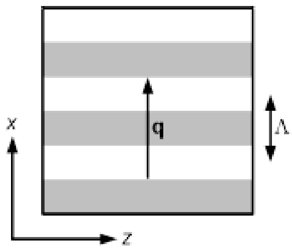
Solution to Problem Set 11
Nonlinear Optics (NLO)

1) Acousto-Optic Modulator

Consider a material in which a sound wave is travelling in x-direction with wave vector $\mathbf{q}$ and frequency Ω. The associated strain induces a periodic change of the refractive index that scatters an incoming optical wave. In Eq. (4.14) of the lecture notes, we derived a coupled-wave relation for the space-dependent amplitudes $\underline{E}(\mathbf{r},\omega_l)$ of the incoming optical wave $(l=0)$ at frequency ω_0 and the various scattered optical waves at frequencies ω_l. Assume that all optical waves are polarized along the y-direction, i.e., $e_l = e_y \ \forall\, l$. The scalar coupled-wave equation can then be written as



$$\sum_l -2\mathrm{j}\mathbf{k}_l \cdot \nabla \underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})} = \frac{2n_0}{c^2} \sum_l \frac{\partial^2}{\partial t^2}\left(\Delta n(\mathbf{r},t)\underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})}\right), \tag{0.1}$$

where the index variation $\Delta n(\mathbf{r},t)$ is given by

$$\Delta n(\mathbf{r},t) = \Delta n_0 \cos(\Omega t - \mathbf{q}\mathbf{r}).$$

1. For a monochromatic incident optical wave at frequency ω_0, the right-hand side of Eq. (0.1) contains frequency components at $\omega_{\pm 1} = \omega_0 \pm \Omega$. In Eq. (1.1), consider only the expressions with $l = 0$ and $l = 1$, and derive two coupled differential equations for the optical wave amplitudes $\underline{E}(\mathbf{r},\omega_0)$ and $\underline{E}(\mathbf{r},\omega_1)$ by comparing the coefficients associated with the same frequency on the left- and right-hand side of Eq. (0.1).

Solution:

In an acousto-optic modulator an incident optical wave (frequency ω_0 and wave vector $\mathbf{k}_0$) interacts with a sound wave (frequency Ω and propagation vector $\mathbf{q}$). The interaction must obey both energy and momentum conservation. For the given example this means that: $\omega_1 = \omega_0 + \Omega$ and $\mathbf{k}_1 = \mathbf{k}_0 + \mathbf{q}$. As a first step let us reformulate the expression for Δn:

$$\Delta n(\mathbf{r},t) = \frac{1}{2}\Delta n_0 \left(e^{\mathrm{j}(\Omega t - \mathbf{q}\mathbf{r})} + e^{-\mathrm{j}(\Omega t - \mathbf{q}\mathbf{r})}\right),$$

and insert it into Eq. (1.1):

$$\sum_l -2\mathrm{j}\mathbf{k}_l \cdot \nabla \underline{E}(\mathbf{r},\omega_l) e^{\mathrm{j}(\omega_l t - \mathbf{k}_l \mathbf{r})} = \frac{n_0}{c^2}\Delta n_0 \sum_l \frac{\partial^2}{\partial t^2}\left(\underline{E}(\mathbf{r},\omega_l)\left(e^{\mathrm{j}((\omega_l+\Omega)t - (\mathbf{k}_l+\mathbf{q})\mathbf{r})} + e^{\mathrm{j}((\omega_l-\Omega)t - (\mathbf{k}_l-\mathbf{q})\mathbf{r})}\right)\right).$$

