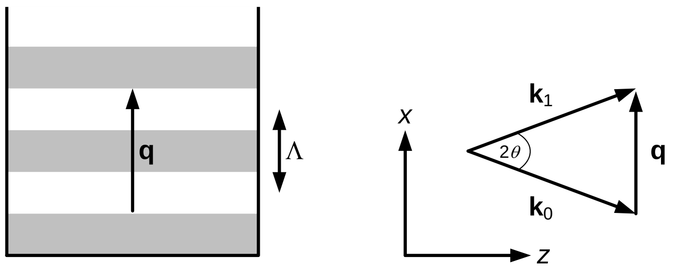
Problem Set 11
Nonlinear Optics (NLO)

Solution

1) Acousto-Optic Modulator



Consider a material in which a sound wave travels in the x-direction with wave vector $\mathbf{q}$ and frequency Ω and induces a refractive index grating that scatters an incoming optical wave. In the lecture notes [Eq. (4.14)] we derived a coupled-wave equation for the space-dependent amplitudes $\underline{E}(\mathbf{r},\omega_l)$ of the incoming wave $(l=0)$ at frequency ω_0 and the various scattered waves at frequencies ω_l. Assume that all waves are polarized along the y-direction, i.e., $e_l = e_y \ \forall\, l$. The scalar coupled-wave equation can then be written as

$$\sum_l -2\mathrm{j}\mathbf{k}_l \cdot \nabla \underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})} = \frac{2n_0}{c^2} \sum_l \frac{\partial^2}{\partial t^2}\left(\Delta n(\mathbf{r},t)\underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})}\right), \tag{0.1}$$

where the index variation $\Delta n(\mathbf{r},t)$ is given by

$$\Delta n(\mathbf{r},t) = \Delta n_0 \cos(\Omega t - \mathbf{q}\mathbf{r}). \tag{0.2}$$

1. For a monochromatic incident optical wave at frequency ω_0 the right-hand side of Eq. (0.1) contains frequency components at $\omega_{\pm 1} = \omega_0 \pm \Omega$. Derive the two coupled differential equations for the wave amplitudes $\underline{E}(\mathbf{r},\omega_0)$ and $\underline{E}(\mathbf{r},\omega_1)$ by comparing the corresponding coefficients associated with the same frequency of Eq. (0.1).
2. Consider the case where both the crystal and the optical waves are infinitely extended in x- and y-direction, which implies $\frac{\partial \underline{E}}{\partial x} = 0$ and $\frac{\partial \underline{E}}{\partial y} = 0$. Assume further that the z-components of the $\mathbf{k}$-vector for both optical waves are equal, i.e. $k_{0z} = k_{1z} = k_z$ and $\omega_0 = \omega_1$. Using these simplifications, show that the two coupled differential equations can be written as:

$$\frac{\partial \underline{E}(z,\omega_1)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_0)e^{-\mathrm{j}\Delta\mathbf{kr}} \tag{0.3}$$

$$\frac{\partial \underline{E}(z,\omega_0)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_1)e^{\mathrm{j}\Delta\mathbf{kr}}$$

with $\kappa = \frac{k_z \Delta n_0}{2n_0}$.

3. Solve the differential equations assuming perfect phase matching, i.e. $\Delta\mathbf{k} = 0$ and using the boundary conditions $\underline{E}(0,\omega_0) = E_0$ and $\underline{E}(0,\omega_1) = 0$. Sketch the evolution of the intensities of the incident and the deflected wave along z. How long should the crystal extend in the z-direction for maximum intensity of the deflected wave?
4. What is the angle of diffraction for light at 632.8 nm in a $LiNbO_3$ cell that is driven at a frequency of 1 GHz? (speed of sound: v_s = 4.1 km/s, refractive index n_0 = 2.3)

Solution:

