Solution to Problem Set 7
Nonlinear Optics (NLO)

1) Pockels cells

Solution

1. Our wave is constructed using two orthogonal waves. The shape resulting from two orthogonal sinusoidal movements are Lissajous-figures. The exact shape is determined by the frequencies and the relative phase difference between the two orthogonal waves.

   In our case, the frequencies and the amplitudes are the same, and the phase differences of 0, π/2, π, and 3π/2 describe the following Lissajous-figures:

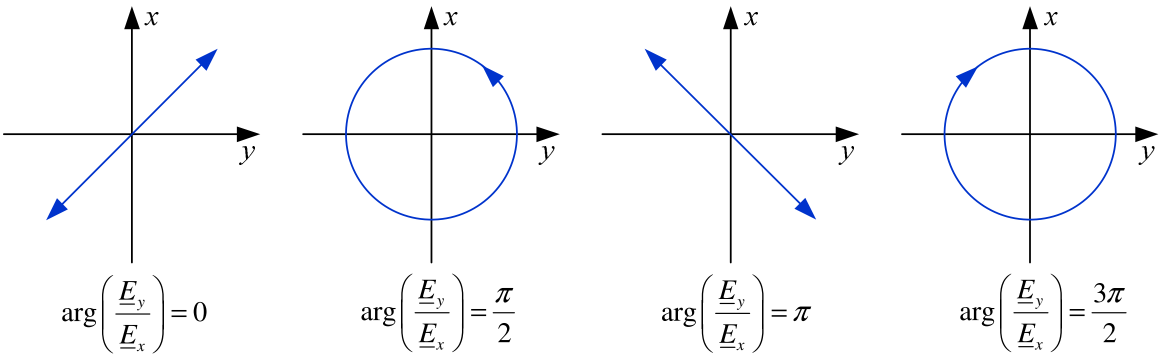
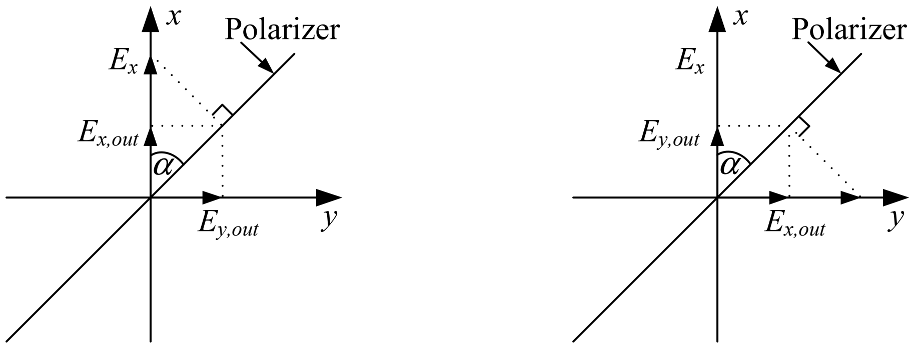


Figure 1: Lissajous-figures described by the endpoint vector $\mathbf{E}(z,t)$ for different values of the phase difference between $\underline{E}_y$ and $\underline{E}_x$.

- $\arg\{\underline{E}_y / \underline{E}_x\} = 0$: $\underline{E}_x$ and $\underline{E}_y$ are in-phase: linear polarization, 45°.
- $\arg\{\underline{E}_y / \underline{E}_x\} = \frac{\pi}{2}$: $\underline{E}_y$ is π/2 ahead: counterclockwise circular polarization.
- $\arg\{\underline{E}_y / \underline{E}_x\} = \pi$: $\underline{E}_x$ and $\underline{E}_y$ are phase-shifted by π: linear polarization, –45°.
- $\arg\{\underline{E}_y / \underline{E}_x\} = \frac{3\pi}{2}$: $\underline{E}_x$ is π/2 ahead: clockwise circular polarization.

