} = \frac{c}{n}$ (c: speed of light) and $k \propto n$.

Reflection and Transmission 4 Comment

The equations outlined in red give the formulas for the reflection and transmission coefficients.

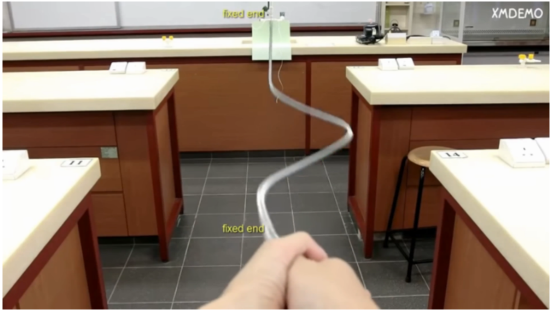
These formulas are not only valid for two connected ropes but also apply in general.

In this lecture I will only use the reflection coefficient of a light wave that hits an interface perpendicularly.

Reflection and Transmission 5: Strings

Reflection at a solid end: With $\mu_2 \to \infty$ and $v_2 \to 0$ results $\lambda_2 \to 0$ and $k_2 \to \infty$.

$$r = \frac{k_1 - k_2}{k_1 + k_2} \to -1$$



(ReflexionSeilwelle.mp4)

Reflection and Transmission 5 — Comment

Two special cases are particularly important.

One special case arises when the rope is attached to a wall. The other special case arises when the end of the rope is attached to a second rob with a negligible small mass per unit length.

If the rope is attached to a wall, the mass per unit length diverges beyond the point of connection.

The propagation speed of the wave in the wall is zero, the wavelength is zero and consequently the wave number k_2 diverges.

The reflection coefficient is -1 and the amplitude of the reflected wave changes sign compared to the incoming wave.

The video shows the effect very nicely.

Reflection and Transmission 6: Strings

Reflection at an open end: With $\mu_2 \to 0$ and $v_2 \to \infty$ one gets $\lambda_2 \to \infty$ and $k_2 \to 0$.

$$r = \frac{k_1 - k_2}{k_1 + k_2} \to +1$$

Reflection and Transmission 6 Comment

The speed of propagation of the transmitted wave becomes very high when the mass per unit length of the second rope is very small.

This makes the wavelength very large and the wave number k_2 tends towards zero.

The reflection coefficient tends towards $+1$, and the amplitude of the reflected wave does not change the sign compared to the incoming wave.

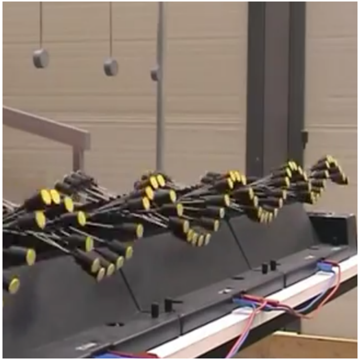
Standing waves

General Wave Phenomena

- Waves in one dimension
- Reflection and transmission
- Standing waves
- Waves in three dimensions
- Electromagnetic waves

Standing waves 1

A standing wave results when a reflected wave is superimposed on the incoming wave.



(laufendeWelleundreflektierteWelle.mp4)

Standing Waves 1: Comment

The video shows how a standing wave forms when the incoming wave is reflected at a fixed end.

Standing waves 2: Reflection at a solid end

reflection at $x = 0$:

$$y = y_{in} + y_r = y_0 \left\{ e^{i(kx-\omega t)} - e^{i(-kx-\omega t)} \right\} = y_0 2i \sin(kx) e^{-i\omega t}$$

and for the real part (real y_0)

$$\boxed{\mathrm{Re}(y) = 2y_0 \sin(kx) \sin(\omega t)}$$

The nodes of the standing wave are

$$\underline{kx = n\pi} = \frac{2\pi}{\lambda} x$$

and the distance to the reflexion point ℓ_n is

$$\boxed{\ell_n = n\frac{\lambda}{2}} \qquad n = 0, 1, 2, ..$$

Standing Waves 2: Comment

As a consequence of the superposition of the incoming and the reflected wave, the time dependency and the spatial variation of the resulting wave is described by the product of two separate functions.

