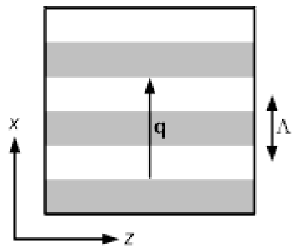
Solution to Problem Set 11
Nonlinear Optics (NLO)

1) Acousto-Optic Modulator

Consider a material in which a sound wave is travelling in *x*-direction with wave vector $\mathbf{q}$ and frequency Ω. The associated strain induces a periodic change of the refractive index that scatters an incoming optical wave. In Eq. (4.14) of the lecture notes, we derived a coupled-wave relation for the space-dependent amplitudes $\underline{E}(\mathbf{r},\omega_l)$ of the incoming optical wave $(l=0)$ at frequency ω_0 and the various scattered optical waves at frequencies ω_l. Assume that all optical waves are polarized along the *y*-direction, i.e., $e_l = e_y \; \forall\, l$. The scalar coupled-wave equation can then be written as



$$\sum_l -2\mathrm{j}\mathbf{k}_l \cdot \nabla \underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})} = \frac{2n_0}{c^2} \sum_l \frac{\partial^2}{\partial t^2}\left(\Delta n(\mathbf{r},t)\underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})}\right), \tag{0.1}$$

where the index variation $\Delta n(\mathbf{r},t)$ is given by

$$\Delta n(\mathbf{r},t) = \Delta n_0 \cos(\Omega t - \mathbf{qr}).$$

1. For a monochromatic incident optical wave at frequency ω_0, the right-hand side of Eq. (0.1) contains frequency components at $\omega_{\pm 1} = \omega_0 \pm \Omega$. In Eq. (1.1), consider only the expressions with $l=0$ and $l=1$, and derive two coupled differential equations for the optical wave amplitudes $\underline{E}(\mathbf{r},\omega_0)$ and $\underline{E}(\mathbf{r},\omega_1)$ by comparing the coefficients associated with the same frequency on the left- and right-hand side of Eq. (0.1).

Solution:

In an acousto-optic modulator an incident optical wave (frequency ω_0 and wave vector $\mathbf{k}_0$) interacts with a sound wave (frequency Ω and propagation vector $\mathbf{q}$). The interaction must obey both energy and momentum conservation. For the given example this means that: $\omega_1 = \omega_0 + \Omega$ and $\mathbf{k}_1 = \mathbf{k}_0 + \mathbf{q}$. As a first step let us reformulate the expression for Δn:

$$\Delta n(\mathbf{r},t) = \frac{1}{2}\Delta n_0 \left(e^{\mathrm{j}(\Omega t - \mathbf{qr})} + e^{-\mathrm{j}(\Omega t - \mathbf{qr})}\right),$$

and insert it into Eq. (1.1):

$$\sum_l -2\mathrm{j}\mathbf{k}_l \cdot \nabla \underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})} = \frac{n_0}{c^2}\Delta n_0 \sum_l \frac{\partial^2}{\partial t^2}\left(\underline{E}(\mathbf{r},\omega_l)\left(e^{\mathrm{j}((\omega_l+\Omega)t - (\mathbf{k}_l+\mathbf{q})\mathbf{r})} + e^{\mathrm{j}((\omega_l-\Omega)t - (\mathbf{k}_l-\mathbf{q})\mathbf{r})}\right)\right).$$

We are now only concerned about terms containing the frequencies ω_0 (the incident wave) and $\omega_1 = \omega_0 + \Omega$ (the deflected wave) in the exponent. On the left hand side of the last equation, these are:

$$\omega_0: \quad -2\mathrm{j}\mathbf{k}_0 \nabla \underline{E}(z,\omega_0) e^{\mathrm{j}(\omega_0 t - \mathbf{k}_0 \mathbf{r})},$$
$$\omega_1: \quad -2\mathrm{j}\mathbf{k}_1 \nabla \underline{E}(z,\omega_1) e^{\mathrm{j}(\omega_1 t - \mathbf{k}_1 \mathbf{r})}.$$

On the right hand side of the same equation, we find:

$$\omega_0: \quad \frac{n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1)\frac{\partial^2}{\partial t^2} e^{\mathrm{j}((\omega_1-\Omega)t-(\mathbf{k}_1-\mathbf{q})\mathbf{r})} = -\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1) e^{\mathrm{j}(\omega_0 t-(\mathbf{k}_1-\mathbf{q})\mathbf{r})}$$
$$\omega_1: \quad \frac{n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0)\frac{\partial^2}{\partial t^2} e^{\mathrm{j}((\omega_0+\Omega)t-(\mathbf{k}_0+\mathbf{q})\mathbf{r})} = -\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0) e^{\mathrm{j}(\omega_1 t-(\mathbf{k}_0+\mathbf{q})\mathbf{r})},$$