1. In an acousto-optic modulator an incident optical wave interacts (frequency ω_0 and propagation vector $\mathbf{k}_0$) with a sound wave (frequency Ω and propagation vector $\mathbf{q}$). The reflected wave must obey both energy and momentum conserved conservation, for the given example this means that $\omega_1 = \omega_0 + \Omega$ and $\mathbf{k}_1 = \mathbf{k}_0 + \mathbf{q}$ must hold. As a first step let's rewrite eq. (0.4) as

$$\Delta n(\mathbf{r},t) = \frac{1}{2}\Delta n_0 \left(e^{\mathrm{j}(\Omega t - \mathbf{qr})} + e^{-\mathrm{j}(\Omega t - \mathbf{qr})}\right) \tag{0.5}$$

and insert it into eq. (0.1):

$$\sum_l -2\mathrm{j}\mathbf{k}_l \cdot \nabla \underline{E}(\mathbf{r},\omega_l)e^{\mathrm{j}(\omega_l t - \mathbf{k}_l\mathbf{r})} = \frac{n_0}{c^2}\Delta n_0 \sum_l \frac{\partial^2}{\partial t^2}\left(\underline{E}(\mathbf{r},\omega_l)\left(e^{\mathrm{j}((\omega_l+\Omega)t-(\mathbf{k}_l+\mathbf{q})\mathbf{r})} + e^{\mathrm{j}((\omega_l-\Omega)t-(\mathbf{k}_l-\mathbf{q})\mathbf{r})}\right)\right) \tag{0.6}$$

We are now only concerned with terms containing the frequencies ω_0 (the incident wave) and $\omega_1 = \omega_0 + \Omega$ (the reflected wave) in the exponent. On the left hand side of eq. (0.6) these are:

$$\begin{aligned} \omega_0: &\quad -2\mathrm{j}\mathbf{k}_0 \nabla \underline{E}(z,\omega_0)e^{\mathrm{j}(\omega_0 t - \mathbf{k}_0\mathbf{r})} \\ \omega_1: &\quad -2\mathrm{j}\mathbf{k}_1 \nabla \underline{E}(z,\omega_1)e^{\mathrm{j}(\omega_1 t - \mathbf{k}_1\mathbf{r})} \end{aligned} \tag{0.7}$$

On the right hand side of eq. (0.6) we find:

$$\begin{aligned} \omega_0: &\quad \frac{n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1)\frac{\partial^2}{\partial t^2}e^{\mathrm{j}((\omega_1-\Omega)t-(\mathbf{k}_1-\mathbf{q})\mathbf{r})} = -\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1)e^{\mathrm{j}(\omega_0 t-(\mathbf{k}_1-\mathbf{q})\mathbf{r})} \\ \omega_1: &\quad \frac{n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0)\frac{\partial^2}{\partial t^2}e^{\mathrm{j}((\omega_0+\Omega)t-(\mathbf{k}_0+\mathbf{q})\mathbf{r})} = -\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0)e^{\mathrm{j}(\omega_1 t-(\mathbf{k}_0+\mathbf{q})\mathbf{r})} \end{aligned} \tag{0.8}$$

where we made use of the fact that $\omega_0 = \omega_1 - \Omega$. We can now write the two coupled differential equations as

$$\mathbf{k}_1 \nabla \underline{E}\left(z,\omega_1\right) = -\mathrm{j}\frac{1}{2}\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0) e^{-\mathrm{j}(\mathbf{k}_0+\mathbf{q}-\mathbf{k}_1)\mathbf{r}} \text{ and} \tag{0.9}$$

$$\mathbf{k}_0 \nabla \underline{E}\left(z,\omega_0\right) = -\mathrm{j}\frac{1}{2}\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1) e^{\mathrm{j}(\mathbf{k}_0+\mathbf{q}-\mathbf{k}_1)\mathbf{r}}. \tag{0.10}$$

