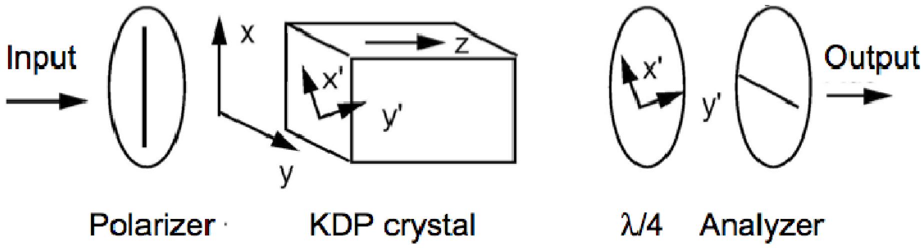
Solution to Problem Set 7
Nonlinear Optics (NLO)

1) Pockels cells



A Pockels cell is a type of optical modulator, where the modulating electric field is applied in the direction of light propagation. Incoming light is first linearly polarized by a polarizer and then launched into a nonlinear crystal (e.g., potassium dihydrogen phosphate, KDP). The light then passes through a quarter-wave plate and a polarization analyzer that is rotated by 90° with respect to the input polarizer. Depending on the applied electric field, the nonlinear crystal changes the polarization state of the light, and the output power after the polarization analyzer is varied. In the following, we assume that the light can be described by a completely polarized plane wave that is propagating in z-direction.

1. A completely polarized plane wave is described by:

$$\mathbf{E}(z,t)=\begin{pmatrix}E_x\\E_y\end{pmatrix}=\mathrm{Re}\left\{\begin{pmatrix}\underline{E}_x\\\underline{E}_y\end{pmatrix}e^{\mathrm{j}(\omega t-kz)}\right\}=\mathrm{Re}\left\{\underline{\mathbf{E}}\,e^{\mathrm{j}(\omega t-kz)}\right\},\ \text{with}\ k=n\omega/c. \tag{1.1}$$

The state of polarization is defined by the relative phase and the amplitudes of the complex values $\underline{E}_x$, $\underline{E}_y$. What kind of pattern does the endpoint of vector $\mathbf{E}(z,t)$ make in (x,y) plane, if $|\underline{E}_x|=|\underline{E}_y|$ and the relative phase difference $\arg\left(\frac{\underline{E}_y}{\underline{E}_x}\right)$ between both components takes a value of 0, π/2, π, and 3π/2?

Solution

Our wave is constructed using two orthogonal waves. The shape resulting from two orthogonal sinusoidal movements are Lissajous-figures. The exact shape is determined by the frequencies and the relative phase difference between the two orthogonal waves.

In our case, the frequencies and the amplitudes are the same, and the phase differences of 0, π/2, π, and 3π/2 describe the following Lissajous-figures:

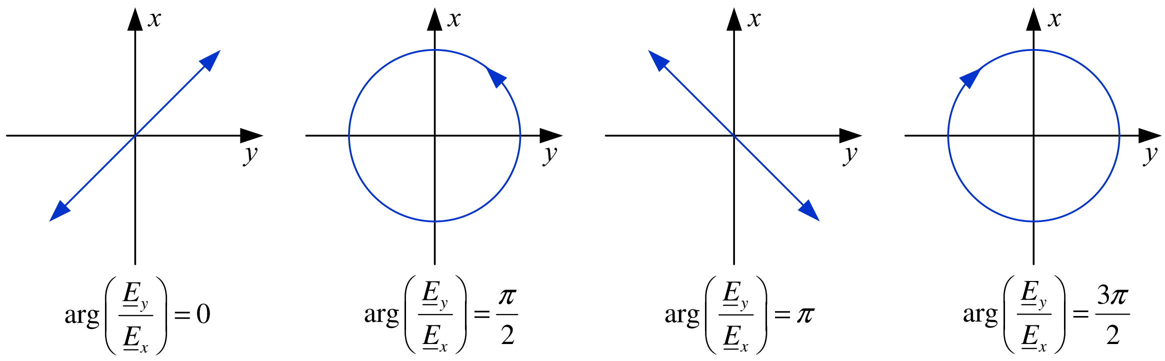


Figure 1: Lissajous-figures described by the endpoint vector $\mathbf{E}(z,t)$ for different values of the phase difference between $\underline{E}_y$ and $\underline{E}_x$.

