Solution to Problem Set 3
Nonlinear Optics (NLO)

1) Nonlinear wave equation

From Maxwell's equations one can derive the following nonlinear wave equation (Eq. (1.80) in the lecture notes):

$$\nabla^2\mathbf{E}(\mathbf{r},t)-\frac{n^2}{c^2}\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2}=\mu_0\frac{\partial^2\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t)}{\partial t^2}. \tag{1.1}$$

In order to solve this second-order differential equation analytically, it is necessary to simplify it. To this end we assume a homogeneous isotropic medium and reduce the representation to plane waves that are polarized along the x-direction.

$$\mathbf{E}(\mathbf{r},t)=E(z,t)\mathbf{e}_x\text{, and} \tag{1.2}$$

$$\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t)=P_{\mathrm{NL}}(z,t)\mathbf{e}_x, \tag{1.3}$$

where $\mathbf{e}_x$ denotes the unit vector along the x-direction. The scalar electric field $E(z,t)$ and the nonlinear polarization $P_{\mathrm{NL}}(z,t)$ can be written as:

$$E(z,t)=\frac{1}{2}\left(\sum_l \underline{E}(z,t,\omega_l)\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)+\text{c.c.}\right)\text{, and} \tag{1.4}$$

$$P_{\mathrm{NL}}(z,t)=\frac{1}{2}\left(\sum_l \underline{P}_{\mathrm{NL}}(z,t,\omega_l)\exp\left(\mathrm{j}(\omega_l t-k_{p,l} z)\right)+\text{c.c.}\right). \tag{1.5}$$

In these relations "c.c." denotes the complex conjugate of the preceding expressions.

1. Insert Eqs. (1.4) and (1.5) into (1.1), and derive an equation that has to be fulfilled for the complex amplitudes $\underline{E}(z,t,\omega_l)=\underline{E}_l$ and $\underline{P}_{\mathrm{NL}}(z,t,\omega_l)=\underline{P}_{\mathrm{NL},l}$ of a particular frequency ω_l.

   Hint: You can simplify the result considering that $k_l=\frac{\omega_l n}{c}$.

Solution

All the terms in the sum oscillate at different frequencies. As all these terms are linearly independent, each frequency component needs to satisfy the differential equation. We start by finding the terms of the respective derivatives for a particular frequency ω_l. Note that in Eqs. (1.4) and (1.5) "c.c." corresponds to negative frequency $-\omega_l$, which is different from ω_l. Therefore, the differential equation must be satisfied for both summands from Eqs. (1.4) and (1.5) separately, and we omit the "c.c." summand in the following calculations.

$$\nabla^2\mathbf{E}(\mathbf{r},t)=\partial_z^2 E(z,t)=\partial_z^2\left(\frac{1}{2}\underline{E}(z,t,\omega_l)\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)\right)=$$
$$\partial_z\left(\frac{1}{2}\frac{\partial\underline{E}}{\partial z}\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)-\mathrm{j}\frac{1}{2}\underline{E}k_l\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)\right)=$$
$$\frac{1}{2}\frac{\partial^2\underline{E}}{\partial z^2}\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)-\mathrm{j}k_l\frac{\partial\underline{E}}{\partial z}\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)-\frac{k_l^2}{2}\underline{E}\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)$$

We obtain similar results for the time derivatives of the electric field and the nonlinear polarization.

$$\partial_t^2 E(z,t)=\partial_t^2\left(\frac{1}{2}\underline{E}(z,t,\omega_l)\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)\right)=$$
$$\frac{1}{2}\frac{\partial^2\underline{E}}{\partial t^2}\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)+\mathrm{j}\omega_l\frac{\partial\underline{E}}{\partial t}\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)-\frac{\omega_l^2}{2}\underline{E}\exp\left(\mathrm{j}(\omega_l t-k_l z)\right)$$

$$\partial_t^2 P_{\mathrm{NL}}(z,t)=\partial_t^2\left(\frac{1}{2}\underline{P}_{\mathrm{NL}}(z,t,\omega_l)\exp\left(\mathrm{j}(\omega_l t-k_{p,l} z)\right)\right)=$$
$$\frac{1}{2}\frac{\partial^2\underline{P}_{\mathrm{NL}}}{\partial t^2}\exp\left(\mathrm{j}(\omega_l t-k_{p,l} z)\right)+\mathrm{j}\omega_l\frac{\partial\underline{P}_{\mathrm{NL}}}{\partial t}\exp\left(\mathrm{j}(\omega_l t-k_{p,l} z)\right)-\frac{\omega_l^2}{2}\underline{P}_{\mathrm{NL}}\exp\left(\mathrm{j}(\omega_l t-k_{p,l} z)\right)$$

