Propagating Electromagnetic Fields

Wave equation in homogeneous dielectric media ($\varepsilon = \text{const}$ and $\mu = \text{const}$):

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0 \qquad \Delta\vec{E} = \begin{pmatrix} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) E_x \\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) E_y \\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) E_z \end{pmatrix}$$

Wanted: $\vec{E}(\vec{r},t)$ solving the differential equation.

Ansatz: (plane wave)

$$\vec{E}(\vec{r},t) = \vec{E}_0\cos(\vec{k}\vec{r}\pm\omega t)$$

Inserting:

$$-\vec{k}^2\,\cancel{\vec{E}_0\cos(\vec{k}\vec{r}\pm\omega t)} + \frac{1}{c^2}\omega^2\,\cancel{\vec{E}_0\cos(\vec{k}\vec{r}\pm\omega t)} = 0$$

Dispersion relation of light

$$c = \frac{\omega}{|\vec{k}|}$$

Propagating Electromagnetic Fields

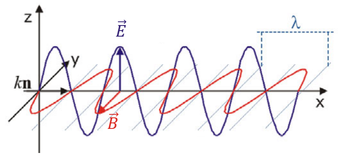
In which way is $\vec{E}$ related to $\vec{B}$ regarding magnitude and direction?

From wave equations:

$$\vec{E}(\vec{r},t) = \vec{E}_0\cos(\vec{k}\vec{r}-\omega t)$$
$$\vec{B}(\vec{r},t) = \vec{B}_0\cos(\vec{k}\vec{r}-\omega t)$$

Induction law: $\operatorname{curl}\vec{E} = -\frac{\partial}{\partial t}\vec{B}$

$$\left.\begin{aligned} \operatorname{curl}\vec{E} &= -\vec{k}\times\vec{E}_0\sin(\vec{k}\vec{r}-\omega t) \\ \frac{\partial}{\partial t}\vec{B} &= \omega\cdot\vec{B}_0\sin(\vec{k}\vec{r}-\omega t) \end{aligned}\right\} \Rightarrow \quad \boxed{\vec{k}\times\vec{E} = \omega\vec{B}} \qquad \boxed{E = c\cdot B}$$



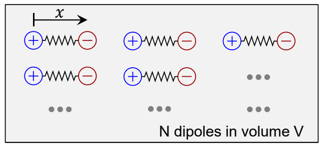
The vectors $\vec{k}, \vec{E}$ and $\vec{B}$ form a **right-handed trihedron**

Light in Matter

Lorentz-Model (one-dimensional)

Simplified: Matter is made of heavy particles of positive charge $+Q$ and light particles of negative charge $-Q$ (electrons). The electric field $\vec{E}$ induces an electric dipole moment $\vec{p}$ through the force $F = -Q\cdot E$ and deflection of the electrons.

$$p = -Q\cdot x$$



Polarisation

$$P = \frac{1}{V}\sum_{i=1}^{N} p_i = \frac{N}{V}\cdot p$$

Newton's equation of motion

$$m\ddot{x} + (D_1 x + \cancel{D_2 x^2 + D_3 x^3 + \ldots}) = -Q\,E$$

with mass m and force constants D_1, D_2, D_3, etc.

Approximation: E is small → x is small → Hooke's law → harmonic oscillator.

Polarisation and Susceptibility

Thus

$$m\ddot{x} + D_1 x = -Q\,E \quad ; \quad E(t) = E_0\cos(\omega t)$$

Ansatz: $x(t) = x_0\cos(\omega t)$

$$x_0 = -\frac{Q}{m}\frac{E_0}{\Omega^2-\omega^2} \qquad \text{with} \quad \Omega = \sqrt{\frac{D_1}{m}} \quad \text{(eigenfrequency)}$$

Polarisation

$$P = \frac{N}{V}(-Q\,x) = \frac{NQ^2}{Vm}\frac{1}{\Omega^2-\omega^2}\cdot E \quad =: \quad \varepsilon_0\chi\cdot E$$

