Solution to Problem Set 10
Nonlinear Optics (NLO)

Optical Parametric Amplifier (OPA)

Optical parametric amplifiers (OPA) exploit difference-frequency generation (DFG) in a second-order nonlinear crystal. In this process, a strong external pump at frequency ω_p is used to amplify a signal at frequency ω_s, thereby generating a new, so-called idler wave at frequency $\omega_i = \omega_p - \omega_s$. All waves are linearly polarized and monochromatic:

$$\mathbf{E}_{\mathrm{p,s,i}}(z,t) = \frac{1}{2}\left(\underline{E}_{\mathrm{p,s,i}}(z)\, e^{\mathrm{j}\left(\omega_{\mathrm{p,s,i}}t - k_{\mathrm{p,s,i}}z\right)}\mathbf{e}_{\mathrm{p,s,i}} + c.c.\right), \tag{0.1}$$

where the subscripts *p*, *s*, and *i* refer to the pump, signal and idler wave. Assuming a fixed set of polarizations $\mathbf{e}_{\mathrm{p,s,i}}$, we can replace the second-order nonlinear tensor by its effective value d_{eff} according to eq. (3.50) in the lecture notes. The interaction of the three waves is then given by the following system of coupled differential equations with $k_m = \dfrac{\omega_m}{cn\left(\omega_m\right)}$:

$$\frac{\partial \underline{E}\left(z,t,\omega_p\right)}{\partial z} = -\mathrm{j}k_p d_{\mathrm{eff}}\, \underline{E}\left(z,t,\omega_i\right)\underline{E}\left(z,t,\omega_s\right)e^{-\mathrm{j}\Delta kz} \tag{0.2}$$

$$\frac{\partial \underline{E}\left(z,t,\omega_s\right)}{\partial z} = -\mathrm{j}k_s d_{\mathrm{eff}}\, \underline{E}\left(z,t,\omega_p\right)\underline{E}^*\left(z,t,\omega_i\right)e^{\mathrm{j}\Delta kz} \tag{0.3}$$

$$\frac{\partial \underline{E}\left(z,t,\omega_i\right)}{\partial z} = -\mathrm{j}k_i d_{\mathrm{eff}}\, \underline{E}\left(z,t,\omega_p\right)\underline{E}^*\left(z,t,\omega_s\right)e^{\mathrm{j}\Delta kz} \tag{0.4}$$

1. Calculate the evolution of the signal and the idler amplitudes along the propagation direction under the assumption of perfect phase matching, $\Delta k = 0$. Assume further that the pump is much stronger than the signal and the idler and that we can therefore neglect pump depletion. Sketch the intensities of signal and idler wave along the propagation direction *z*. Derive an expression for the amplification (power gain) of the signal wave.

2. As a second-order nonlinear material we use Beta Barium Borate (BBO), a negative-uniaxial crystal for which we can achieve phase matching by exploiting birefringence. Let us assume that all three waves propagate in the same direction, which defines an angle θ with the optical axis. For type-2 phase matching, the signal and the idler waves propagate in different normal modes with orthogonal polarization states. Formulate the phase matching condition for the refractive indices of the various waves, assuming that the signal is propagating in ordinary polarization, whereas the idler and the pump are in extraordinary polarization and experience angle-dependent refractive indices.

3. For BBO the dispersion relations for the ordinary and extraordinary in the range of 0.64 µm to 3.18 µm are given by (λ in µm):

$$
\begin{aligned}
n_o^2(\lambda) &= 2.7359 + \frac{0.01878}{\lambda^2 - 0.01822} - 0.01471\lambda^2, \\
n_e^2(\lambda) &= 2.3753 + \frac{0.01224}{\lambda^2 - 0.01667} - 0.01627\lambda^2.
\end{aligned}
\tag{0.5}
$$

   The pump wavelength is 800 nm. Assuming the configuration described in question 2, plot the phase matching angle θ as a function of the signal wavelength in the region between 1100 nm and 1600 nm. Use a math software package (e.g. MATLAB) for this plot.

The following two parts are slightly more challenging. They will not be considered for the bonus system, but provide additional insight.

4. The effective nonlinear coefficient d_{eff} for BBO (symmetry group 3m or C_{3V}) as a function of the angles φ and θ is given in the following paper:

   D. N. Nikogosyan. *Beta Barium Borate (BBO) - A Review of Its Properties and Applications*. Appl. Phys. A **52** (1991) 359-386.