The equation outlined in red gives the result of the superposition.

The running wave becomes a standing wave.

The nodes of the standing wave are determined by the underlined equation.

The distance between to nodes is half a wavelength.

Standing waves 3: Reflection at an open end

reflection at $x = 0$:

$$y = y_{\text{in}} + y_r = y_0 \left\{ e^{i(kx-\omega t)} + e^{i(-kx-\omega t)} \right\} = y_0 2\cos(kx) e^{-i\omega t}$$

and for the real part (real y_0)

$$\boxed{\text{Re}(y) = 2y_0 \cos(kx)\cos(\omega t)}$$

The nodes of the standing wave determined by the condition

$$\underline{kx = \frac{\pi}{2} + n\pi = \frac{\pi}{2}(1+2n)}$$

and the distance to the reflexion point ℓ_n is

$$kx = \frac{2\pi}{\lambda} x = \frac{\pi}{2}(1+2n) \rightarrow \boxed{\ell_n = \frac{\lambda}{4}(1+2n)} \quad n = 0, 1, 2, ..$$

Standing Waves 3 Comment

The first equation gives the superposition of the reflected and the incoming wave when the reflection occurs at an open end.

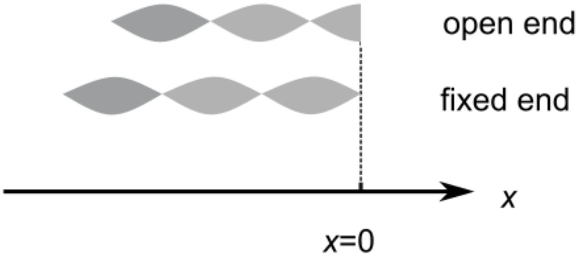
The time dependence and the spatial variation of the wave are described by the product of two cosine functions.

The underlined equations give the conditions for the nodes of the standing wave.

The nodes are again separated by half the wavelength.

The first node is observed a quarter wavelength away from the place where the reflection occurs.

Standing waves 4



Standing Waves 4 Comment

The figure illustrates these results.

The shaded areas indicate the local oscillations with the spatial variation of the amplitude.

As has been assumed in the calculations, the incoming wave comes from the left side and is reflected at $x = 0$.

Standing waves 5: resonances

(StehendeSeilwelle2.mp4)

Standing Waves 5 Comment

The energy of the wave cannot escape when the reflection occurs at two opposite ends.

The video shows a string with two fixed ends.

The waves are excited by a motor with a frequency of 50 Hz.

Depending on the tension of the strings, resonances occur in the form of standing waves.

When the resonance condition is not met, the string moves chaotically.

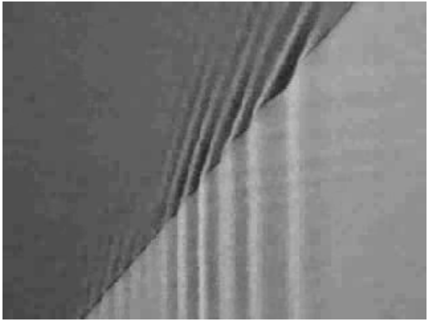
3D-waves

General Wave Phenomena

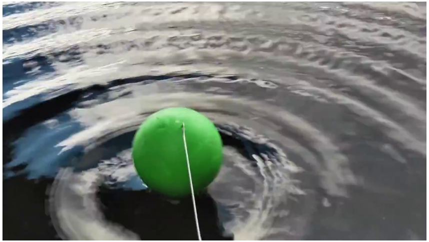
- Waves in one dimension
- Reflection and transmission
- Standing waves
- Waves in three dimensions
- Electromagnetic waves

Waves in three dimensions 1

The figures show surface waves on water



linear wavefront



circular wavefront

The phase of an harmonic wave is constant along a wavefront.

Waves in three dimensions 1 Comment

The figures show surface waves on water.

In the left figure, a linear paddle is used to excite parallel wave fronts.

In the right figure, a ball is used to stimulate angular wave fronts.

With harmonic excitation the phase of the wave increases from wave front to wave front by 2π.

The phase is constant along the linear and angular wavefronts.

Waves in three dimensions 2

The linear wavefront in two dimensions becomes a plane in three-dimensional space.