We now set these terms back into the nonlinear wave equation. Exploiting the fact that the dispersion relation of plane waves gives us $k_l=\frac{\omega_l n}{c}$, and multiplying both sides of the equation by $\exp\left(-\mathrm{j}(\omega_l t-k_l z)\right)$, we get:

$$\frac{1}{2}\frac{\partial^2\underline{E}}{\partial z^2}-\mathrm{j}\frac{\omega_l n}{c}\frac{\partial\underline{E}}{\partial z}-\frac{\omega_l^2 n^2}{2c^2}\underline{E}-\frac{1}{2}\frac{n^2}{c^2}\frac{\partial^2\underline{E}}{\partial t^2}-\mathrm{j}\omega_l\frac{n^2}{c^2}\frac{\partial\underline{E}}{\partial t}+\frac{\omega_l^2 n^2}{2c^2}\underline{E}=$$
$$\mu_0\left(\frac{1}{2}\frac{\partial^2\underline{P}_{\mathrm{NL}}}{\partial t^2}+\mathrm{j}\omega_l\frac{\partial\underline{P}_{\mathrm{NL}}}{\partial t}-\frac{\omega_l^2}{2}\underline{P}_{\mathrm{NL}}\right)e^{-\mathrm{j}(k_{p,l}-k_l)z}$$

To further simplify the equation, we introduce the so-called slowly-varying envelope approximation (SVEA). This SVEA is based on the fact that the complex envelope functions $\underline{E}(z,t,\omega_l)$ and $\underline{P}_{\mathrm{NL}}(z,t,\omega_l)$ do not change significantly on the length scale of one wavelength and the time scale of one optical oscillation period. This can be mathematically expressed as:

$$\left|\frac{\partial^2\underline{E}(z,t,\omega_l)}{\partial t^2}\right|<<\omega_l\left|\frac{\partial\underline{E}(z,t,\omega_l)}{\partial t}\right|, \tag{1.6}$$

$$\left|\frac{\partial^2\underline{E}(z,t,\omega_l)}{\partial z^2}\right|<<k_l\left|\frac{\partial\underline{E}(z,t,\omega_l)}{\partial z}\right|, \text{ and} \tag{1.7}$$

$$\left|\frac{\partial^2 \underline{P}_{\mathrm{NL}}(z,t,\omega_l)}{\partial t^2}\right| << \omega_l \left|\frac{\partial \underline{P}_{\mathrm{NL}}(z,t,\omega_l)}{\partial t}\right| << \omega_l^2 \left|\underline{P}_{\mathrm{NL}}(z,t,\omega_l)\right|. \tag{1.8}$$

2. Simplify the nonlinear wave equation using the SVEA, and express the resulting equation in the following form (Eq. (1.95) in the lecture notes):

$$\frac{\partial \underline{E}(z,t,\omega_l)}{\partial z} + \frac{n}{c}\frac{\partial \underline{E}(z,t,\omega_l)}{\partial t} = -\mathrm{j}\frac{\omega_l}{2\varepsilon_0 cn}\underline{P}_{\mathrm{NL}}(z,t,\omega_l)\, e^{-\mathrm{j}(k_{p,l}-k_l)z}. \tag{1.9}$$