We are now only concerned about terms containing the frequencies ω_0 (the incident wave) and $\omega_1 = \omega_0 + \Omega$ (the deflected wave) in the exponent. On the left hand side of the last equation, these are:

$$\begin{aligned}\omega_0 :\quad & -2\mathrm{j}\mathbf{k}_0 \nabla \underline{E}(z,\omega_0) e^{\mathrm{j}(\omega_0 t - \mathbf{k}_0 \mathbf{r})},\\ \omega_1 :\quad & -2\mathrm{j}\mathbf{k}_1 \nabla \underline{E}(z,\omega_1) e^{\mathrm{j}(\omega_1 t - \mathbf{k}_1 \mathbf{r})}.\end{aligned}$$

On the right hand side of the same equation, we find:

$$\begin{aligned}\omega_0 :\quad & \frac{n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1)\frac{\partial^2}{\partial t^2} e^{\mathrm{j}((\omega_1-\Omega)t-(\mathbf{k}_1-\mathbf{q})\mathbf{r})} = -\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1) e^{\mathrm{j}(\omega_0 t-(\mathbf{k}_1-\mathbf{q})\mathbf{r})}\\ \omega_1 :\quad & \frac{n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0)\frac{\partial^2}{\partial t^2} e^{\mathrm{j}((\omega_0+\Omega)t-(\mathbf{k}_0+\mathbf{q})\mathbf{r})} = -\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0) e^{\mathrm{j}(\omega_1 t-(\mathbf{k}_0+\mathbf{q})\mathbf{r})},\end{aligned}$$

where we made use of the fact that $\omega_0 = \omega_1 - \Omega$. We can now write the two coupled differential equations as:

$$\mathbf{k}_1 \nabla \underline{E}(z,\omega_1) = -\mathrm{j}\frac{1}{2}\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0) e^{-\mathrm{j}(\mathbf{k}_0+\mathbf{q}-\mathbf{k}_1)\mathbf{r}}, \text{ and} \tag{1.2}$$

$$\mathbf{k}_0 \nabla \underline{E}(z,\omega_0) = -\mathrm{j}\frac{1}{2}\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1) e^{\mathrm{j}(\mathbf{k}_0+\mathbf{q}-\mathbf{k}_1)\mathbf{r}}. \tag{1.3}$$

2. Consider the case where both the crystal and the optical waves are infinitely extended in x- and y-direction, which implies $\frac{\partial \underline{E}}{\partial x} = 0$ and $\frac{\partial \underline{E}}{\partial y} = 0$. Assume further that the z-components of the $\mathbf{k}$-vector for both optical waves are equal, i.e. $k_{0z} = k_{1z} = k_z$. Using these simplifications, show that the two coupled differential equations can be written as:

$$\frac{\partial \underline{E}(z,\omega_1)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_0) e^{-\mathrm{j}\Delta\mathbf{k}\mathbf{r}}$$

$$\frac{\partial \underline{E}(z,\omega_0)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_1) e^{\mathrm{j}\Delta\mathbf{k}\mathbf{r}}$$

with $\kappa = \frac{k_z \Delta n_0}{2n_0}$ and $\Delta\mathbf{k} = \mathbf{k}_0 + \mathbf{q} - \mathbf{k}_1$.

Solution:

1. The term $\mathbf{k}\nabla\underline{E}$ can be written as $k_x \frac{\partial \underline{E}}{\partial x} + k_y \frac{\partial \underline{E}}{\partial y} + k_z \frac{\partial \underline{E}}{\partial z}$. Using this we can rewrite Eqs. (1.2) and (1.3) as:

$$\frac{\partial \underline{E}(\mathbf{r},\omega_1)}{\partial z} = -\mathrm{j}\frac{1}{2k_{1z}}\frac{\omega_1^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_0)e^{-\mathrm{j}\Delta\mathbf{kr}}\text{, and}$$

$$\frac{\partial \underline{E}(\mathbf{r},\omega_0)}{\partial z} = -\mathrm{j}\frac{1}{2k_{0z}}\frac{\omega_0^2 n_0}{c^2}\Delta n_0 \underline{E}(\mathbf{r},\omega_1)e^{\mathrm{j}\Delta\mathbf{kr}}.$$