The circular wavefront in two dimensions becomes a sphere in three-dimensional space.

The phase of a spherical wave is

$$\varphi = kr - \omega t$$

r denotes the distance to the center.

Waves in three dimensions 2 Comment

The linear wavefront in two dimensions becomes a plane in three-dimensional space.

The resulting wave is called a plane wave.

The circular wavefront in two dimensions becomes a sphere in three-dimensional space.

The resulting wave is called a spherical wave.

The phase of a spherical wave corresponds to the phase of a one-dimensional wave, since the only free parameter is the radius of the sphere.

The coordinate x in the phase of the one-dimensional wave is replaced by the radius r of the sphere.

Waves in three dimensions 3

The phase of a plane wave is a plane equation

$$\varphi = \vec{k}\vec{r} - \omega t$$

The wave number k is replaced by the wave vector $\vec{k}$.

The wave vector is perpendicular to the planes of constant phase

$$\varphi_0 = \vec{k}\vec{r}_0 - \omega t_0 = \vec{k}\vec{r} - \omega t_0 \quad \text{and} \quad \vec{r} = \vec{r}_0 + \vec{r}_\perp$$

The magnitude of the wave vector is the wave number

$$|\vec{k}| = k = \frac{2\pi}{\lambda}$$

Waves in three dimensions 3 — Comment

The phase of a plane wave is the plane equation.

The coordinate x of the one dimensional wave is replaced by the vector $\vec{r}$ and the wave number k by the wave vector $\vec{k}$.

The orientation of the wave vector is perpendicular to the wave front.

The phase does not change if a vector is added to the vector r_0 that is perpendicular to the wave vector.

The magnitude of the wave vector corresponds to the wave number, since the phase has to change by 2π from wave front to wave front.

Plane waves are important because any wave front can always be approximated locally by a plane wave.

Waves in three dimensions 4

The wave function of a plane wave is

$$\psi(\vec{r},t) = \psi_0 e^{i\left(\vec{k}\vec{r}-\omega t\right)}$$

The wave equation in three dimension is

$$\frac{\partial^2\psi}{\partial t^2} = v^2\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right) \rightarrow \frac{\partial^2\psi}{\partial t^2} = v^2\nabla^2\psi$$

The definition of the nabla operator is

$$\nabla = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix} \quad \text{and} \quad \nabla^2 = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

Waves in three dimensions 4 Comment

The first formula outlined in red gives the function of a plane wave.

The next line shows the generalization of the one-dimensional wave equation.

To the partial derivative of the coordinate *x*, the partial derivatives of the coordinates *y* and *z* have to be added.

Usually this equation is abbreviated with the Nabla operator.

The last equation outlined in red gives the definition of the Nabla operator.

The square of the Nabla operator is the Laplace operator which is usually abbreviated by the symbol Δ.

Waves in three dimensions 5

With

$$\psi(\vec{r},t) = \psi_0 e^{i\left(\vec{k}\vec{r}-\omega t\right)}$$

and

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi$$

results the dispersion relation

$$\boxed{\omega^2 = v^2(k_x^2 + k_y^2 + k_z^2) = v^2 \vec{k}^2}$$

Waves in three dimensions 5 Comment

The relationship between the wave vector and the angular frequency of the wave results when the three-dimensional wave equation is applied to the wave function of a plane wave.

The relationship between the angular frequency and the wavenumber is called the dispersion relationship.

The three-dimensional wave equation is reduced to the one-dimensional wave equation when a plane wave propagates along one of the coordinate axes.

Waves in three dimensions 6: Sound waves

Sound waves: In addition to the random motion of the particles in a gas, there is also the coherent motion due to the wave

$\rightarrow$The displacement due to the wave

$$\vec{s}(\vec{r}, t)$$

is added to the motion of the particles

$\rightarrow$ and the sound pressure p_s is added to the static pressure p_0

$$p(\vec{r}, t) = p_0 + p_s(\vec{r}, t)$$

Waves in three dimensions 6

Comment

Sound waves are a typical example for three-dimensional waves.

The thermal motion of the gas particles is superimposed by the coherent motion due to the sound wave.

