Solution to Problem Set 13
Nonlinear Optics (NLO)

1) Nonlinear Schrödinger Equation

The nonlinear Schrödinger-equation (NLSE) of an optical fiber was derived in the lecture:

$$\frac{\partial}{\partial z}\underline{A}(z,t)+\beta_c^{(1)}\frac{\partial}{\partial t}\underline{A}(z,t)-\frac{1}{2}\mathrm{j}\beta_c^{(2)}\frac{\partial^2}{\partial t^2}\underline{A}(z,t)=-\frac{\alpha}{2}\underline{A}(z,t)-\mathrm{j}\gamma\left|\underline{A}(z,t)\right|^2\underline{A}(z,t) \quad (1.1)$$

1. Explain the parameters $\beta_c^{(1)}$, $\beta_c^{(2)}$, α and γ.

Solution

$\beta_c^{(1)}$: Reciprocal of group velocity

$\beta_c^{(2)}$: Group velocity dispersion (GVD): Frequency dependence of the group velocity

α: Loss due to material absorption and scattering

γ: Third-order nonlinearity parameter

2. For optical fibers, the parameter $D=\mathrm{d}\beta_c^{(1)}/\mathrm{d}\lambda$ is usually specified instead of $\beta_c^{(2)}$. What is the connection between D and $\beta_c^{(2)}$? A standard single-mode fiber (SSMF) has D = 18 ps/(nm·km) at the vacuum wavelength of λ = 1.55 µm. Calculate $\beta_c^{(2)}$ and explain the meaning of D.

Solution

$$\beta_c^{(2)}=\frac{\mathrm{d}\beta_c^{(1)}}{\mathrm{d}\omega}$$

In order to find $\beta_c^{(2)}$, we need to find the operator $\frac{\mathrm{d}}{\mathrm{d}\omega}$. Since $\omega=\frac{2\pi c}{\lambda}$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}=\frac{\mathrm{d}\omega}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\omega}=-\frac{2\pi c}{\lambda^2}\frac{\mathrm{d}}{\mathrm{d}\omega}$$

$$D=\frac{\mathrm{d}\beta_c^{(1)}}{\mathrm{d}\lambda}=-\frac{2\pi c}{\lambda^2}\frac{\mathrm{d}\beta_c^{(1)}}{\mathrm{d}\omega}=-\frac{2\pi c}{\lambda^2}\beta_c^{(2)}$$

$$\beta_c^{(2)}=-\frac{\lambda^2}{2\pi c}D=-22.96\frac{\mathrm{ps}}{\mathrm{km}^2}$$

The dispersion value $D=18\frac{\mathrm{ps}}{\mathrm{km}\cdot\mathrm{nm}}$ at the vacuum wavelength of $1.55\mu\mathrm{m}$ means that, after 1km of transmission, two signals with a center wavelength difference of 1nm will be delayed with respect to one another by 18nm.

3. Consider the new coordinate system t', z' generated by the following transformation as well as the new function $\underline{A}'$:

$$t' = t - \beta_c^{(1)} z$$
$$z' = z$$
$$\underline{A}'(z', t') = \underline{A}(z, t)$$

Imagine for the moment that $\underline{A}(z,t)$ represents a pulse moving along z with velocity $1/\beta_c^{(1)}$. Sketch the functions $\underline{A}(z,t)$ and $\underline{A}'(z',t')$ as a function of t and t' for two different positions of z and z'. Explain why the (z',t') coordinate system is usually referred to as a retarded time frame.

Solution

The (z',t') coordinate system is usually referred to as retarded time frame, because for a certain coordinate z', the time of the pulse arrival is always $t'=0$. While in the (z,t) coordinate system z and t are independent coordinates, in the (z',t') coordinate system t' depends on z'. The counting of time at coordinate z' is delayed until the pulse reaches it (delayed = retarded). The $(z,t) \rightarrow (z',t')$ coordinate transform is introduced in order to simplify our equations.

The sketches of $\underline{A}(z,t)$ and $\underline{A}'(z',t')$ are provided in Fig. 1.