   From within the KIT network you can download this paper here[1]. Identify the relevant relations and determine d_{eff} for the calculated type-2 phase matching for a signal wavelength of 1500 nm. Choose an angle φ that maximizes d_{eff}.

5. A signal at a wavelength of 1500 nm propagates through the crystal along with a pump wave of intensity $I_{\text{p}} = 40\,\frac{\text{GW}}{\text{cm}^2}$. Calculate the amplification that the signal experiences if the interaction length within the nonlinear crystal amounts to 2 mm.

Solution

1. We are interested in how the signal and the idler intensities evolve. To solve the differential equations, we decouple eq. (0.3) and eq. (0.4) from each other by taking the respective 2nd derivatives. We directly take into account the assumed phase matching ($\Delta k = 0$) and the non-depletion constraint of the pump wave ($\frac{\partial \underline{E}_{\text{p}}(z,t,\omega_{\text{p}})}{\partial z} = 0 \;\Rightarrow\; \underline{E}_{\text{p}}(z,t,\omega_{\text{p}}) = \underline{E}_{\text{p},0} = \text{const.}$):

[1] http://link.springer.com/article/10.1007/BF00323647

$$\frac{\partial^2 \underline{E}_{\mathrm{s}}\left(z,t,\omega_{\mathrm{s}}\right)}{\partial z^2} = -\mathrm{j}\frac{\omega_{\mathrm{s}}}{cn\left(\omega_{\mathrm{s}}\right)} d_{\mathrm{eff}} \left(\frac{\partial \underline{E}_{\mathrm{p}}\left(z,t,\omega_{\mathrm{p}}\right)}{\partial z} \underline{E}_{\mathrm{i}}^*\left(z,t,\omega_{\mathrm{i}}\right) + \frac{\partial \underline{E}_{\mathrm{i}}^*\left(z,t,\omega_{\mathrm{i}}\right)}{\partial z} \underline{E}_{\mathrm{p}}\left(z,t,\omega_{\mathrm{p}}\right) \right)$$
$$= -\mathrm{j}\frac{\omega_{\mathrm{s}}}{cn\left(\omega_{\mathrm{s}}\right)} d_{\mathrm{eff}} \frac{\partial \underline{E}_{\mathrm{i}}^*\left(z,t,\omega_{\mathrm{i}}\right)}{\partial z} \underline{E}_{\mathrm{p},0} \tag{0.6}$$

$$\frac{\partial^2 \underline{E}_{\mathrm{i}}\left(z,t,\omega_{\mathrm{s}}\right)}{\partial z^2} = -\mathrm{j}\frac{\omega_{\mathrm{i}}}{cn\left(\omega_{\mathrm{i}}\right)} d_{\mathrm{eff}} \left(\frac{\partial \underline{E}_{\mathrm{p}}\left(z,t,\omega_{\mathrm{p}}\right)}{\partial z} \underline{E}_{\mathrm{s}}^*\left(z,t,\omega_{\mathrm{s}}\right) + \frac{\partial \underline{E}_{\mathrm{s}}^*\left(z,t,\omega_{\mathrm{s}}\right)}{\partial z} \underline{E}_{\mathrm{p}}\left(z,t,\omega_{\mathrm{p}}\right) \right)$$
$$= -\mathrm{j}\frac{\omega_{\mathrm{i}}}{cn\left(\omega_{\mathrm{i}}\right)} d_{\mathrm{eff}} \frac{\partial \underline{E}_{\mathrm{s}}^*\left(z,t,\omega_{\mathrm{s}}\right)}{\partial z} \underline{E}_{\mathrm{p},0} \tag{0.7}$$

We obtain the following decoupled differential equations for signal and idler amplitudes by inserting the respective first derivatives into the second order differential equation:

$$\frac{\partial^2 \underline{E}_{\mathrm{s}}\left(z,t,\omega_{\mathrm{s}}\right)}{\partial z^2} = -\mathrm{j}\frac{\omega_{\mathrm{s}}}{cn\left(\omega_{\mathrm{s}}\right)} d_{\mathrm{eff}} \underline{E}_{\mathrm{p},0} \left(-\mathrm{j}\frac{\omega_{\mathrm{i}}}{cn\left(\omega_{\mathrm{i}}\right)} d_{\mathrm{eff}} \underline{E}_{\mathrm{p},0} \underline{E}_{\mathrm{s}}^*\left(z,t,\omega_{\mathrm{s}}\right) \right)^*$$
$$= \frac{\omega_{\mathrm{s}}}{cn\left(\omega_{\mathrm{s}}\right)} \frac{\omega_{\mathrm{i}}}{cn\left(\omega_{\mathrm{i}}\right)} d_{\mathrm{eff}}^2 \left|\underline{E}_{\mathrm{p},0}\right|^2 \underline{E}_{\mathrm{s}}\left(z,t,\omega_{\mathrm{s}}\right) \tag{0.8}$$