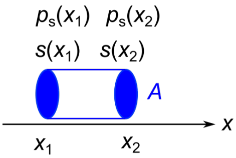
The sound wave causes an additional displacement of the particles.

This coherent additional shift locally changes the pressure of the gas.

The sound pressure is usually many orders of magnitude smaller than the static pressure of the gas.

Waves in three dimensions 7: Sound waves

For a plane sound wave propagating along the x axis, consider a small volume between the coordinates x_1 and x_2.



$p_s(x_1)$ and $p_s(x_2)$ denote the sound pressure and $s(x_1)$ and $s(x_2)$ denote the displacement of the end faces A caused by the sound wave.

With an isothermal variation of the volume, the Boyle-Mariotte law applies

$$pV = constant$$

Waves in three dimensions 7 Comment

To find the wave equation of sound waves, one can consider a plane wave propagating along the x-axis.

The sketch shows a small volume between the coordinates x_1 and x_2.

The forces on the end faces A due to the static pressure cancel each other out.

The sound wave causes the end faces to shift.

The volume between the coordinates x_1 and x_2 changes due to the displacement of the end faces, which causes a pressure change in the volume.

Newton was the first to calculate the speed of sound.

He assumed that the temperature of a gas would not be changed by a sound wave and used Boyle-Mariotte's law. But that's not true.

Waves in three dimensions 8: Sound waves

For an adiabatic variation of the volume holds the equation

$$pV^{\gamma} = \text{constant}$$

with $\gamma = 1.4$ for air

The variation of $pV^{\gamma} = \text{constant}$ is

$$\delta p V^{\gamma} + p\gamma V^{\gamma-1}\delta V = 0$$

with $p = p_0$, $\delta p = p_s$, i.e. $p_s = -\gamma \frac{p_0}{V}\delta V$ and

$$\delta V = A(s(x_2) - s(x_1)) = A(s(x_1) + \frac{\partial s}{\partial x}(x_2 - x_1) - s(x_1))$$

The relation between shift and sound pressure is

$$p_s = -\gamma p_0 \frac{\partial s}{\partial x}$$

Waves in three dimensions 8 Comment 1

A sound wave is an adiabatic process.

The rapid pressure variation changes the temperature locally.

The first equation outlined in red gives the relationship between volume and pressure in an adiabatic process.

The adiabatic coefficient γ of air is 1.4.

To find the relationship between volume change and pressure change, the adiabatic equation outlined in red must be varied.

If the pressure is varied, then the first term results and the second term results when the volume is varied.

Waves in three dimensions 8 — Comment 2

The variation of the constant on the right is zero.

The underlined equation shows the relationship between sound pressure and volume change δV.

The change in volume is due to the displacement of the end faces, which is assumed to be small.

The application of a Taylor series shows the relationship between sound pressure and the displacement of the end faces.

The sound pressure is proportional to the derivative of the displacement.

Waves in three dimensions 9: Sound waves

Newton's equation of motion

$$m\frac{\partial^2 s}{\partial t^2} = A(p_S(x_1) - p_S(x_2))$$

with

$$p_s(x_2) = p_s(x_1) + \frac{\partial p_s}{\partial x}(x_2 - x_1)$$

results

$$m\frac{\partial^2 s}{\partial t^2} = -A(x_2 - x_1)\frac{\partial p_s}{\partial x} \quad \text{and with} \quad \frac{\partial p_s}{\partial x} = -\gamma p_0 \frac{\partial^2 s}{\partial x^2}$$

with the volume $V = A(x_2 - x_1)$ and the mass density $\rho = m/V$ results the wave equation for the displacement s, which is caused by a sound wave

$$\frac{\partial^2 s}{\partial t^2} = \frac{\gamma p_0}{\rho}\frac{\partial^2 s}{\partial x^2}$$

Waves in three dimensions 9 Comment

The first equation outlined in red gives Newton’s equation of motion for the mass that is enclosed in the volume element.

The product of mass and acceleration results from the force acting on the volume element.

It does not matter whether the shift *s* of the left or right end face or an average is used, since the variation in the shift over the volume is small.

The force that acts on the mass in the volume element results from the difference between the forces that act on the left and right end face.

The underlined equation results from a Taylor expansion.

The wave equation for sound results from the derivative of the sound pressure.