- $\arg\{\underline{E}_y/\underline{E}_x\}=0$: $\underline{E}_x$ and $\underline{E}_y$ are in-phase: linear polarization, 45°.
- $\arg\{\underline{E}_y/\underline{E}_x\}=\frac{\pi}{2}$: $\underline{E}_y$ is π/2 ahead: anti-clockwise circular polarization.
- $\arg\{\underline{E}_y/\underline{E}_x\}=\pi$: $\underline{E}_x$ and $\underline{E}_y$ are phase-shifted by π: linear polarization, –45°.
- $\arg\{\underline{E}_y/\underline{E}_x\}=\frac{3\pi}{2}$: $\underline{E}_x$ is π/2 ahead: clockwise circular polarization.

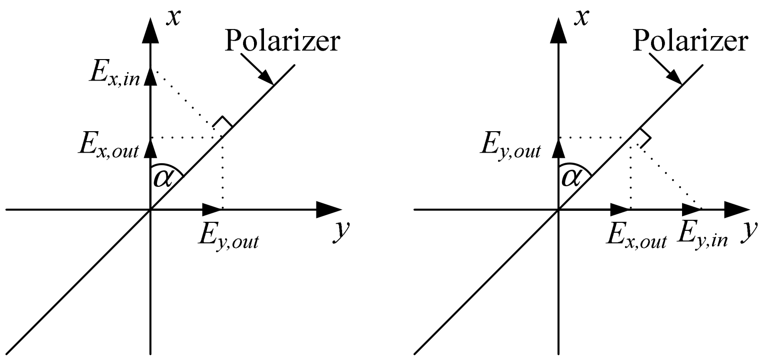
Please note that there are two opposing conventions for polarization states, namely observer is at the point of view of the source, or of the receiver. Circularly polarized light can be referred clockwise or anti-clockwise. The clockwise circular polarization in one convention corresponds to the anti-clockwise in the other, and vice versa. In this solution, we have used the convention of the observer at the point of view of the receiver.

2. Only the components of the light that are polarized parallel to the transmission axis of a polarizer can pass. Show that the complex field vector $\mathbf{E}_{out}$ at the output of a polarizer is given by $\mathbf{E}_{out}=\mathbf{M}_\alpha\mathbf{E}_{in}$, where the following matrix $\mathbf{M}_\alpha$ (also called Jones matrix) describes a polarizer with a transmission axis that forms an angle of α with the x-axis:

$$\mathbf{M}_\alpha=\begin{pmatrix}\cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha\end{pmatrix}. \tag{1.2}$$

Solution

Let us consider each component E_x and E_y of the incident wave separately.



$$\mathbf{E}_{out}\Big|_{E_{y,in}=0} = \begin{pmatrix} E_{x,out} \\ E_{y,out} \end{pmatrix} = \begin{pmatrix} \cos^2\alpha \\ \sin\alpha\cos\alpha \end{pmatrix} E_{x,in} \qquad \mathbf{E}_{out}\Big|_{E_{x,in}=0} = \begin{pmatrix} E_{x,out} \\ E_{y,out} \end{pmatrix} = \begin{pmatrix} \sin\alpha\cos\alpha \\ \sin^2\alpha \end{pmatrix} E_{y,in}$$

Figure 1: Output of a polarizer for different angles of the polarized light at the input.

The wave at the output of the linear polarizer whose axis forms an angle α with *x*-axis can be expressed as the summation of the above derived fields:

$$\mathbf{E}_{out} = \mathbf{E}_{out}\Big|_{E_{y,in}=0} + \mathbf{E}_{out}\Big|_{E_{x,in}=0} = \underbrace{\begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix}}_{\mathbf{M}_\alpha} \mathbf{E}_{in}\,.$$

When we apply a voltage U between the input and output facet of the KDP crystal, the nonlinear medium becomes birefringent. The orientation of the crystal is such that the principal axes x' and y' of the index ellipsoid are rotated by -45° with respect to x and y. This leads to a phase difference between the x'- and y'-components of the electric field (see Eq. (3.35) in the lecture notes):

$$\Delta\Phi = r_{63} n_o^3 E_z k_0 L = r_{63} n_o^3 k_0 U\ , \tag{1.3}$$

where L is the length of the crystal, and E_z is the applied field along the crystal (induced by the voltage U).

3. Derive the transfer matrix $\mathbf{M}_{\mathrm{KDP}}$ of the KDP crystal that relates the complex field vector $\mathbf{E}_{out}$ to the input field $\mathbf{E}_{in}$. Use the $x'y'$ coordinate system as your basis.