$$\frac{\partial^2 \underline{E}_{\mathrm{i}}\left(z,t,\omega_{\mathrm{s}}\right)}{\partial z^2} = \frac{\omega_{\mathrm{s}}}{cn\left(\omega_{\mathrm{s}}\right)} \frac{\omega_{\mathrm{i}}}{cn\left(\omega_{\mathrm{i}}\right)} d_{\mathrm{eff}}^2 \left|\underline{E}_{\mathrm{p},0}\right|^2 \underline{E}_{\mathrm{i}}\left(z,t,\omega_{\mathrm{s}}\right) \tag{0.9}$$

With $\kappa = \sqrt{\frac{\omega_{\mathrm{s}}}{cn\left(\omega_{\mathrm{s}}\right)} \frac{\omega_{\mathrm{i}}}{cn\left(\omega_{\mathrm{i}}\right)} d_{\mathrm{eff}}^2 \left|\underline{E}_{\mathrm{p},0}\right|^2}$, the general solution to this problem is given by:

$$\underline{E}_{\mathrm{s}}\left(z,t,\omega_{\mathrm{s}}\right) = A\cosh(\kappa z) + B\sinh(\kappa z) \tag{0.10}$$

$$\underline{E}_{\mathrm{i}}\left(z,t,\omega_{\mathrm{s}}\right) = C\cosh(\kappa z) + D\sinh(\kappa z) \tag{0.11}$$

In an amplification scheme, the most common situation would be that initially there is only a pump wave and a signal wave to be amplified. The idler amplitude is initially zero and is then generated by the interaction of pump and signal wave. So we end up with the boundary conditions $\underline{E}_{\mathrm{s}}\left(0,t,\omega_{\mathrm{s}}\right) = \underline{E}_{\mathrm{s},0}$ and $\underline{E}_{\mathrm{i}}\left(0,t,\omega_{\mathrm{i}}\right) = 0$. We immediately see that $A = \underline{E}_{\mathrm{s},0}$; $B = 0$ because $\frac{\partial \underline{E}_{\mathrm{s}}\left(0,t,\omega_{\mathrm{s}}\right)}{\partial z} \sim \underline{E}_{\mathrm{i}}\left(0,t,\omega_{\mathrm{i}}\right) = 0$ and therefore:

$$\underline{E}_{\mathrm{s}}\left(z,t,\omega_{\mathrm{s}}\right) = \underline{E}_{\mathrm{s},0}\cosh(\kappa z) \tag{0.12}$$

We can find $\underline{E}_i$ by using equation (0.3):

$$\underline{E}_\mathrm{i} = \left(\mathrm{j}\frac{cn(\omega_\mathrm{s})}{\omega_\mathrm{s} d_\mathrm{eff}\underline{E}_\mathrm{p,0}} \frac{\partial \underline{E}_\mathrm{s}}{\partial z} \right)^* = -\mathrm{j}\frac{\kappa cn(\omega_\mathrm{s})}{\omega_\mathrm{s} d_\mathrm{eff}\underline{E}^*_\mathrm{p,0}} \underline{E}^*_\mathrm{s,0}\sinh(\kappa z)$$

$$= -\mathrm{j}\sqrt{\frac{\omega_\mathrm{s}}{cn(\omega_\mathrm{s})}\frac{\omega_\mathrm{i}}{cn(\omega_\mathrm{i})} d^2_\mathrm{eff}\left|\underline{E}_\mathrm{p,0}\right|^2}\, \frac{cn(\omega_\mathrm{s})}{\omega_\mathrm{s} d_\mathrm{eff}\underline{E}^*_\mathrm{p,0}} \underline{E}^*_\mathrm{s,0}\sinh(\kappa z) \qquad (0.13)$$

$$= -\frac{\mathrm{j}}{\underline{E}^*_\mathrm{p,0} \big/ \left|\underline{E}_\mathrm{p,0}\right|}\sqrt{\frac{n(\omega_\mathrm{s})}{n(\omega_\mathrm{i})}}\sqrt{\frac{\omega_\mathrm{i}}{\omega_\mathrm{s}}}\underline{E}^*_\mathrm{s,0}\sinh(\kappa z)$$