2. The term $\mathbf{k}\nabla\underline{E}$ can be written as $k_x \frac{\partial \underline{E}}{\partial x} + k_y \frac{\partial \underline{E}}{\partial y} + k_z \frac{\partial \underline{E}}{\partial z}$. Using this we can rewrite eq. (0.9) and (0.10) as

$$\frac{\partial \underline{E}\left(\mathbf{r},\omega_1\right)}{\partial z} = -\mathrm{j}\frac{1}{2k_{1z}}\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0) e^{-\mathrm{j}\Delta\mathbf{kr}} \text{ and} \tag{0.11}$$

$$\frac{\partial \underline{E}\left(\mathbf{r},\omega_0\right)}{\partial z} = -\mathrm{j}\frac{1}{2k_{0z}}\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1) e^{\mathrm{j}\Delta\mathbf{kr}} \tag{0.12}$$

Using the relation $\frac{1}{k_{1z}}\frac{\omega_1^2 n_0}{c^2} = \frac{1}{k_{1z}}\frac{k_{1z}^2}{n_0} = \frac{k_{1z}}{n_0}$ we get the given differential equations.

$$\frac{\partial \underline{E}\left(z,\omega_1\right)}{\partial z} = -\mathrm{j}\kappa \underline{E}\left(z,\omega_0\right) e^{-\mathrm{j}\Delta\mathbf{kr}} \tag{0.13}$$

$$\frac{\partial \underline{E}\left(z,\omega_0\right)}{\partial z} = -\mathrm{j}\kappa \underline{E}\left(z,\omega_1\right) e^{\mathrm{j}\Delta\mathbf{kr}} \tag{0.14}$$

with $\kappa = \frac{k_z \Delta n_0}{2n_0}$.

3. Assuming perfect phase matching, i.e. $\Delta\mathbf{k} = 0$, taking the derivative of eq. (0.13) and inserting eq. (0.14) we can decouple the two differential equations and get

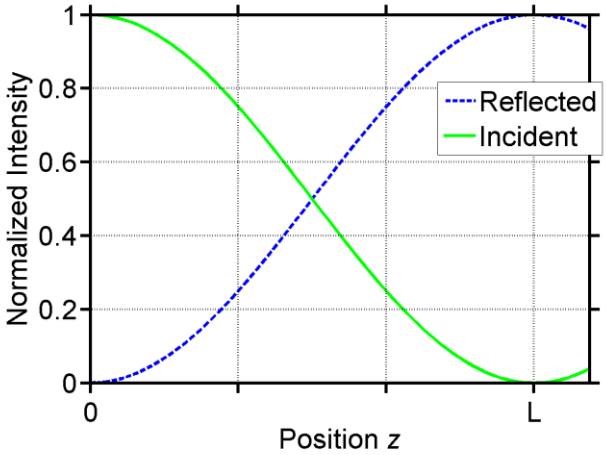
$$\frac{\partial^2 \underline{E}\left(z,\omega_1\right)}{\partial z^2} = -\kappa^2 \underline{E}\left(z,\omega_1\right) \tag{0.15}$$

Using the given boundary conditions we get the solutions

$$\underline{E}\left(z,\omega_1\right) = -\mathrm{j}E_0 \sin\kappa z \tag{0.16}$$

$$\underline{E}\left(z,\omega_0\right) = E_0 \cos\kappa z \tag{0.17}$$

Theoretically it is possible to transfer the total power of the incident wave to the reflected wave. This would be the case after a distance $L = \frac{\pi n_0}{k_z \Delta n_0}$.



4. Using the figure on the first page of the exercise sheet you can immediately see that

$$\sin\theta = \frac{\frac{1}{2}|\mathbf{q}|}{|\mathbf{k}_1|} = 1.95°$$

Questions and Comments:

Pablo Marin
Building: 30.10, Room: 2.23
Phone: 0721/608-42487
pablo.marin@kit.edu

Philipp Trocha
Room: 2.32/2
42480
philipp.trocha@kit.edu