Solution

The KDP is a uniaxial crystal that becomes biaxial under the influence of an externally applied electric field. This is called the Pockels effect or linear electro-optic effect. When a wave enters the crystal, it can be decomposed into two components that are polarized parallel to the principal axes x' and y'. The phase difference between the x' and y' component of the electric field is given by:

$$\Delta\Phi = \left(n_{x'} - n_{y'}\right) k_0 L = r_{63} n_o^3 E_z k_0 L = r_{63} n_o^3 k_0 U\ . \tag{1.3a}$$

Expressing the input and output electric field in the following way:

$$\vec{E}_{in} = E_{x',in}\vec{e}_{x'} + E_{y',in}\vec{e}_{y'}$$
$$\vec{E}_{out} = E_{x',in}e^{-\mathrm{j}n_{x'}k_0L}\vec{e}_{x'} + E_{y',in}e^{-\mathrm{j}n_{y'}k_0L}\vec{e}_{y'} = E_{x',out}\vec{e}_{x'} + E_{y',out}\vec{e}_{y'},$$

and using matrix notation, we get:

$$\begin{pmatrix} E_{x',out} \\ E_{y',out} \end{pmatrix} = \underbrace{e^{-\mathrm{j}\frac{n_{x'}+n_{y'}}{2}k_0L}}_{\text{common phase}} \begin{pmatrix} e^{-\mathrm{j}\frac{\Delta\phi}{2}} & 0 \\ 0 & e^{\mathrm{j}\frac{\Delta\phi}{2}} \end{pmatrix} \begin{pmatrix} E_{x',in} \\ E_{y',in} \end{pmatrix}.$$

The common phase term can be neglected, as it only influences the 'global' phase of the waves and has no influence over the phase difference. As it can be seen from Eq. (1.3), the phase difference does not depend on L, and is only influenced by changing the voltage U.

4. The λ/4 wave plate causes an additional phase shift of π/2 (λ/4 light path difference) between x'- and y'-components of the field. Its slow x' and fast y' axes have the same orientation as the respective principal axes of KDP. Derive the corresponding transfer matrix $\mathbf{M}_{\lambda/4}$ in the $x'y'$ coordinate system.

Solution

The quarter-wave plate delays the x' component (slow axis) by π/2 (λ/4 light path difference) relative to y' component (fast axis). The matrix is:

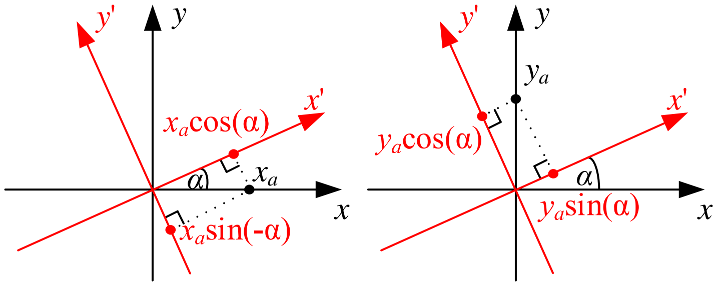
$$\mathbf{M}_{\lambda/4} = \begin{pmatrix} e^{-\mathrm{j}\frac{\pi}{4}} & 0 \\ 0 & e^{\mathrm{j}\frac{\pi}{4}} \end{pmatrix}.$$

5. Derive the transfer matrix of the complete system.

   Hint: The coordinate transformations between $x'y'$ and xy can be formulated by 2x2 rotation matrices.

Solution

To calculate the overall system behavior, we need the transformation between our different coordinate systems. In general, if the original coordinate system xy is rotated at an angle α about the origin, in order to create the new coordinate system $x'y'$, the matrix relation that describes the coordinate transformation from xy to $x'y'$ can be derived by separately analyzing mapping of the x and y coordinates of a point in the xy system to the coordinates of the $x'y'$ system. This has been illustrated in Fig. 1.



$$\begin{pmatrix}x'_a\\y'_a\end{pmatrix}_{y=0}=\begin{pmatrix}\cos(\alpha)\\\sin(-\alpha)\end{pmatrix}x_a \qquad \begin{pmatrix}x'_a\\y'_a\end{pmatrix}_{x=0}=\begin{pmatrix}\cos(\alpha)\\\sin(\alpha)\end{pmatrix}y_a$$

By superposition of the two cases we get the matrix that describes the coordinate transformation by rotation by α°: $\begin{pmatrix}x'_a\\y'_a\end{pmatrix}=\begin{pmatrix}\cos(\alpha)&\sin(\alpha)\\\sin(-\alpha)&\cos(\alpha)\end{pmatrix}\begin{pmatrix}x_a\\y_a\end{pmatrix}$.

