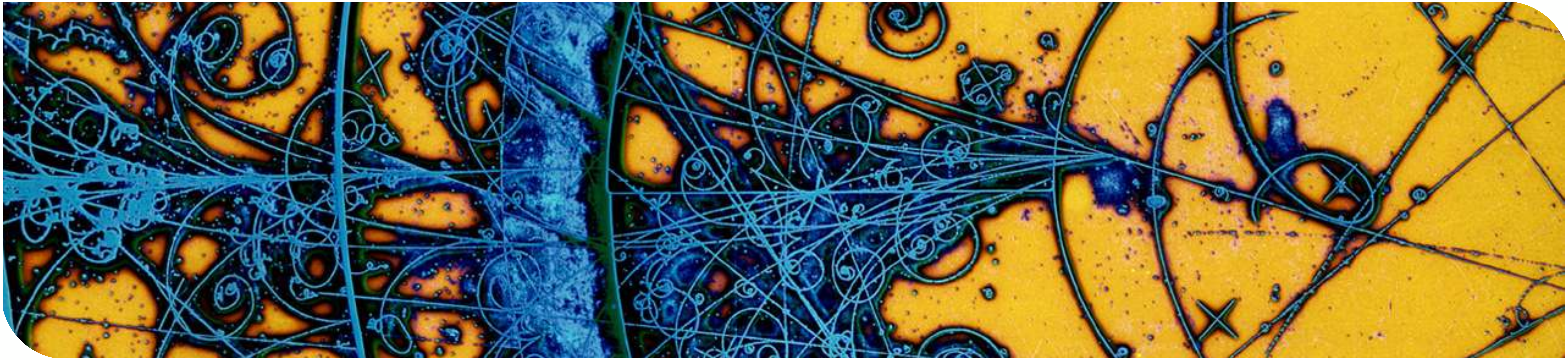
Karlsruhe Institute of Technology

Particle Physics 1
Lecture 2: Dirac equation

Prof. Dr. Markus Klute (markus.klute@kit.edu), **Dr. Pablo Goldenzweig** (pablo.goldenzweig@kit.edu)
Institute of Experimental Particle Physics (ETP)
Winter 2024/2025



Credit: CERN

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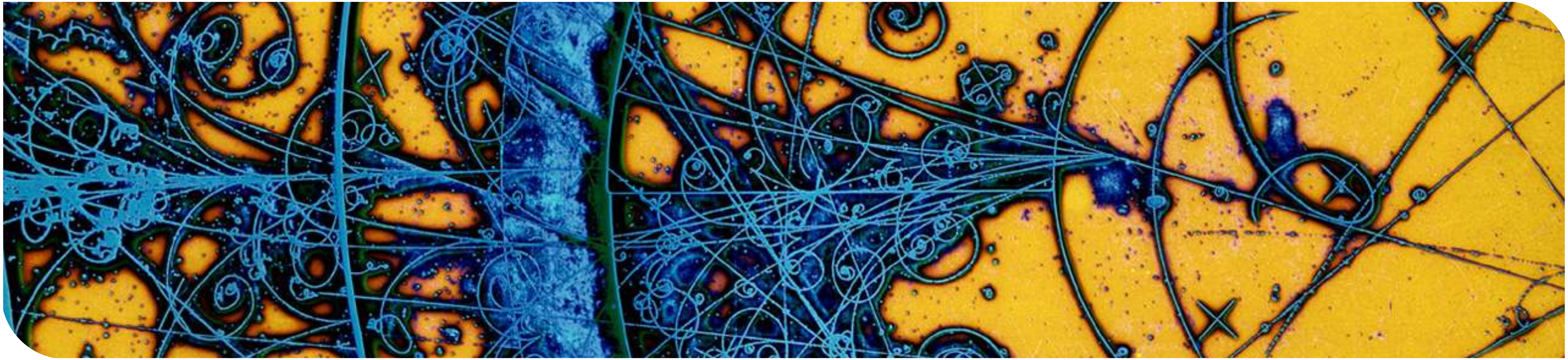


Credit: CERN

First exercise

- Shift forward by one week:
  - Will be assigned on 31. Oct.
  - Reviewed on 07. Nov.
- No tutorial session next week (31. Oct.)

Hardware exercise: New slots

Two slots clash with the Tuesday 14:00-15:30 lecture

We will NOT have lecture on these days

We will use the contingency days in 2025 to make up the lectures

Sensor characterization	Donnerstag, 5. Dezember 2024 - 14:00	(nicht verfügbar)
Sensor characterization	Dienstag, 10. Dezember 2024 - 14:00	BUCHEN
Sensor characterization	Donnerstag, 12. Dezember 2024 - 14:00	(nicht verfügbar)
Sensor characterization	Dienstag, 17. Dezember 2024 - 14:00	BUCHEN
Sensor characterization	Donnerstag, 19. Dezember 2024 - 14:00	(nicht verfügbar)
Strip sensor readout	Donnerstag, 5. Dezember 2024 - 14:00	BUCHEN
Strip sensor readout	Freitag, 6. Dezember 2024 - 14:00	BUCHEN
Strip sensor readout	Freitag, 13. Dezember 2024 - 14:00	BUCHEN
Strip sensor readout	Donnerstag, 19. Dezember 2024 - 14:00	(nicht verfügbar)
Strip sensor readout	Freitag, 20. Dezember 2024 - 14:00	(nicht verfügbar)
Module readout	Donnerstag, 5. Dezember 2024 - 14:00	(nicht verfügbar)
Module readout	Montag, 9. Dezember 2024 - 14:00	BUCHEN
Module readout	Donnerstag, 12. Dezember 2024 - 14:00	(nicht verfügbar)
Module readout	Montag, 16. Dezember 2024 - 14:00	BUCHEN
Module readout	Donnerstag, 19. Dezember 2024 - 14:00	(nicht verfügbar)