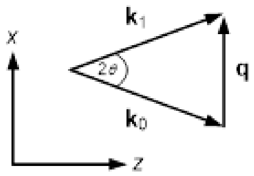
Using the relation $\frac{1}{k_{1z}}\frac{\omega_1^2 n_0}{c^2} = \frac{1}{k_{1z}}\frac{k_{1z}^2}{n_0} = \frac{k_{1z}}{n_0}$, we get the given differential equations:

$$\frac{\partial \underline{E}(z,\omega_1)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_0)e^{-\mathrm{j}\Delta\mathbf{kr}}\text{, and} \tag{1.4}$$

$$\frac{\partial \underline{E}(z,\omega_0)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_1)e^{\mathrm{j}\Delta\mathbf{kr}}, \tag{1.5}$$

with $\kappa = \frac{k_z \Delta n_0}{2n_0}$.

3. Solve the differential equations assuming perfect phase matching, i.e. $\Delta\mathbf{k} = 0$ and using the boundary conditions $\underline{E}(0,\omega_0) = E_0$ and $\underline{E}(0,\omega_1) = 0$. Sketch the evolution of the intensities of the incident and the deflected wave along z. How long should the crystal extend in the z-direction for maximum intensity of the deflected wave?



Solution:

Assuming perfect phase matching, i.e., $\Delta\mathbf{k} = 0$, and taking the derivative of Eq. (1.5) with respect to z, we get:

$$\frac{\partial^2 \underline{E}(z,\omega_0)}{\partial z^2} = -\mathrm{j}\kappa\frac{\partial \underline{E}(z,\omega_1)}{\partial z}.$$

By substituting this into Eq. (1.4), we get:

$$\frac{\partial^2 \underline{E}(z,\omega_0)}{\partial z^2} = -\kappa^2 \underline{E}(z,\omega_0). \tag{1.6}$$

In a similar fashion, by first taking the derivative of Eq. (1.4) with respect to z, and then substituting the result into Eq. (1.5), we get:

$$\frac{\partial^2 \underline{E}(z,\omega_1)}{\partial z^2} = -\kappa^2 \underline{E}(z,\omega_1). \tag{1.7}$$

The general solutions of Eq. (1.6) and (1.7) are respectively:

$$\underline{E}(z,\omega_0) = \underline{c}_1 \cos(\kappa z) + \underline{c}_2 \sin(\kappa z),$$

$$\underline{E}(z,\omega_1) = \underline{c}_3 \cos(\kappa z) + \underline{c}_4 \sin(\kappa z).$$

From the boundary conditions we find that $\underline{c}_1 = E_0$, and $\underline{c}_3 = 0$, therefore:

$$\underline{E}(z,\omega_0) = E_0 \cos(\kappa z) + \underline{c}_2 \sin(\kappa z),$$

$$\underline{E}(z,\omega_1) = \underline{c}_4 \sin(\kappa z).$$

Plugging-in the last two equations into Eq. (1.4), and having in mind that $\Delta\mathbf{k} = 0$, we get:

$$\frac{\partial \underline{E}(z,\omega_1)}{\partial z} = -\mathrm{j}\kappa \underline{E}(z,\omega_0)$$

$$\frac{\partial\left(\underline{c}_4 \sin(\kappa z)\right)}{\partial z} = -\mathrm{j}\kappa\left(E_0 \cos(\kappa z) + \underline{c}_2 \sin(\kappa z)\right)$$

$$\kappa \underline{c}_4 \cos(\kappa z) = -\mathrm{j}\kappa E_0 \cos(\kappa z) - \mathrm{j}\kappa \underline{c}_2 \sin(\kappa z)$$