As we assumed phase matching, it means that the refractive indices of signal and idler cannot differ too much from each other (careful, only true for this particular phase matching condition!), such that $\frac{n(\omega_s)}{n(\omega_i)} \approx 1$. The intensities of signal and idler wave are then:

$$I_\mathrm{s}(z) = I_\mathrm{s,0}\cosh(\kappa z)^2 \qquad (0.14)$$

$$I_\mathrm{i}(z) = \frac{n(\omega_\mathrm{s})}{n(\omega_\mathrm{i})}\frac{\omega_\mathrm{i}}{\omega_\mathrm{s}} I_\mathrm{s,0}\sinh(\kappa z)^2 \approx \frac{\omega_\mathrm{i}}{\omega_\mathrm{s}} I_\mathrm{s,0}\sinh(\kappa z)^2 \qquad (0.15)$$

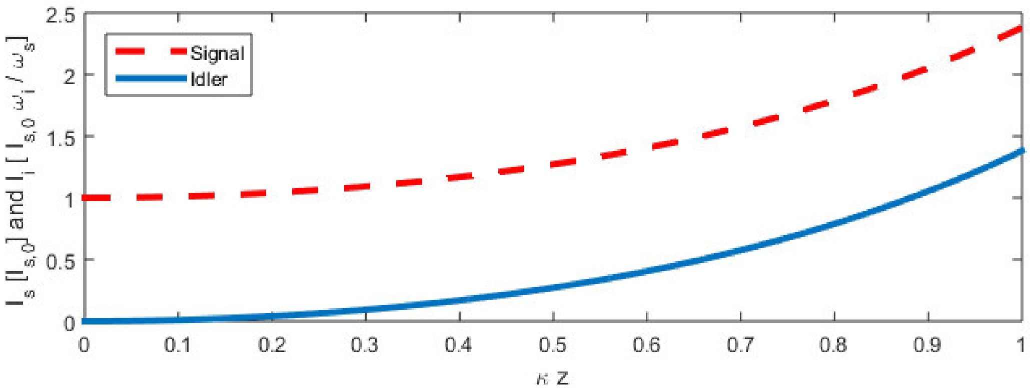


Figure 1: Signal and idler intensity over propagation through the OPA. The signal has an initial value unequal zero and is being further amplified, while the idler starts from zero.

2. In the given set of orientations, we can write the phase matching condition as:

$$\Delta k = k_\mathrm{i} + k_\mathrm{s} - k_\mathrm{p} = 0 \Leftrightarrow$$
$$\frac{\omega_\mathrm{i}}{c} n_\mathrm{e'}(\omega_i, \theta) + \frac{\omega_\mathrm{s}}{c} n_\mathrm{o}(\omega_\mathrm{s}) - \frac{\omega_\mathrm{p}}{c} n_\mathrm{e'}(\omega_\mathrm{p}, \theta) = 0 \Leftrightarrow \qquad (0.16)$$
$$\frac{1}{\lambda_\mathrm{i}} n_\mathrm{e'}(\omega_i, \theta) + \frac{1}{\lambda_\mathrm{s}} n_\mathrm{o}(\omega_\mathrm{s}) - \frac{1}{\lambda_\mathrm{p}} n_\mathrm{e'}(\omega_\mathrm{p}, \theta) = 0$$

The conservation of energy gives a further constraint:

$$\frac{1}{\lambda_i}+\frac{1}{\lambda_s}=\frac{1}{\lambda_p}\Rightarrow\lambda_i=\frac{\lambda_s\lambda_p}{\lambda_s-\lambda_p} \tag{0.17}$$

3. The relation for the the angle-dependent extraordinary refractive index is:

$$\frac{1}{n_{e'}^2(\omega,\theta)}=\frac{\sin^2\theta}{n_e^2(\omega)}+\frac{\cos^2\theta}{n_o^2(\omega)} \tag{0.18}$$

The phase matching angle can be implicitly solved and plotted. We use $\lambda_s\in[1100\,\text{nm},1600\,\text{nm}]$ and $\lambda_p=800\,\text{nm}$. Together with the boundary condition for the idler wavelength in eq. (0.17), eq (0.16) can then be numerically evaluated (e.g. with the ‘fzero’-function in Matlab):