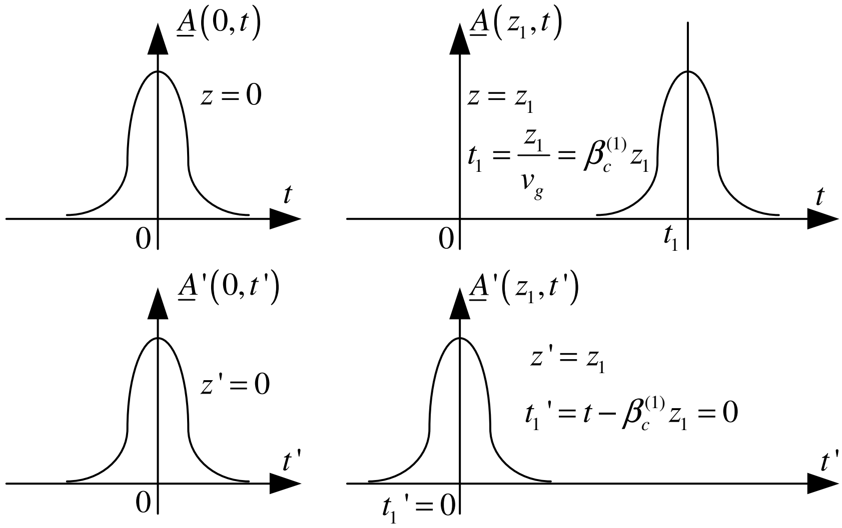

Fig. 1: Sketches of the functions $\underline{A}(z,t)$ and $\underline{A}'(z',t')$ as a function of t and t' for two different positions of z and z'

4. Find a formulation of the NLSE for $\underline{A}'$. Notice that the term $\beta_c^{(1)} \frac{\partial}{\partial t'} \underline{A}'(z',t')$ does not longer appear in the differential equation. In the following, we will omit the primes

keeping in mind that the time dependence is given with respect to a retarded reference frame.

Solution

In order to reformulate the NLSE, we must find the following derivatives: $\frac{\partial}{\partial z}\underline{A}(z,t)$, $\frac{\partial}{\partial t}\underline{A}(z,t)$, and $\frac{\partial^2}{\partial t^2}\underline{A}(z,t)$. By applying the chain rule, we get:

$$\frac{\partial \underline{A}}{\partial z} = \frac{\partial \underline{A}'}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial \underline{A}'}{\partial t'}\frac{\partial t'}{\partial z} = \frac{\partial \underline{A}'}{\partial z'} - \beta_c^{(1)}\frac{\partial \underline{A}'}{\partial t'}$$

$$\frac{\partial \underline{A}}{\partial t} = \frac{\partial \underline{A}'}{\partial z'}\frac{\partial z'}{\partial t} + \frac{\partial \underline{A}'}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial \underline{A}'}{\partial t'}$$

$$\frac{\partial \underline{A}^2}{\partial t^2} = \frac{\partial^2 \underline{A}'}{\partial t'^2}$$

From here it follows:

$$\frac{\partial \underline{A}'}{\partial z'} \cancel{-\beta_c^{(1)}\frac{\partial \underline{A}'}{\partial t'}} \cancel{+\beta_c^{(1)}\frac{\partial \underline{A}'}{\partial t'}} - \frac{1}{2}\mathrm{j}\beta_c^{(2)}\frac{\partial^2 \underline{A}'}{\partial t'^2} = -\frac{\alpha}{2}\underline{A}' - \mathrm{j}\gamma\left|\underline{A}'\right|^2\underline{A}', \text{ where: } \underline{A}' = \underline{A}'(z',t').$$

5. We will now assume that there are no losses ($\alpha = 0$) and search for solutions describing fundamental solitons, i.e., waveforms which do not change their shape as they propagate along *z*. We therefore require the magnitude of the complex amplitude $\underline{A}'(z,t)$ to be independent of *z*, but still allow for a *z*-dependent phase shift. Substitute the ansatz $\underline{A}'(z,t) = A_0(t)\exp(-\mathrm{j}Kz)$ in the NLSE. Assuming further that $A_0(t)$ is a real function, show that the following differential equation holds for $A_0(t)$:

$$\frac{1}{2}\beta_c^{(2)}\frac{1}{A_0(t)}\frac{\partial^2 A_0(t)}{\partial t^2} - \gamma A_0^2(t) = -K \quad . \tag{1.2}$$

Solution

By assuming that there are no losses ($\alpha = 0$), and inserting the ansatz for solitons $\underline{A}'(z,t) = A_0(t)e^{-\mathrm{j}Kz}$, we get the following relations:

$$\frac{\partial A_0(t)e^{-\mathrm{j}Kz}}{\partial z} - \frac{1}{2}\mathrm{j}\beta_c^{(2)}\frac{\partial^2}{\partial t^2}\left(A_0(t)e^{-\mathrm{j}Kz}\right) = -\mathrm{j}\gamma A_0(t)e^{-\mathrm{j}Kz}A_0(t)e^{\mathrm{j}Kz}A_0(t)e^{-\mathrm{j}Kz}$$

$$\cancel{-\mathrm{j}}KA_0(t)\cancel{e^{-\mathrm{j}Kz}}\cancel{-\mathrm{j}}\frac{1}{2}\beta_c^{(2)}\frac{\partial^2 A_0(t)}{\partial t^2}\cancel{e^{-\mathrm{j}Kz}} = \cancel{-\mathrm{j}}\gamma A_0^2(t)A_0(t)\cancel{e^{-\mathrm{j}Kz}}$$

$$\frac{\beta_c^{(2)}}{2}\frac{\partial^2 A_0(t)}{\partial t^2} - \gamma A_0^2(t)A_0(t) = -KA_0(t).$$

By dividing the last equation by $A_0(t)$ we get Eq. (1.2).