Solution

We immediately see that the terms proportional to $\underline{E}$ cancel. As a next step, we exploit the fact that the magnitudes of envelope-functions $\underline{E}$ and $\underline{P}_{\mathrm{NL}}$ vary only slowly with time and space (SVEA), which allows us canceling all higher order derivatives (i.e., we will only consider the respective derivatives of lowest order for $\underline{E}$ and $\underline{P}_{\mathrm{NL}}$):

$$-\mathrm{j}\frac{2\omega_l n}{c}\frac{\partial \underline{E}}{\partial z} - \mathrm{j}2\omega_l\frac{n^2}{c^2}\frac{\partial \underline{E}}{\partial t} = -\mu_0\omega_l^2 \underline{P}_{\mathrm{NL}} e^{-\mathrm{j}(k_{p,l}-k_l)z}.$$

After multiplying by $\mathrm{j}\frac{c}{2\omega_l n}$ and using $\mu_0 = \frac{1}{\varepsilon_0 c^2}$ we get the demanded form of the equation:

$$\frac{\partial \underline{E}(z,t,\omega_l)}{\partial z} + \frac{n}{c}\frac{\partial \underline{E}(z,t,\omega_l)}{\partial t} = -\mathrm{j}\frac{\omega_l}{2\varepsilon_0 cn}\underline{P}_{\mathrm{NL}}(z,t,\omega_l)\, e^{-\mathrm{j}(k_{p,l}-k_l)z}.$$

3. Introduce a retarded time frame that is propagating along with the optical signal, i.e., $z' = z$ and $t' = t - \frac{zn}{c}$, and reformulate Eq. (1.9). Explain the physical meaning of the retarded time frame.

Solution

What now is left to do is the transformation of the equation into a co-propagating time frame ("retarded time frame") canceling the first derivative of the electric field with respect to time. The new independent variables in the transformed system are given by:

$$z' = z$$

$$t' = t - \frac{zn}{c}$$

The electric field and the nonlinear polarization in the transformed system are given by new functions $\underline{E}'(z',t',\omega_l)$ and $\underline{P}'_{\mathrm{NL}}(z',t',\omega_l)$, which are related to the fields in the old system:

$$\underline{E}'(z',t',\omega_l) = \underline{E}'\left(z, t-\frac{zn}{c}, \omega_l\right) = \underline{E}(z,t,\omega_l)$$

$$\underline{P}'_{\text{NL}}(z',t',\omega_l) = \underline{P}'_{\text{NL}}\left(z, t-\frac{zn}{c}, \omega_l\right) = \underline{P}_{\text{NL}}(z,t,\omega_l)$$

Transforming the differential equation into the new coordinate system can be achieved by finding expressions for the derivatives of the electric field in the new system using the chain rule:

$$\frac{\partial \underline{E}(z,t,\omega_l)}{\partial z} = \frac{\partial \underline{E}'(z',t',\omega_l)}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial \underline{E}'(z',t',\omega_l)}{\partial t'}\frac{\partial t'}{\partial z} = \frac{\partial \underline{E}'(z',t',\omega_l)}{\partial z'} - \frac{n}{c}\frac{\partial \underline{E}'(z',t',\omega_l)}{\partial t'}$$

$$\frac{\partial \underline{E}(z,t,\omega_l)}{\partial t} = \frac{\partial \underline{E}'(z',t',\omega_l)}{\partial z'}\frac{\partial z'}{\partial t} + \frac{\partial \underline{E}'(z',t',\omega_l)}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial \underline{E}'(z',t',\omega_l)}{\partial t'}$$

Now, the time derivative of the electric field in the differential equation cancels out. The differential equation has been simplified to:

$$\frac{\partial \underline{E}'(z',t',\omega_l)}{\partial z'} = -\mathrm{j}\frac{\omega_l}{2\varepsilon_0 cn}\underline{P}'_{\text{NL}}(z',t',\omega_l)\,e^{-\mathrm{j}(k_{p,l}-k_l)z'}.$$

In the retarded time frame, we only observe how the field amplitude changes with space.

