Karlsruher Institut für Technologie

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# Computerpraktikum zur Vorlesung Moderne Methoden der Datenanalyse - Blatt 7

## Exercise 7: Confidence intervals

In the exercises of this sheet we consider a measured sample of events after applying quality cuts to separate signal from background events. The total number of observed events follows a Poisson distribution:

$$P(n_0|\nu_t) = \frac{\nu_t^{n_0} e^{-\nu_t}}{n_0!}, \tag{1}$$

$n_0$ is the number of observed events after the quality cuts, and $\nu_t = \nu_{t,S} + \nu_{t,B}$ is the true number of events expected to pass the cuts, where $\nu_{t,S}$ is the contribution from true signal events and $\nu_{t,B}$ the background remaining after the cuts.

- **Exercise 7.1: Significance** **voluntary**

  For a specific experiment, the background is expected to be $\nu_{t,B} = 1$ while the measurement is $n_0 = 3$.

  **a)** How often do you expect to measure $n_0$ or more events if you expect only background?

  **b)** What is the corresponding significance[1]?

  **c)** Is the measurement compatible with a statistical fluctuation of the background, or is there an evidence (defined as significance $\geq 3\,\sigma$ ) or a discovery of the signal ($\geq 5\,\sigma$)?

  **d)** How many events would have to be observed to reach the evidence and discovery thresholds?

- **Exercise 7.2: Confidence intervals** **obligatory**

  **a)** **Frequentist approach (Neyman construction)** This classical approach can be used to determine a confidence interval or lower/upper limit, respectively, at a given confidence level (CL).

[1] In physics, the significance is often expressed as "the number of sigmas" of a Gaussian distribution (= Z-value) corresponding to the p-value of observing a given signal under the hypothesis that there is only background. You can use, e.g., the function `scipy.stats.norm.ppf` for the calculation.

First, let us assume that the background is negligible: $\nu_{t,B} = 0$. Using the frequentist approach, we will compute a 90% CL upper limit on $\nu_t = \nu_{t,S}$ if the measured number of events is $n_0 = 3$.

Please inspect the related functions provided in the notebook for this exercise sheet. They implement the following confidence level definitions:

$$
\begin{aligned}
CL_{SB} &= \sum_{n=0}^{n_0} P(n|\nu_{t,S} + \nu_{t,B}), && (2)\\
CL_B &= \sum_{n=0}^{n_0} P(n|\nu_{t,B}), && (3)\\
CL_S &= CL_{SB}/CL_B, && (4)
\end{aligned}
$$

as well as the function `get_x`. The latter allows to determine a confidence limit using the definitions above. Use these predefined functions to solve the following part of the exercise.

- Determine the 90 % CL upper limit $\nu_{UL}$ such that:

$$
\sum_{n=0}^{n_0} P(n|\nu_{UL}) = 0.10. \qquad (5)
$$

- Now, consider the realistic case with background, i.e., $\nu_{t,B} > 0$:
  - Compute the 90% CL upper limits on $\nu_{t,S}$ as function of $\nu_{t,B}$. Assume values for the expected number of background events to be between 0 and 7. By convention, the upper limit on $\nu_{t,S}$ is the limit which would be obtained for $\nu_{t,S}$ without background minus the number of expected background events, i.e., $\nu_{t,B}$, ($CL_{SB}$-limit). This is implemented in the function `get_upper_poisson_limit` with $\nu_{t,B} > 0$.
    $CL_{SB}$ is the confidence level with respect to the signal-plus-background hypothesis and a measure for the compatibility of the experiment with this hypothesis. What makes this procedure inconvenient?
  - Make a plot of the $CL_{SB}$-limit on $\nu_{t,S}$ as a function of $\nu_{t,B}$ varying $n_0$ from 0 to 5 (draw all six curves in the same plot).
  - Using the predefined function `get_upper_poisson_limit_normalized` you can calculate the limit using the $CL_S$-method: $CL_S = CL_{SB}/CL_B$. $CL_B$ is a measure for the compatibility with the background-only hypothesis. The $CL_S$-method provides a useful limit, even if the number of observed events is much smaller than the expected background. Compute the 90% CL upper limits on $\nu_{t,S}$ now using this method and again as function of $\nu_{t,B}$ with $\nu_{t,B} \in [0, 7]$.
  - Also plot the $CL_S$-limit on $\nu_{t,S}$ as a function of $\nu_{t,B}$ varying $n_0$ from 0 to 5.

**b) Likelihood approach**

The likelihood function for a Poisson process for one single measurement is:

$$
L(n_0|\nu_t) = \frac{\nu_t^{n_0} e^{-\nu_t}}{n_0!}. \qquad (6)
$$

where $n_0 = 3$ is the number of measured events.

- Draw the curve $-2 \ln L$ as function of $\nu_t$, performing a scan over a meaningful range of values. Where is the minimum, $-2 \ln L_{min}$, of this curve?
- Calculate the boundaries of the confidence interval at 68% CL. These are the points for which $2 \cdot \Delta \ln L = 1$, where $\Delta \ln L$ is defined as $\ln L_{min} - \ln L$.
- Determine the 90% CL upper limit, i.e. the point with $\nu_t > n_0$ for which $2 \cdot \Delta \ln L = (1.28)^2$.
  Note: To translate a CL into the proper $\Delta \ln L$ cut for a **one-sided** interval, the following relation is used:

$$2 \cdot \Delta \ln L = (\sqrt{2} \cdot \texttt{erfinv}(2 \cdot \mathrm{CL} - 1))^2, \tag{7}$$

  where `erfinv` is the inverse of the error function provided for instance by the `scipy` method scipy.special.erfinv. The term which is passed on as argument to the `erfinv` function is equal to $1 - 2 \cdot (1 - CL) = 2 \cdot CL - 1$.
- Plot the likelihood function and the convidence intervals of interest.

**c) Bayesian approach**

The Bayesian posterior probability $P(\nu_t|n_0)$ is given by the Bayes theorem:

$$P(\nu_t|n_0) = \frac{L(n_0|\nu_t)\ \Pi(\nu_t)}{\int_{\text{all } \nu_t} L(n_0|\nu_t)\ \Pi(\nu_t) d\nu_t}. \tag{8}$$

$\Pi(\nu_t)$ is called the prior probability on $\nu_t$ and describes our prior belief about the distribution of this parameter. Try two different priors and compare the results:

(i) $\Pi(\nu_t) = const.$ for $\nu_t > 0$ and 0 otherwise,

(ii) $\Pi(\nu_t)$ proportional to $1/\nu_t$ for $\nu_t > 0$ and 0 otherwise.

- Compute and draw the posterior probability for $n_0 = 3$ and $\nu_{t,B} = 0$.
- What are the 90% credibility upper limits?

**Finally, compare the upper limits of the methods a), b), c) for $n_0 = 3$ and $\nu_{t,B} = 0$.**