The last can be true for all z, only if the sine function on the left disappears, meaning that $\underline{c}_2 = 0$. Finally we get:

$$\kappa \underline{c}_4 \cos(\kappa z) = -\mathrm{j}\kappa E_0 \cos(\kappa z),$$

and from here it follows that $\underline{c}_4 = -\mathrm{j}E_0$. Therefore:

$$\underline{E}(z,\omega_0) = E_0 \cos(\kappa z),$$

$$\underline{E}(z,\omega_1) = -\mathrm{j}E_0 \sin(\kappa z).$$

The corresponding intensities are proportional to squares of the electric fields. Since both fields have the same magnitude, the same will hold for the intensities:

$$I(z,\omega_0) = I_0 \cos^2(\kappa z),$$

$$I(z,\omega_1) = I_0 \sin^2(\kappa z).$$

The maximum intensity of the deflected wave will occur at $z = L$ that gives: $\sin^2(\kappa L) = 1$. From here we can calculate L as:

$$L = \frac{\pi}{2\kappa} = \frac{\pi n_0}{k_z \Delta n_0}.$$

The intensity evolution of the two waves is displayed in Fig. 1.

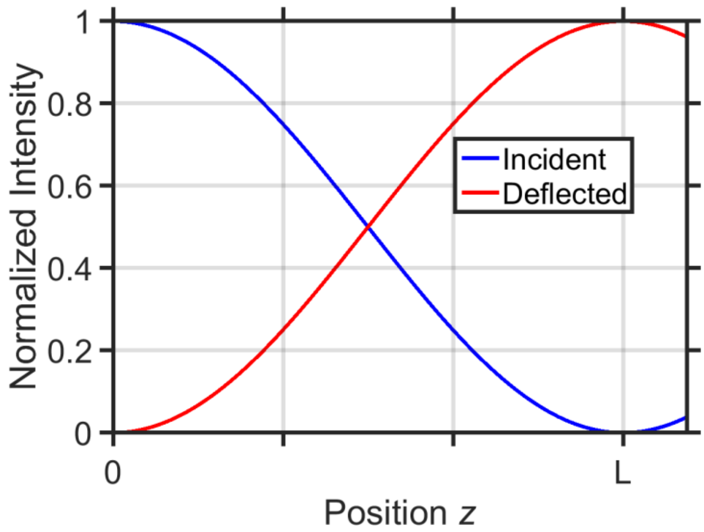


Fig. 1: Normalized intensity evolution of the incident and the deflected wave.

4. What is the angle of diffraction for light at 632.8 nm in a $LiNbO_3$ cell that is driven at a frequency of 1 GHz? (speed of sound: $v_s = 4.1$ km/s, refractive index $n_0 = 2.3$)

Solution:

Using the figure associated to Part 3. of this problem set, we can see that $\sin\theta = \frac{|\mathbf{q}|}{2k_z}$.

We can calculate $|\mathbf{q}| = \frac{2\pi}{\Lambda} = \frac{2\pi}{v_s/(\Omega/2\pi)} = \frac{\Omega}{v_s}$. Also: $k_z = \frac{2\pi n_0}{\lambda}$. By plugging-in the given numerical values, we can calculate $\sin\theta$, and from here we get: $\theta \approx 1.92°$.

2) Side note / Bonus exercise: Nonlinear absorption in a material

Semiconductors such as silicon show losses beyond regular linear losses. This is caused by multi-photon absorption, i.e., the simultaneous absorption of multiple photons, where the overall photon energy is enough to overcome the bandgap and hence to transfer electrons from the valence to the conduction band. In the following, we consider only the case of two-photon absorption. It can be described by the imaginary part of a complex third order susceptibility $\underline{\chi}^{(3)}(\omega:\omega,-\omega,\omega) = \chi^{(3)} + \mathrm{j}\chi_i^{(3)}$ that affects the propagation of an electric field in a material as follows (see also Lecture Notes Eq. (1.116)):

$$\frac{\partial \underline{E}(z)}{\partial z} = -\mathrm{j}\frac{3\omega\underline{\chi}^{(3)}(\omega:\omega,-\omega,\omega)}{8cn_0}\left|\underline{E}(z)\right|^2 \underline{E}(z). \tag{0.2}$$