2. Let us consider each component E_x and E_y of the incident wave separately.



$$\mathbf{E}_{out}\Big|_{E_{y,in}=0} = \begin{pmatrix} E_{x,out} \\ E_{y,out} \end{pmatrix} = \begin{pmatrix} \cos^2\alpha \\ \sin\alpha\cos\alpha \end{pmatrix} E_{x,in} \qquad \mathbf{E}_{out}\Big|_{E_{x,in}=0} = \begin{pmatrix} E_{x,out} \\ E_{y,out} \end{pmatrix} = \begin{pmatrix} \sin\alpha\cos\alpha \\ \sin^2\alpha \end{pmatrix} E_{y,in}$$

Figure 1: Output of a polarizer for different angles of the polarized light at the input.

The wave at the output of the linear polarizer whose axis forms an angle α with x-axis can be expressed as the summation of the above derived fields:

$$\mathbf{E}_{out} = \mathbf{E}_{out}\Big|_{E_{y,in}=0} + \mathbf{E}_{out}\Big|_{E_{x,in}=0} = \underbrace{\begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix}}_{\mathbf{M}_\alpha} \mathbf{E}_{in}\,. \tag{1}$$

3. The KDP is a uniaxial crystal that becomes biaxial under the influence of an externally applied electric field. This is called the Pockels effect or linear electro-optic effect. When a wave enters the crystal, it can be decomposed into two components that are polarized parallel to the principal axes x' and y'. The phase difference between the x' and y' component of the electric field is given by:

$$\Delta\Phi = \left(n_{x'} - n_{y'}\right) k_0 L = r_{63} n_o^3 E_z k_0 L = r_{63} n_o^3 k_0 U\ . \tag{2}$$

Expressing the input and output electric field in the following way:

$$\begin{aligned} \vec{E}_{in} &= E_{x',in}\vec{e}_{x'} + E_{y',in}\vec{e}_{y'} \\ \vec{E}_{out} &= E_{x',in}e^{-\mathrm{j}n_{x'}k_0L}\vec{e}_{x'} + E_{y',in}e^{-\mathrm{j}n_{y'}k_0L}\vec{e}_{y'} = E_{x',out}\vec{e}_{x'} + E_{y',out}\vec{e}_{y'} \end{aligned}, \tag{3}$$

and using matrix notation, we get:

$$\begin{pmatrix} E_{x',out} \\ E_{y',out} \end{pmatrix} = \underbrace{e^{-\mathrm{j}\frac{n_{x'}+n_{y'}}{2}k_0L}}_{\text{common phase}} \begin{pmatrix} e^{-\mathrm{j}\frac{\Delta\phi}{2}} & 0 \\ 0 & e^{\mathrm{j}\frac{\Delta\phi}{2}} \end{pmatrix} \begin{pmatrix} E_{x',in} \\ E_{y',in} \end{pmatrix}. \tag{4}$$

The common phase term can be neglected, as it only influences the 'global' phase of the waves and has no influence over the phase difference. As it can be seen from

Eq. (2), the phase difference does not depend on L, and is only influenced by changing the voltage U.

4. The quarter-wave plate delays the x' component (slow axis) by π/2 (λ/4 light path difference) relative to y' component (fast axis). The matrix is:

$$\mathbf{M}_{\lambda/4} = \begin{pmatrix} e^{-\mathrm{j}\frac{\pi}{4}} & 0 \\ 0 & e^{\mathrm{j}\frac{\pi}{4}} \end{pmatrix}. \tag{5}$$

5. To calculate the overall system behavior, we need the transformation between our different coordinate systems.

Rotation by –45° $(x,y)\rightarrow(x',y')$: $\begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix}$

Rotation by 45° $(x',y')\rightarrow(x,y)$: $\begin{pmatrix} \cos 45° & \sin 45° \\ -\sin 45° & \cos 45° \end{pmatrix}$