where we made use of the fact that $\omega_0 = \omega_1 - \Omega$. We can now write the two coupled differential equations as:

$$\mathbf{k}_1 \nabla \underline{E}(z,\omega_1) = -\mathrm{j}\frac{1}{2}\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0) e^{-\mathrm{j}(\mathbf{k}_0+\mathbf{q}-\mathbf{k}_1)\mathbf{r}}, \text{ and} \tag{1.2}$$

$$\mathbf{k}_0 \nabla \underline{E}(z,\omega_0) = -\mathrm{j}\frac{1}{2}\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1) e^{\mathrm{j}(\mathbf{k}_0+\mathbf{q}-\mathbf{k}_1)\mathbf{r}}. \tag{1.3}$$

2. Consider the case where both the crystal and the optical waves are infinitely extended in *x*- and *y*-direction, which implies $\frac{\partial \underline{E}}{\partial x} = 0$ and $\frac{\partial \underline{E}}{\partial y} = 0$. Assume further that the *z*-components of the **k**-vector for both optical waves are equal, i.e. $k_{0z} = k_{1z} = k_z$. Using these simplifications, show that the two coupled differential equations can be written as:

$$\frac{\partial \underline{E}(z,\omega_1)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_0) e^{-\mathrm{j}\Delta\mathbf{k}\mathbf{r}}$$

$$\frac{\partial \underline{E}(z,\omega_0)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_1) e^{\mathrm{j}\Delta\mathbf{k}\mathbf{r}}$$

with $\kappa = \frac{k_z \Delta n_0}{2n_0}$ and $\Delta\mathbf{k} = \mathbf{k}_0 + \mathbf{q} - \mathbf{k}_1$.

Solution:

1. The term $\mathbf{k}\nabla\underline{E}$ can be written as $k_x \frac{\partial \underline{E}}{\partial x} + k_y \frac{\partial \underline{E}}{\partial y} + k_z \frac{\partial \underline{E}}{\partial z}$. Using this we can rewrite Eqs. (1.2) and (1.3) as:

$$\frac{\partial \underline{E}(\mathbf{r},\omega_1)}{\partial z} = -\mathrm{j}\frac{1}{2k_{1z}}\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0)e^{-\mathrm{j}\Delta\mathbf{k}\mathbf{r}}\text{, and}$$

$$\frac{\partial \underline{E}(\mathbf{r},\omega_0)}{\partial z} = -\mathrm{j}\frac{1}{2k_{0z}}\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1)e^{\mathrm{j}\Delta\mathbf{k}\mathbf{r}}\,.$$

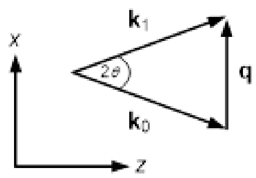
Using the relation $\frac{1}{k_{1z}}\frac{\omega_1^2 n_0}{c^2} = \frac{1}{k_{1z}}\frac{k_{1z}^2}{n_0} = \frac{k_{1z}}{n_0}$, we get the given differential equations:

$$\frac{\partial \underline{E}(z,\omega_1)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_0)e^{-\mathrm{j}\Delta\mathbf{k}\mathbf{r}}\text{, and} \tag{1.4}$$

$$\frac{\partial \underline{E}(z,\omega_0)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_1)e^{\mathrm{j}\Delta\mathbf{k}\mathbf{r}}\,, \tag{1.5}$$

with $\kappa = \frac{k_z \Delta n_0}{2n_0}$.

3. Solve the differential equations assuming perfect phase matching, i.e. $\Delta\mathbf{k} = 0$ and using the boundary conditions $\underline{E}(0,\omega_0) = E_0$ and $\underline{E}(0,\omega_1) = 0$. Sketch the evolution of the intensities of the incident and the deflected wave along *z*. How long should the crystal extend in the *z*-direction for maximum intensity of the deflected wave?



Solution:

Assuming perfect phase matching, i.e., $\Delta\mathbf{k} = 0$, and taking the derivative of Eq. (1.5) with respect to *z*, we get:

$$\frac{\partial^2 \underline{E}(z,\omega_0)}{\partial z^2} = -\mathrm{j}\kappa\frac{\partial \underline{E}(z,\omega_1)}{\partial z}\,.$$

By substituting this into Eq. (1.4), we get:

$$\frac{\partial^2 \underline{E}(z,\omega_0)}{\partial z^2} = -\kappa^2 \underline{E}(z,\omega_0)\,. \tag{1.6}$$

In a similar fashion, by first taking the derivative of Eq. (1.4) with respect to *z*, and then substituting the result into Eq. (1.5), we get:

$$\frac{\partial^2 \underline{E}(z,\omega_1)}{\partial z^2} = -\kappa^2 \underline{E}(z,\omega_1)\,. \tag{1.7}$$