4. For self-phase modulation (SPM) and cross-phase modulation (XPM) the nonlinear polarization is given by

$$P_{\text{NL,SPM}} = \frac{3}{4}\varepsilon_0\chi^{(3)}\left|\underline{E}(z,t,\omega_1)\right|^2\underline{E}(z,t,\omega_1)\text{, and} \tag{1.10}$$

$$P_{\text{NL,XPM}} = \frac{6}{4}\varepsilon_0\chi^{(3)}\left|\underline{E}(z,t,\omega_2)\right|^2\underline{E}(z,t,\omega_1), \tag{1.11}$$

respectively. Insert these relations into the simplified nonlinear wave equation, i.e., the result from part 3, which should be equivalent to Eq. (1.99) in the lecture notes. Explain why these processes do not affect the magnitude $\left|\underline{E}(z,t,\omega_1)\right|$ of the complex amplitude if $\chi^{(3)}$ is a real number. Hint: Consider the change $\frac{\partial \underline{E}}{\partial z}$ of the phasor $\underline{E}$ in the complex plane. Explain the term **self**-phase and **cross**-phase modulation.

Solution

For self-phase Modulation (SPM), the differential equation becomes:

$$\frac{\partial \underline{E}'(z',t',\omega_1)}{\partial z'} = -\mathrm{j}\frac{\omega_1}{2\varepsilon_0 cn}\frac{3}{4}\varepsilon_0\chi^{(3)}\left|\underline{E}'(z',t',\omega_1)\right|^2\underline{E}'(z',t',\omega_1)\,e^{-\mathrm{j}(k_{p,l}-k_1)z'},$$

where $k_{p,l} = k_1 - k_1 + k_1 = k_1$ (see Eq. (1.106) in the lecture notes).

In the complex plane of the phasor $\underline{E}$, a multiplication by the imaginary unit j means a phase rotation by 90° (see Figure 1). As the spatial derivative $\frac{\partial \underline{E}'(z',t',\omega_1)}{\partial z'}$ is only proportional to j and $\underline{E}$ itself, it is always perpendicular to $\underline{E}$ and does not alter its magnitude but its phase. The change of phase is proportional to the intensity of the component $\omega_1 : \left|\underline{E}'(z',t',\omega_1)\right|^2$ and we call the process self-phase modulation.

For cross-phase modulation (XPM) the differential equation would be:

$$\frac{\partial \underline{E}'(z',t',\omega_1)}{\partial z'} = -\mathrm{j}\frac{\omega_1}{2\varepsilon_0 cn}\frac{3}{4}\varepsilon_0\chi^{(3)}\left|\underline{E}'(z',t',\omega_2)\right|^2 \underline{E}'(z',t',\omega_1)\,e^{-\mathrm{j}(k_{p,l}-k_1)z'},$$

where $k_{p,l} = k_2 - k_2 + k_1 = k_1$ (see Eq. (1.106) in the lecture notes). Analogously, for XPM the phase of the field $\underline{E}'(z',t',\omega_1)$ is modulated by the intensity of another field $\underline{E}'(z',t',\omega_2)$.

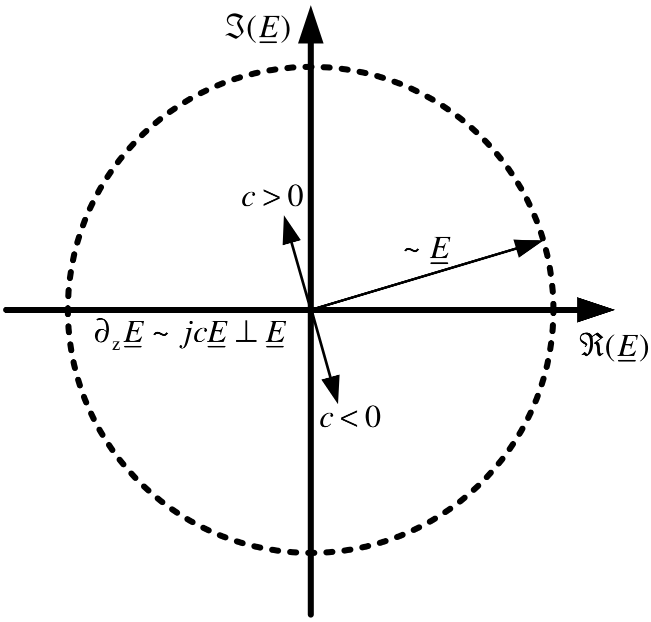

Figure 1: Phasor representation of the complex field amplitude. The derivative is always perpendicular to the field in the complex plane.

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