6. Show that $A_0(t) = A_1 \operatorname{sech}\left(\frac{t}{T}\right) = A_1/\cosh\left(\frac{t}{T}\right)$ is a valid solution ansatz for the differential equation (1.2). Remember that $\cosh^2 - \sinh^2 = 1$, and that the derivative of sinh(x) is cosh(x) and vice versa. Show in particular that the pulse amplitude A_1 and the pulse duration T must fulfill the following relations:

$$K = \frac{1}{2}\gamma A_1^2 \tag{1.3}$$

$$A_1^2 = -\frac{\beta_c^{(2)}}{\gamma T^2}\ . \tag{1.4}$$

Solution

By inserting the ansatz into Eq. (1.2), we get:

$$\frac{\partial^2 A_0(t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial A_0(t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(\frac{A_1}{\cosh\left(\frac{t}{T}\right)}\right)\right) = \frac{\partial}{\partial t}\left(\frac{-A_1\sinh\left(\frac{t}{T}\right)}{\cosh^2\left(\frac{t}{T}\right)}\cdot\frac{1}{T}\right) =$$

$$= -\frac{A_1}{T}\cdot\frac{\frac{1}{T}\cosh\left(\frac{t}{T}\right)\cosh^2\left(\frac{t}{T}\right) - 2\cosh\left(\frac{t}{T}\right)\sinh\left(\frac{t}{T}\right)\sinh\left(\frac{t}{T}\right)}{\cosh^4\left(\frac{t}{T}\right)} =$$

$$= \frac{A_1}{T^2}\cdot\frac{2\sinh^2\left(\frac{t}{T}\right) - \cosh^2\left(\frac{t}{T}\right)}{\cosh^3\left(\frac{t}{T}\right)} = \frac{A_1}{T^2}\cdot\frac{2\left(\cosh^2\left(\frac{t}{T}\right) - 1\right) - \cosh^2\left(\frac{t}{T}\right)}{\cosh^3\left(\frac{t}{T}\right)} =$$

$$= \frac{A_1}{T^2}\cdot\frac{\cosh^2\left(\frac{t}{T}\right) - 2}{\cosh^3\left(\frac{t}{T}\right)} = \frac{A_1}{T^2}\cdot\frac{1}{\cosh\left(\frac{t}{T}\right)}\left(1 - \frac{2}{\cosh^2\left(\frac{t}{T}\right)}\right).$$

From here it follows:

$$\frac{\beta_c^{(2)}}{2T^2}\cdot\frac{1}{A_0(t)}\cdot\frac{A_1}{\cosh\left(\frac{t}{T}\right)}\left(1 - \frac{2}{\cosh^2\left(\frac{t}{T}\right)}\right) - \gamma A_o^2(t) = -K$$

$$\frac{\beta_c^{(2)}}{2T^2} + K = \gamma\frac{A_1^2}{\cosh^2\left(\frac{t}{T}\right)} + \frac{\beta_c^{(2)}}{T^2\cosh^2\left(\frac{t}{T}\right)} = \frac{1}{\cosh^2\left(\frac{t}{T}\right)}\left(\gamma A_1^2 + \frac{\beta_c^{(2)}}{T^2}\right).$$

This means the following: $\text{const}_1 = f(t)\cdot\text{const}_2$, and it is only possible if both constants are equal to 0, meaning: $\frac{\beta_c^{(2)}}{2T^2}+K=0$, and $\gamma A_1^2+\frac{\beta_c^{(2)}}{T^2}=0$. From here it follows: $\frac{\beta_c^{(2)}}{T^2}=-2K$, and $\frac{\beta_c^{(2)}}{T^2}=-\gamma A_1^2$. Finally, we get: $K=\frac{\gamma A_1^2}{2}$, and $A_1^2=-\frac{\beta_c^{(2)}}{\gamma T^2}$.

7. Eq. 1.2 can be reformulated as:

$$\frac{1}{2}\beta_c^{(2)}\frac{\partial^2 A_0(t)}{\partial t^2}=\left(\gamma A_0^2(t)-K\right)A_0(t). \tag{1.5}$$

How can this relation be interpreted taking into account the interplay of dispersion and self-phase modulation? If a pulse gets shorter, do you expect that it must have a larger or smaller peak intensity for building a soliton? Check your answer with the help of Eq. (1.4).

Solution

If we take a look at Eq. (1.4), it becomes clear that in order for the equation to be true, if the pulse gets shorter (T gets smaller), the pulse peak intensity A_1 must become larger.

Questions and Comments:

Aleksandar Nesic
Building: 30.10, Room: 2.32-2
Phone: 0721/608-42480

Matthias Blaicher
Building: 30.10, Room: 2.22
Phone: 0721/608-41934 (26824)

nlo@ipq.kit.edu