The general solutions of Eq. (1.6) and (1.7) are respectively:

$$\underline{E}\left(z,\omega_0\right)=\underline{c}_1\cos(\kappa z)+\underline{c}_2\sin(\kappa z)\,,$$

$$\underline{E}\left(z,\omega_1\right)=\underline{c}_3\cos(\kappa z)+\underline{c}_4\sin(\kappa z)\,.$$

From the boundary conditions we find that $\underline{c}_1=E_0$, and $\underline{c}_3=0$, therefore:

$$\underline{E}\left(z,\omega_0\right)=E_0\cos(\kappa z)+\underline{c}_2\sin(\kappa z)\,,$$

$$\underline{E}\left(z,\omega_1\right)=\underline{c}_4\sin(\kappa z)\,.$$

Plugging-in the last two equations into Eq. (1.4), and having in mind that $\Delta\mathbf{k}=0$, we get:

$$\frac{\partial\underline{E}\left(z,\omega_1\right)}{\partial z}=-\mathrm{j}\kappa\underline{E}\left(z,\omega_0\right)$$

$$\frac{\partial\left(\underline{c}_4\sin(\kappa z)\right)}{\partial z}=-\mathrm{j}\kappa\left(E_0\cos(\kappa z)+\underline{c}_2\sin(\kappa z)\right)$$

$$\kappa\underline{c}_4\cos(\kappa z)=-\mathrm{j}\kappa E_0\cos(\kappa z)-\mathrm{j}\kappa\underline{c}_2\sin(\kappa z)$$

The last can be true for all z, only if the sine function on the left disappears, meaning that $\underline{c}_2=0$. Finally we get:

$$\kappa\underline{c}_4\cos(\kappa z)=-\mathrm{j}\kappa E_0\cos(\kappa z)\,,$$

and from here it follows that $\underline{c}_4=-\mathrm{j}E_0$. Therefore:

$$\underline{E}\left(z,\omega_0\right)=E_0\cos(\kappa z)\,,$$

$$\underline{E}\left(z,\omega_1\right)=-\mathrm{j}E_0\sin(\kappa z)\,.$$

The corresponding intensities are proportional to squares of the electric fields. Since both fields have the same magnitude, the same will hold for the intensities:

$$I\left(z,\omega_0\right)=I_0\cos^2(\kappa z)\,,$$

$$I\left(z,\omega_1\right)=I_0\sin^2(\kappa z)\,.$$

The maximum intensity of the deflected wave will occur at $z=L$ that gives: $\sin^2(\kappa L)=1$. From here we can calculate L as:

$$L=\frac{\pi}{2\kappa}=\frac{\pi n_0}{k_z\Delta n_0}\,.$$

The intensity evolution of the two waves is displayed in Fig. 1.

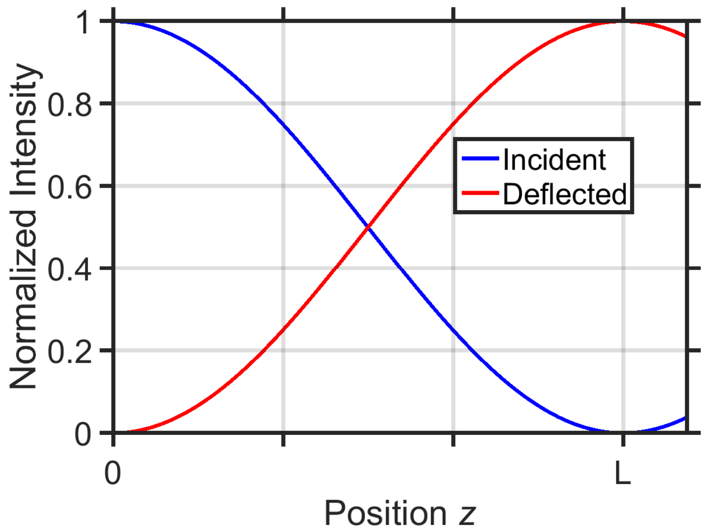


Fig. 1: Normalized intensity evolution of the incident and the deflected wave.

4. What is the angle of diffraction for light at 632.8 nm in a $LiNbO_3$ cell that is driven at a frequency of 1 GHz? (speed of sound: $v_s = 4.1$ km/s, refractive index $n_0 = 2.3$)

Solution:

Using the figure associated to Part 3. of this problem set, we can see that $\sin\theta = \frac{|\mathbf{q}|}{2k_z}$.

We can calculate $|\mathbf{q}| = \frac{2\pi}{\Lambda} = \frac{2\pi}{v_s/(\Omega/2\pi)} = \frac{\Omega}{v_s}$. Also: $k_z = \frac{2\pi n_0}{\lambda}$. By plugging-in the given numerical values, we can calculate $\sin\theta$, and from here we get: $\theta \approx 1.92°$.

Questions and Comments:

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