Phenomenologically, two photon absorption is described by the two-photon absorption (TPA) coefficient β_{TPA} that is defined through the decay of the intensity $I(z)=\frac{1}{2}\frac{n_0}{Z_0}\left|\underline{E}(z)\right|^2$ according to

$$\frac{\partial I(z)}{\partial z}=-\beta_{TPA}I(z)^2. \tag{0.3}$$

1. How is the susceptibility $\underline{\chi}^{(3)}$ related to the TPA coefficient?
2. Solve the differential equation (0.3) with the initial condition $I(0)=I_0$ and sketch the intensity as a function of the input intensity I_0 at a fixed position z.

Solution:

Compute the complex conjugated of Eq. (1.2):

$$\frac{\partial \underline{E}^*(z)}{\partial z}=\mathrm{j}\frac{3\omega\left(\underline{\chi}^{(3)}(\omega:\omega,-\omega,\omega)\right)^*}{8cn_0}\left|\underline{E}(z)\right|^2\underline{E}^*(z)$$

Add this expression to equation (1.2) according to

$$\begin{aligned}\underline{E}^*(z)\frac{\partial \underline{E}(z)}{\partial z}+\underline{E}(z)\frac{\partial \underline{E}^*(z)}{\partial z}&=-\mathrm{j}\frac{3\omega\underline{\chi}^{(3)}(\omega:\omega,-\omega,\omega)}{8cn_0}\left|\underline{E}(z)\right|^4\\&\quad+\mathrm{j}\frac{3\omega\left(\underline{\chi}^{(3)}(\omega:\omega,-\omega,\omega)\right)^*}{8cn_0}\left|\underline{E}(z)\right|^4\\&=\frac{3\omega}{8cn_0}\left|\underline{E}(z)\right|^4 j\left(\left(\underline{\chi}^{(3)}(\omega:\omega,-\omega,\omega)\right)^*-\underline{\chi}^{(3)}(\omega:\omega,-\omega,\omega)\right)\\&=\frac{3\omega}{4cn_0}\left|\underline{E}(z)\right|^4\operatorname{Im}\left\{\underline{\chi}^{(3)}\right\}\\&=\frac{\partial\left|\underline{E}(z)\right|^2}{\partial z}\end{aligned}$$

This gives an expression for the intensity:

$$\frac{\partial I(z)}{\partial z} = \frac{1}{2}\frac{n_0}{Z_0}\frac{\partial \left|\underline{E}(z)\right|^2}{\partial z} = -\frac{1}{2}\frac{n_0}{Z_0}\frac{3\omega}{4cn_0}\left|\underline{E}(z)\right|^4 \operatorname{Im}\left\{\underline{\chi}^{(3)}\right\}$$

$$= -\beta_{TPA} I(z)^2 = -\beta_{TPA}\frac{1}{4}\frac{n_0^2}{Z_0^2}\left|\underline{E}(z)\right|^4$$

$$-\frac{1}{2}\frac{n_0}{Z_0}\frac{3\omega}{4cn_0}\operatorname{Im}\left\{\underline{\chi}^{(3)}\right\} = \beta_{TPA}\frac{1}{4}\frac{n_0^2}{Z_0^2}$$

$$\beta_{TPA} =$$

$$-\frac{1}{2}\frac{Z_0}{n_0^2}\frac{3\omega}{c}\operatorname{Im}\left\{\underline{\chi}^{(3)}\right\} = \beta_{TPA}$$

The solution of the differential equation can be found simply by separation of variables. With $z_0 = 0$, the solution reads as

$$I(z) = \frac{I_0}{I_0 \beta_{TPA} z + 1}$$

For an increasing initial intensity, this expression will start from $I(z) = I_0$ and convert to $I(z) = \left(\beta_{TPA} z\right)^{-1}$, i.e. become independent of the initial intensity. SKETCH TBD

Questions and Comments:

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