Optical Susceptibility

$$\chi(\omega) = \frac{NQ^2}{Vm\varepsilon_0}\frac{1}{\Omega^2-\omega^2}$$

Dielectric Function

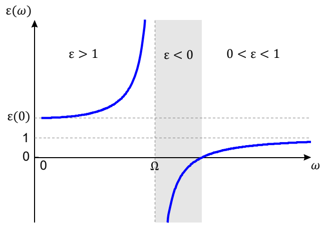
$$\begin{aligned} D &= \varepsilon_0 E + P \\ &= \varepsilon_0 E + \varepsilon_0 \chi E \\ &= \varepsilon_0 (1+\chi) E \quad =: \quad \varepsilon_0 \varepsilon E \end{aligned}$$

then

$$\varepsilon(\omega) = 1 + \chi(\omega)$$

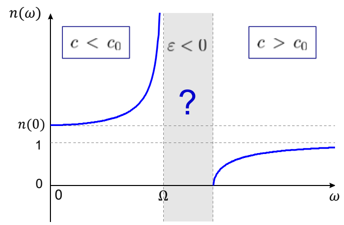
and therefore

$$\boxed{\varepsilon(\omega) = 1 + \frac{NQ^2}{V m \varepsilon_0} \frac{1}{\Omega^2 - \omega^2}}$$



Refractive Index Lorentz-Oscillator

$$\boxed{n(\omega) = \sqrt{\varepsilon(\omega)}}$$



Plane wave (complex)

$$\begin{aligned} E(x,t) &= E_0 \cos(kx - \omega t) + i\, E_0 \sin(kx - \omega t) \\ &= E_0\, e^{i(kx - \omega t)} \qquad ; \quad \frac{c_0}{n} = \frac{\omega}{k} \quad \rightarrow \quad k = \frac{\omega}{c_0} n \end{aligned}$$

$$(\ n \text{ is complex: } n = n' + in''\ )$$

Insert:

$$E(x,t) = E_0\, e^{i(\frac{\omega}{c_0} n x - \omega t)} = E_0\, e^{i(\frac{\omega}{c_0} n' x - \omega t)} \cdot e^{-\frac{\omega}{c_0} n'' x}$$

Real part:

$$\Re\{E(x,t)\} = E_0 \cdot \underbrace{\cos(k_0\, n' x - \omega t)}_{\text{propagating wave}} \cdot \underbrace{e^{-k_0\, n'' x}}_{\text{decaying}} \quad ; \quad k_0 = \frac{\omega}{c_0} = \frac{2\pi}{\lambda_0}$$

In a space of negative permittivity and thus imaginary refractive index the amplitude of an electromagnetic wave decays exponentially [$\sim \exp(-\frac{\alpha}{2} x)$]. The **penetration depth *d*** of the intensity is given by

$$d = \frac{1}{\alpha} = \frac{c_0}{2\omega\, n''} = \frac{1}{4\pi} \frac{\lambda_0}{n''}$$

Imaginary Part of Refractive Index

An imaginary refractive index may have two different physical causes:

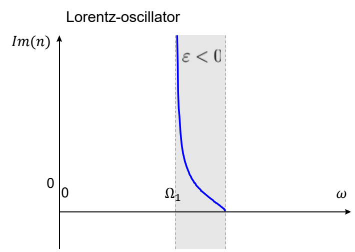
1) Absorption

Then α is called **absorption coefficient** and it is

$$S(x) = S(0)\, e^{-\alpha x}$$

2) Total reflection

This is the case for the discussed Lorentz-oscillator (without damping!). The decaying wave is called **evanescent field**.



Complex Refractive Index

How is real and imaginary part of $n = \sqrt{\varepsilon}$ is calculated, if $\varepsilon = \varepsilon' + i\varepsilon''$ is given?

$$
\begin{aligned}
n &= n' + in'' \\
n^2 &= n'^2 - n''^2 + i\,2n'n'' \quad \overset{!}{=} \quad \varepsilon' + i\varepsilon''
\end{aligned}
$$

Then

$$
\begin{aligned}
\varepsilon' &= n'^2 - n''^2 \\
\varepsilon'' &= 2n'n''
\end{aligned}
$$

Solving for n' and n'' with boundary conditions $n'^2 > 0$ and $n''^2 > 0$

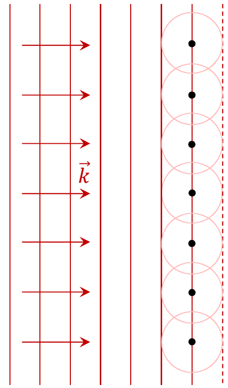
$$
\begin{aligned}
n'^2 &= \frac{1}{2}\left(\sqrt{\varepsilon'^2 + \varepsilon''^2} + \varepsilon'\right) \\
n''^2 &= \frac{1}{2}\left(\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon'\right)
\end{aligned}
$$

Diffraction and Huygens‘ Principle

The Fresnel-Huygens principle

At a given time each point of a wavefront is the source of a secondary wave (≙ spherical wave) with the same frequency as the primary wave. The new location of the wavefront results from the interference of all secondary waves.

Huygens (1678); Fresnel (1816)



Diffraction and Huygens‘ Principle