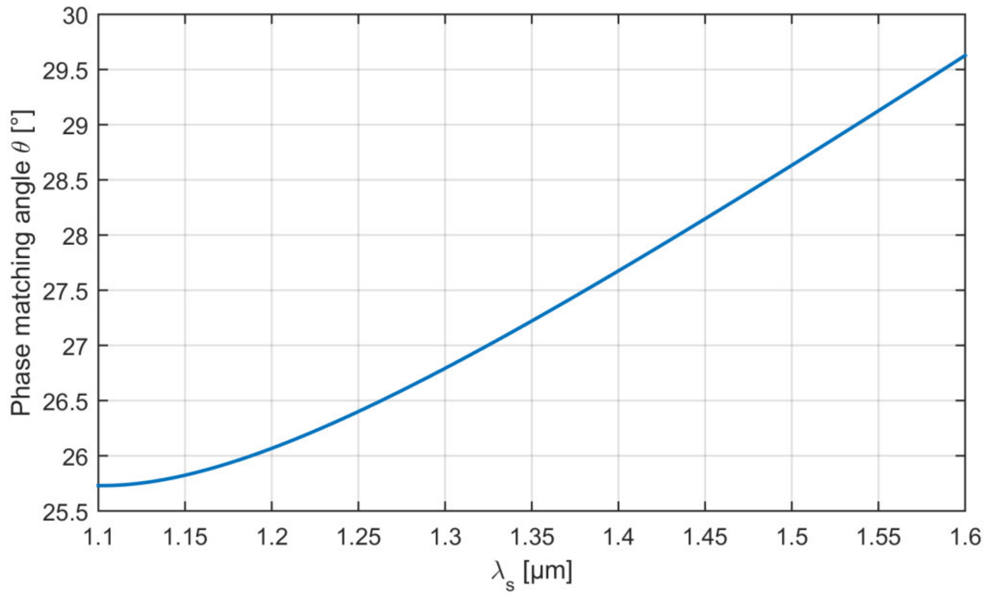


Figure 2: Phase matching angle over signal wavelength.

4. The relevant relation is eq. 8 in the given reference:

$d_{\text{eff}}^{\text{eoe}}=d_{\text{eff}}^{\text{oee}}=d_{22}\cos^2\theta\cos(3\varphi)$

The factor of three in the cosine emerges from a three-fold rotational symmetry of the BBO crystal. We can find the value of d_{22} in the paper on page 362, where it says that $d_{22}=2.22\pm0.09\frac{\text{pm}}{\text{V}}$. One of the azimuthal angles that maximize $d_{\text{eff}}^{\text{eoe}}$ is $\varphi=0$. For $\lambda_s=1500\text{nm}$, the phase matching angle is given by $\theta=27.9°$. This results into $d_{\text{eff}}^{\text{eoe}}=2.22\frac{\text{pm}}{\text{V}}\cos^2(27.9°)\cos(3\cdot0)=1.74\frac{\text{pm}}{\text{V}}$.

5. From eq. (0.14) in can be seen that the power gain is given by:

$$G = \cosh^2(\kappa L) \tag{1.18}$$

where $L = 2\,\text{mm}$ is the interaction length. Because the electric field strength of the pump wave enters into κ, we need to calculate it first. We know that $I_p = \frac{c n_p \varepsilon_0}{2} \left| \underline{E}_{p,0} \right|^2$, which gives us a field strength of $\left| \underline{E}_{p,0} \right| = 432\,\text{MV/m}$. The gain is then $G = 323 = 25.1\,\text{dB}$.

Contents

Part 3 Computation

Define n_o and n_e

```
n_o= @(x) sqrt(2.7359 + 0.01878 / (x.^2-0.01822) - 0.01471 * x.^2);
n_e= @(x) sqrt(2.3753 + 0.01224 / (x.^2-0.01667) - 0.01627 * x.^2);

lambda_p = 0.8;
lambda_s = 1.1:0.001:1.6;

lambda_i = zeros(1,length(lambda_s));
theta_pm = zeros(1,length(lambda_s));

tic
for n=1:length(lambda_s)

    lambda_i(n) = lambda_p * lambda_s(n) / (lambda_s(n) - lambda_p);

    % Introduce x = cos(theta)^2, for which f(x) = 0 is solved faster
    f = @(x) 1/lambda_p    * ((1-x)/n_e(lambda_p   )^2 + x/n_o(lambda_p   )^2)^(-1/2) ...
            -1/lambda_s(n) * n_o(lambda_s(n)) ...
            -1/lambda_i(n) * ((1-x)/n_e(lambda_i(n))^2 + x/n_o(lambda_i(n))^2)^(-1/2);

    sol = fzero(f,1/sqrt(2));

    theta_pm(n)=acos(sqrt(sol))* 180 / pi;
end
toc
```