Waves in three dimensions 10: Sound waves

speed of sound

$$c_S = \sqrt{\frac{\gamma p_0}{\rho}}$$

Wave equations of the shift and the sound pressure

$$\frac{\partial^2 \vec{s}}{\partial t^2} = c_S^2 \nabla^2 \vec{s}$$

$$\frac{\partial^2 p_s}{\partial t^2} = c_S^2 \nabla^2 p_s$$

Waves in three dimensions 10 Comment

The first equation outlined in red gives the speed of sound.

If this equation is extended with the area A of the end faces of the volume element, then formally the formula for the speed of a wave on a rope results.

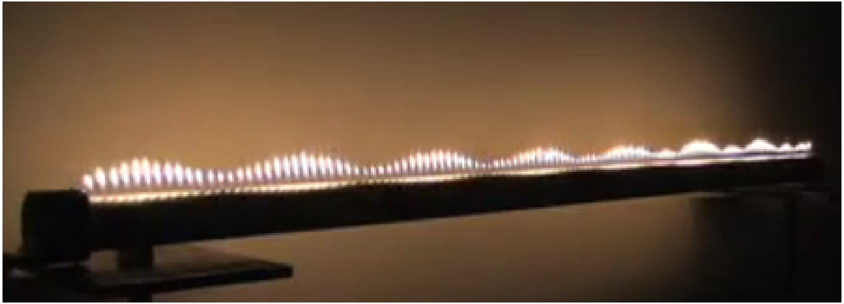
The second equations outlined in red give the general wave equations of sound waves.

The direction of propagation of the sound wave is no longer restricted to the x-axis.

There are three wave equations for the displacement since the displacement is a vector.

A little calculation gives the equation for the sound pressure.

Waves in three dimensions 11: Sound waves



(Flammenrohr.mp4)

Waves in three dimensions 11 Comment

As with waves on a rope, standing waves can also form with sound.

The standing waves can be amplified by resonance if the energy of the sound wave is limited to a finite range.

The video shows a standing wave that forms in a metal pipe.

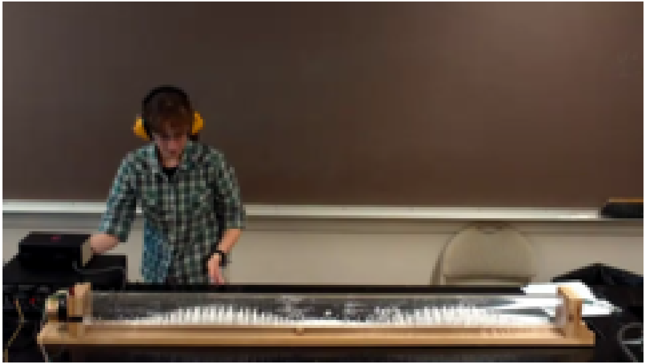
The sound waves are stimulated by a loudspeaker in front of the tube.

The tube is filled with a flammable gas.

Small flames over openings from which the gas can escape indicate the local gas pressure.

The flames make it visible that standing waves are forming for certain frequencies.

Waves in three dimensions 12: Sound waves



(KundtsTube.mp4)

Waves in three dimensions 12

Comment

The video shows a glass tube filled with small styrofoam balls.

The sound wave whirls up the small styrofoam balls, thereby indicating the coherent movement of the gas.

In addition, a strong high-frequency sound wave is applied to overcome the electrostatic forces between the small styrofoam balls.

Revision

Summary in questions

1. Write down the wave function for a general one-dimensional wave.
2. Write down the wave equation for a one-dimensional wave.
3. Write down the wave function for a one-dimensional harmonic wave.
4. Write down the formulas for the wave number and angular frequency.
5. Write down the formula for the speed of a rope wave.
6. Write down the formula for the reflection and transmission coefficient of the amplitude.
7. Give the reflection coefficient for the case that the end of the rope is firmly attached to a wall.
8. Give the reflection coefficient for the case that the rope is connected to a very light rope.
9. What is a standing wave and when do they arise?

Summary in questions

10. Write down the three-dimensional wave equation.
11. Write down the wave function of a plane wave.
12. Explain the properties of the wave vector $\vec{k}$.