

Verantwortlich für Vorlesung bzw. Computerpraktikum:

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Computerpraktikum zur Vorlesung Moderne Methoden der Datenanalyse - Blatt 7

Exercise 7: Confidence intervals

In the exercises of this sheet we consider a measured sample of events after applying quality cuts to separate signal from background events. The total number of observed events follows a Poisson distribution:

$$P(n_0|\nu_t) = \frac{\nu_t^{n_0} e^{-\nu_t}}{n_0!}, \quad (1)$$

n_0 is the number of observed events after the quality cuts, and $\nu_t = \nu_{t,S} + \nu_{t,B}$ is the true number of events expected to pass the cuts, where $\nu_{t,S}$ is the contribution from true signal events and $\nu_{t,B}$ the background remaining after the cuts.

• Exercise 7.1: Significance**voluntary**

For a specific experiment, the background is expected to be $\nu_{t,B} = 1$ while the measurement is $n_0 = 3$.

- How often do you expect to measure n_0 or more events if you expect only background?
- What is the corresponding significance¹?
- Is the measurement compatible with a statistical fluctuation of the background, or is there an evidence (defined as significance $\geq 3\sigma$) or a discovery of the signal ($\geq 5\sigma$)?
- How many events would have to be observed to reach the evidence and discovery thresholds?

• Exercise 7.2: Confidence intervals**obligatory**

- Frequentist approach (Neyman construction)** This classical approach can be used to determine a confidence interval or lower/upper limit, respectively, at a given confidence level (CL).

¹In physics, the significance is often expressed as “the number of sigmas” of a Gaussian distribution (= Z-value) corresponding to the p-value of observing a given signal under the hypothesis that there is only background. You can use, e.g., the function `scipy.stats.norm.ppf` for the calculation.

First, let us assume that the background is negligible: $\nu_{t,B} = 0$. Using the frequentist approach, we will compute a 90% CL upper limit on $\nu_t = \nu_{t,S}$ if the measured number of events is $n_0 = 3$.

Please inspect the related functions provided in the notebook for this exercise sheet. They implement the following confidence level definitions:

$$CL_{SB} = \sum_{n=0}^{n_0} P(n|\nu_{t,S} + \nu_{t,B}), \quad (2)$$

$$CL_B = \sum_{n=0}^{n_0} P(n|\nu_{t,B}), \quad (3)$$

$$CL_S = CL_{SB}/CL_B, \quad (4)$$

as well as the function `get_x`. The latter allows to determine a confidence limit using the definitions above. Use these predefined functions to solve the following part of the exercise.

- Determine the 90 % CL upper limit ν_{UL} such that:

$$\sum_{n=0}^{n_0} P(n|\nu_{UL}) = 0.10. \quad (5)$$

- Now, consider the realistic case with background, i.e., $\nu_{t,B} > 0$:
 - ◊ Compute the 90% CL upper limits on $\nu_{t,S}$ as function of $\nu_{t,B}$. Assume values for the expected number of background events to be between 0 and 7. By convention, the upper limit on $\nu_{t,S}$ is the limit which would be obtained for $\nu_{t,S}$ without background minus the number of expected background events, i.e., $\nu_{t,B}$, (CL_{SB} -limit). This is implemented in the function `get_upper_poisson_limit` with $\nu_{t,B} > 0$.
 CL_{SB} is the confidence level with respect to the signal-plus-background hypothesis and a measure for the compatibility of the experiment with this hypothesis. What makes this procedure inconvenient?
 - ◊ Make a plot of the CL_{SB} -limit on $\nu_{t,S}$ as a function of $\nu_{t,B}$ varying n_0 from 0 to 5 (draw all six curves in the same plot).
 - ◊ Using the predefined function `get_upper_poisson_limit_normalized` you can calculate the limit using the CL_S -method: $CL_S = CL_{SB}/CL_B$. CL_B is a measure for the compatibility with the background-only hypothesis. The CL_S -method provides a useful limit, even if the number of observed events is much smaller than the expected background. Compute the 90% CL upper limits on $\nu_{t,S}$ now using this method and again as function of $\nu_{t,B}$ with $\nu_{t,B} \in [0, 7]$.
 - ◊ Also plot the CL_S -limit on $\nu_{t,S}$ as a function of $\nu_{t,B}$ varying n_0 from 0 to 5.

b) Likelihood approach

The likelihood function for a Poisson process for one single measurement is:

$$L(n_0|\nu_t) = \frac{\nu_t^{n_0} e^{-\nu_t}}{n_0!}. \quad (6)$$

where $n_0 = 3$ is the number of measured events.

- Draw the curve $-2 \ln L$ as function of ν_t , performing a scan over a meaningful range of values. Where is the minimum, $-2 \ln L_{min}$, of this curve?
- Calculate the boundaries of the confidence interval at 68% CL. These are the points for which $2 \cdot \Delta \ln L = 1$, where $\Delta \ln L$ is defined as $\ln L_{min} - \ln L$.
- Determine the 90% CL upper limit, i.e. the point with $\nu_t > n_0$ for which $2 \cdot \Delta \ln L = (1.28)^2$.

Note: To translate a CL into the proper $\Delta \ln L$ cut for a **one-sided** interval, the following relation is used:

$$2 \cdot \Delta \ln L = (\sqrt{2} \cdot \text{erfinv}(2 \cdot \text{CL} - 1))^2, \quad (7)$$

where `erfinv` is the inverse of the error function provided for instance by the `scipy` method `scipy.special.erfinv`. The term which is passed on as argument to the `erfinv` function is equal to $1 - 2 \cdot (1 - \text{CL}) = 2 \cdot \text{CL} - 1$.

- Plot the likelihood function and the confidence intervals of interest.

c) Bayesian approach

The Bayesian posterior probability $P(\nu_t | n_0)$ is given by the Bayes theorem:

$$P(\nu_t | n_0) = \frac{L(n_0 | \nu_t) \Pi(\nu_t)}{\int_{\text{all } \nu_t} L(n_0 | \nu_t) \Pi(\nu_t) d\nu_t}. \quad (8)$$

$\Pi(\nu_t)$ is called the prior probability on ν_t and describes our prior belief about the distribution of this parameter. Try two different priors and compare the results:

- $\Pi(\nu_t) = \text{const.}$ for $\nu_t > 0$ and 0 otherwise,
 - $\Pi(\nu_t)$ proportional to $1/\nu_t$ for $\nu_t > 0$ and 0 otherwise.
- Compute and draw the posterior probability for $n_0 = 3$ and $\nu_{t,B} = 0$.
 - What are the 90% credibility upper limits?

Finally, compare the upper limits of the methods a), b), c) for $n_0 = 3$ and $\nu_{t,B} = 0$.