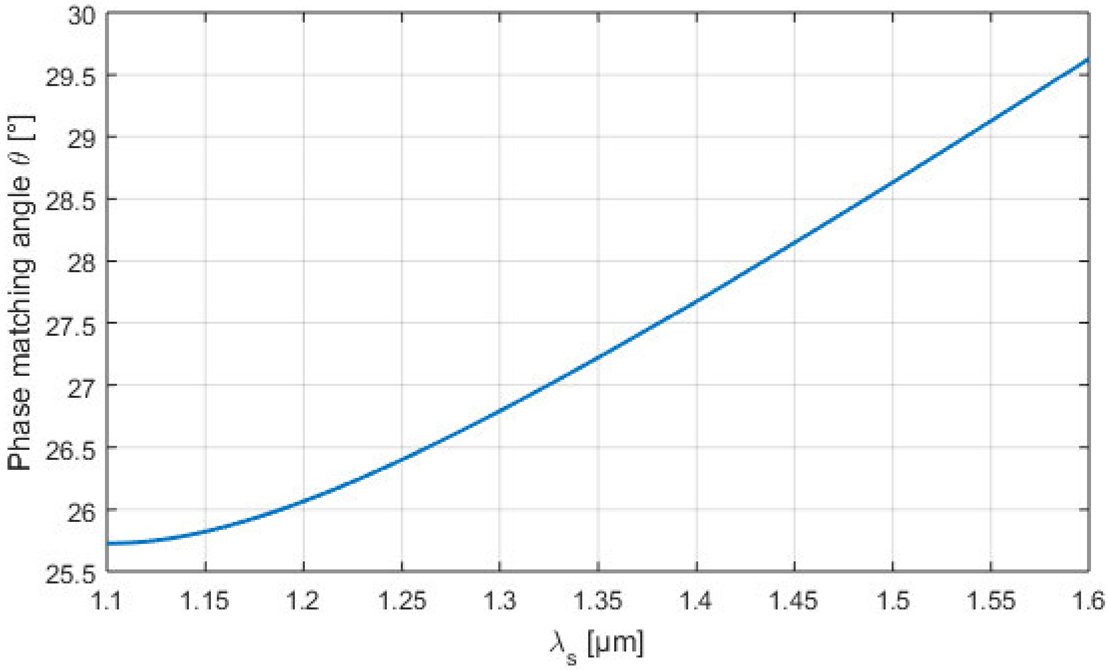
```
Elapsed time is 0.765943 seconds.
```

Part 3 Plot

```
close all
figure('Units','normalized','position',[0.33 0.33 0.33 0.33])
set(gca,'Fontsize',20)
box on

plot(lambda_s,theta_pm,'Linewidth',1.5)
xlabel('\lambda_s [µm]')
ylabel('Phase matching angle \theta [°]')
grid on

set(gcf, 'PaperPositionMode', 'auto')
print -dpng -r300 phase_matching
```



Part 5

```
[~,I]  = find(lambda_s == 1.5);
thetaG = theta_pm(I) / 180 * pi;

ns     = (sin(thetaG)^2 / n_e(lambda_s(I))^2 + cos(thetaG)^2 / n_o(lambda_s(I))^2)^-(1/2);
ni     = (sin(thetaG)^2 / n_e(lambda_i(I))^2 + cos(thetaG)^2 / n_o(lambda_i(I))^2)^-(1/2);
d      = 2.22E-12 * cos(thetaG)^2;

c      = 299792458;
e0     = 8.854E-12;

E      = sqrt(2 / (ns* c * e0) * 4E14);

kappa  = 2 * pi * sqrt(1 / lambda_i(I) / ni / lambda_s(I) / ns) * 1E6 * d * E;

gain   = cosh(kappa*2e-3).^2;
gain   = 10*log10(gain);

display(gain)
```

```
gain =

   25.0821
```

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