The coordinate transformation from xy to $x'y'$ corresponds to the rotation of the coordinate axes by -45°; the reverse transformation is the rotation by 45°. Remember that the matrices are arranged from right to left with respect to the light propagation:

$$\begin{pmatrix}E_{x,out}\\E_{y,out}\end{pmatrix}=\underbrace{\begin{pmatrix}0&0\\0&1\end{pmatrix}}_{\text{Pol2, }\alpha=90^\circ}\underbrace{\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}}_{(x',y')\to(x,y)}\underbrace{\begin{pmatrix}e^{-j\frac{\pi}{4}}&0\\0&e^{j\frac{\pi}{4}}\end{pmatrix}}_{\lambda/4\text{ plate}}\underbrace{\begin{pmatrix}e^{-j\frac{\Delta\phi}{2}}&0\\0&e^{j\frac{\Delta\phi}{2}}\end{pmatrix}}_{\text{KDP}}\underbrace{\begin{pmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}}_{(x,y)\to(x',y')}\underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix}}_{\text{Pol1, }\alpha=0^\circ}\begin{pmatrix}E_{x,in}\\E_{y,in}\end{pmatrix}=$$

$$=\frac{1}{2}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}e^{-j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}&0\\0&e^{j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}E_{x,in}\\E_{y,in}\end{pmatrix}=$$

$$=\frac{1}{2}\begin{pmatrix}0&0\\-1&1\end{pmatrix}\begin{pmatrix}e^{-j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}&0\\0&e^{j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix}\begin{pmatrix}E_{x,in}\\E_{y,in}\end{pmatrix}=$$

$$=\frac{1}{2}\begin{pmatrix}0&0\\-1&1\end{pmatrix}\begin{pmatrix}e^{-j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}&0\\e^{j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}&0\end{pmatrix}\begin{pmatrix}E_{x,in}\\-E_{y,in}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0&0\\e^{j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}-e^{-j\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)}&0\end{pmatrix}\begin{pmatrix}E_{x,in}\\E_{y,in}\end{pmatrix}=$$

$$=\begin{pmatrix}0&0\\j\sin\left(\frac{\pi}{4}+\frac{\Delta\phi}{2}\right)&0\end{pmatrix}\begin{pmatrix}E_{x,in}\\E_{y,in}\end{pmatrix}.$$

$$E_{x,out} = 0$$
$$E_{y,out} = \mathrm{j}\sin\left(\frac{\pi}{4} + \frac{\Delta\phi}{2}\right) E_{x,in}.$$

6. Calculate the power transmission

$$T = \frac{I_{out}}{I_{in}} = \frac{|\mathbf{E}_{out}|^2}{|\mathbf{E}_{in}|^2} \tag{1.4}$$

of a plane wave that is linearly polarized in x-direction.

Solution

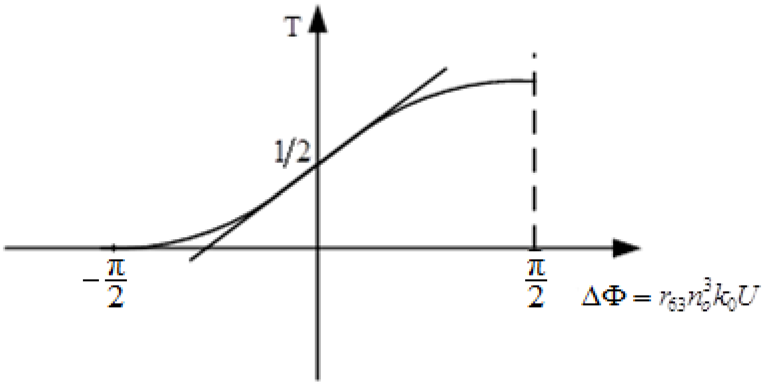
The power transmission is maximum for $\vec{E}_{in} = E_{x,in}\vec{e}_x$. It can be written as:

$$T = \frac{I_{out}}{I_{in}} = \frac{|E_{out}|^2}{|E_{in}|^2} = \frac{|E_{y,out}|^2}{|E_{x,in}|^2} = \sin^2\left(\frac{\Delta\phi}{2} + \frac{\pi}{4}\right).$$

7. Sketch the dependence of the transmission T on the voltage U. What is the purpose of the λ/4 wave plate?

Solution

Sketch:



Without applied voltage, half of the power is transmitted. The transmission increases with a positive voltage and decreases with a negative voltage. In this operating point, the transmission-voltage dependence can be approximated by a linear function, and the slope (the first derivative) of this dependence is here the largest. That means that a large transmission variation can be achieved for a relatively small voltage variation.

Without the quarter-wave plate, the transmitted power would depend quadratically on the applied voltage, and larger voltage variations (and more time) would be necessary to get the same extinction ratio (the ratio between two transmission levels, which is an important parameter in telecommunications).

Questions and Comments:

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