Remember that the matrices are arranged from right to left with respect to the light propagation:

$$\begin{pmatrix} E_{x,out} \\ E_{y,out} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Pol2, }\alpha=90°} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{(x',y')\rightarrow(x,y)} \underbrace{\begin{pmatrix} e^{-\mathrm{j}\frac{\pi}{4}} & 0 \\ 0 & e^{\mathrm{j}\frac{\pi}{4}} \end{pmatrix}}_{\lambda/4\text{ plate}} \underbrace{\begin{pmatrix} e^{-\mathrm{j}\frac{\Delta\phi}{2}} & 0 \\ 0 & e^{\mathrm{j}\frac{\Delta\phi}{2}} \end{pmatrix}}_{\text{KDP}} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{(x,y)\rightarrow(x',y')} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{Pol1, }\alpha=0°} \begin{pmatrix} E_{x,in} \\ E_{y,in} \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-\mathrm{j}\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)} & 0 \\ 0 & e^{\mathrm{j}\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_{x,in} \\ E_{y,in} \end{pmatrix} =$$

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$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-\mathrm{j}\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)} & 0 \\ e^{\mathrm{j}\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)} & 0 \end{pmatrix}\begin{pmatrix} E_{x,in} \\ E_{y,in} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ e^{\mathrm{j}\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)} - e^{-\mathrm{j}\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)} & 0 \end{pmatrix}\begin{pmatrix} E_{x,in} \\ E_{y,in} \end{pmatrix} =$$

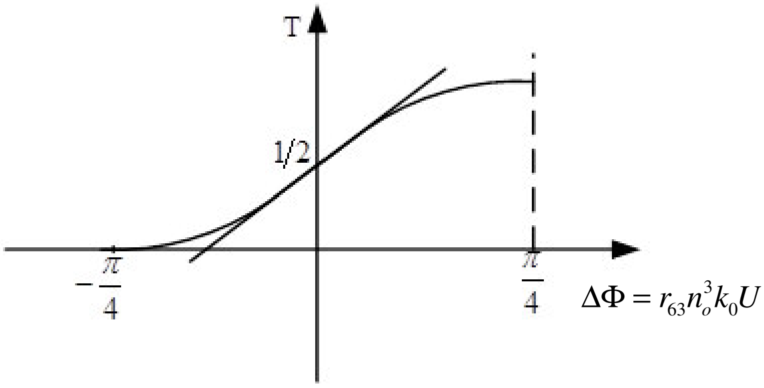
$$= \begin{pmatrix} 0 & 0 \\ \mathrm{j}\sin\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right) & 0 \end{pmatrix}\begin{pmatrix} E_{x,in} \\ E_{y,in} \end{pmatrix}$$

$$E_{x,out} = 0$$
$$E_{y,out} = \mathrm{j}\sin\left(\frac{\pi}{4} + \frac{\Delta\phi}{2}\right) E_{x,in}. \tag{6}$$

6. The power transmission is maximum for $\vec{E}_{in} = E_{x,in}\vec{e}_x$. It can be written as:

$$T = \frac{I_{out}}{I_{in}} = \frac{|E_{out}|^2}{|E_{in}|^2} = \frac{|E_{y,out}|^2}{|E_{x,in}|^2} = \sin^2\left(\frac{\Delta\phi}{2} + \frac{\pi}{4}\right). \tag{7}$$

7. Sketch:



Without applied voltage, half of the power is transmitted. The transmission increases with a positive voltage and decreases with a negative voltage. In this operating point, the transmission-voltage dependence can be approximated by a linear function, and the slope (the first derivative) of this dependence is here the largest. That means that a large transmission variation can be achieved for a relatively small voltage variation.

Without the quarter-wave plate, the transmitted power would depend quadratically on the applied voltage, and larger voltage variations (and more time) would be necessary to get the same extinction ratio (the ratio between two transmission levels, which is an important parameter in telecommunications).