Problem Set 3
Nonlinear Optics (NLO)

Due: May 16, 2018, 08:00 AM

1) Nonlinear wave equation

From Maxwell's equations, one can derive the following nonlinear wave equation (Eq. (1.80) in the lecture notes):

$$\nabla^2 \mathbf{E}(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_{\mathrm{NL}}(\mathbf{r},t)}{\partial t^2}. \tag{1.1}$$

In order to solve this second-order differential equation analytically, it is necessary to simplify it. To this end, we assume a homogeneous isotropic medium and reduce the representation to plane waves that are polarized along the x-direction:

$$\mathbf{E}(\mathbf{r},t) = E(z,t)\mathbf{e}_x, \text{ and} \tag{1.2}$$

$$\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t) = P_{\mathrm{NL}}(z,t)\mathbf{e}_x, \tag{1.3}$$

where $\mathbf{e}_x$ denotes the unit vector along the x-direction. The scalar electric field $E(z,t)$ and the nonlinear polarization $P_{\mathrm{NL}}(z,t)$ can be written as:

$$E(z,t) = \frac{1}{2}\left(\sum_l \underline{E}(z,t,\omega_l)\exp\left(\mathrm{j}(\omega_l t - k_l z)\right) + \text{c.c.}\right), \text{ and} \tag{1.4}$$

$$P_{\mathrm{NL}}(z,t) = \frac{1}{2}\left(\sum_l \underline{P}_{\mathrm{NL}}(z,t,\omega_l)\exp\left(\mathrm{j}(\omega_l t - k_{p,l} z)\right) + \text{c.c.}\right). \tag{1.5}$$

In these relations "c.c." denotes the complex conjugate of the preceding expressions.

1. Insert Eqs. (1.4) and (1.5) into Eq. (1.1), and derive an equation that has to be fulfilled for the complex amplitudes $\underline{E}(z,t,\omega_l) = \underline{E}_l$ and $\underline{P}_{\mathrm{NL}}(z,t,\omega_l) = \underline{P}_{\mathrm{NL},l}$ of a particular frequency ω_l.

   Hint: You can simplify the result considering that $k_l = \dfrac{\omega_l n}{c}$.

To further simplify the equation, we introduce the so-called slowly-varying envelope approximation (SVEA). The SVEA is based on the fact that the complex envelope functions $\underline{E}(z,t,\omega_l)$ and $\underline{P}_{\mathrm{NL}}(z,t,\omega_l)$ do not change significantly on the length scale of one wavelength and the time scale of one optical oscillation period. This can be mathematically expressed as:

$$\left|\frac{\partial^2 \underline{E}(z,t,\omega_l)}{\partial t^2}\right| << \omega_l \left|\frac{\partial \underline{E}(z,t,\omega_l)}{\partial t}\right|, \tag{1.6}$$

$$\left|\frac{\partial^2 \underline{E}(z,t,\omega_l)}{\partial z^2}\right| << k_l \left|\frac{\partial \underline{E}(z,t,\omega_l)}{\partial z}\right|, \text{ and} \tag{1.7}$$

$$\left|\frac{\partial^2 \underline{P}_{\text{NL}}(z,t,\omega_l)}{\partial t^2}\right| << \omega_l \left|\frac{\partial \underline{P}_{\text{NL}}(z,t,\omega_l)}{\partial t}\right| << \omega_l^2 \left|\underline{P}_{\text{NL}}(z,t,\omega_l)\right|. \tag{1.8}$$

2. Simplify the nonlinear wave equation using the SVEA, and express the resulting equation in the following form (Eq. (1.95) in the lecture notes):

$$\frac{\partial \underline{E}(z,t,\omega_l)}{\partial z} + \frac{n}{c}\frac{\partial \underline{E}(z,t,\omega_l)}{\partial t} = -\mathrm{j}\frac{\omega_l}{2\varepsilon_0 cn}\underline{P}_{\text{NL}}(z,t,\omega_l)\, e^{-\mathrm{j}(k_{p,l}-k_l)z}. \tag{1.9}$$

3. Introduce a retarded time frame that is propagating along with the optical signal, i.e., $z' = z$ and $t' = t - \frac{zn}{c}$, and reformulate Eq. (1.9). Explain the physical meaning of the retarded time frame.

4. For self-phase modulation (SPM) and cross-phase modulation (XPM) the nonlinear polarization is given by

$$\underline{P}_{\text{NL,SPM}} = \frac{3}{4}\varepsilon_0 \chi^{(3)} \left|\underline{E}(z,t,\omega_1)\right|^2 \underline{E}(z,t,\omega_1), \text{ and} \tag{1.10}$$

$$\underline{P}_{\text{NL,XPM}} = \frac{6}{4}\varepsilon_0 \chi^{(3)} \left|\underline{E}(z,t,\omega_2)\right|^2 \underline{E}(z,t,\omega_1), \tag{1.11}$$

respectively. Insert these relations into the simplified nonlinear wave equation, i.e., the result from part 3, which should be equivalent to Eq. (1.99) in the lecture notes. Explain why these processes do not affect the magnitude $\left|\underline{E}(z,t,\omega_1)\right|$ of the complex amplitude if $\chi^{(3)}$ is a real number. Hint: Consider the change $\frac{\partial \underline{E}}{\partial z}$ of the phasor $\underline{E}$ in the complex plane. Explain the terms **self**-phase and **cross**-phase modulation.

Questions and Comments:

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