Preparatory meeting today

Advanced seminar

Hauptseminar “Teilchenphysik”
(Husemann, Ferber, Klute, Müller)
4013214 (SS and WS)

Preparatory meeting and topic assignment:
Friday, 25.10.2023, 14:00-15:00, room 9/1 in 30.23

https://ilias.studium.kit.edu/ilias.php?baseClass=ilrepositorygui&cmdNode=xe:lq&cmdClass=ilObjCourseGUI&ref_id=2498057&redirectSource=ilCourseRegistrationGUI

What do you expect from this lecture?

Thanks for letting us know!

Please keep adding responses

I am expectont from this lecture to get a better understanding about particle physics, in especific, about the theoretical background and how does modern particle detectors work and the different types of detectors there are and what are their uses. Personally, another thing I expect from this lecture is to get a better understanding about the CMS detector at CERN and to see which are some of the experiments that the KIT and the CERN are now.

I expect an introduction particle phyiscs which should help me to choose if I want to do my master thesis in this field. Furthermore, I hope to get a brief introduction on the different experiments in order to get a clearer picture of where I wanna persue my future.

I hear this lecture as addition for TTP1 to learn about the particle physics from the experimental perspective as well, so I hopefully will be able afterwards to connect the learned knowledge of both lectures.

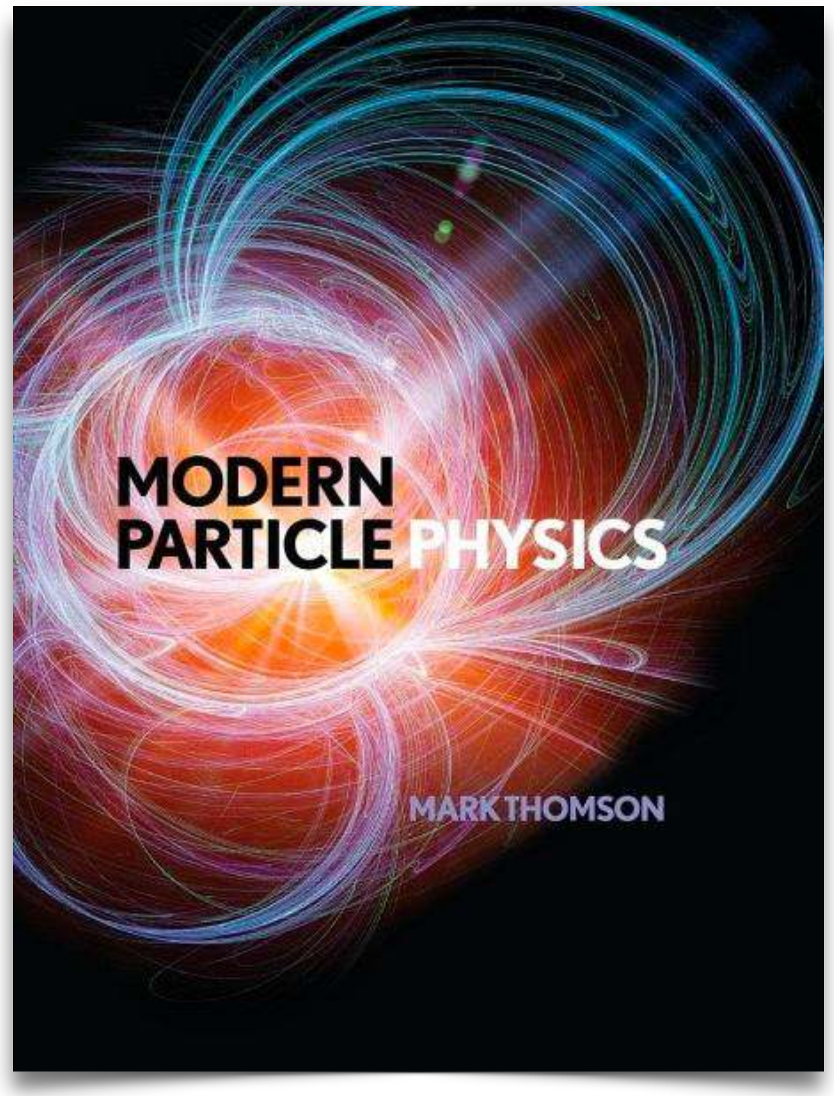
I can expect to learn about the design and operation of particle detectors, data collection, and analysis techniques used in high-energy physics experiments like those at CERN. The focus will be on understanding how experimental setups test theories from the Standard Model and beyond.

Further improving the knowledge on the entire chain happening at particle physics experiments to gather information about elementary particles from the „simple" measurement of signals to the complex understanding of particles we have. I.e. how to derive the behavior of particles from the experiments

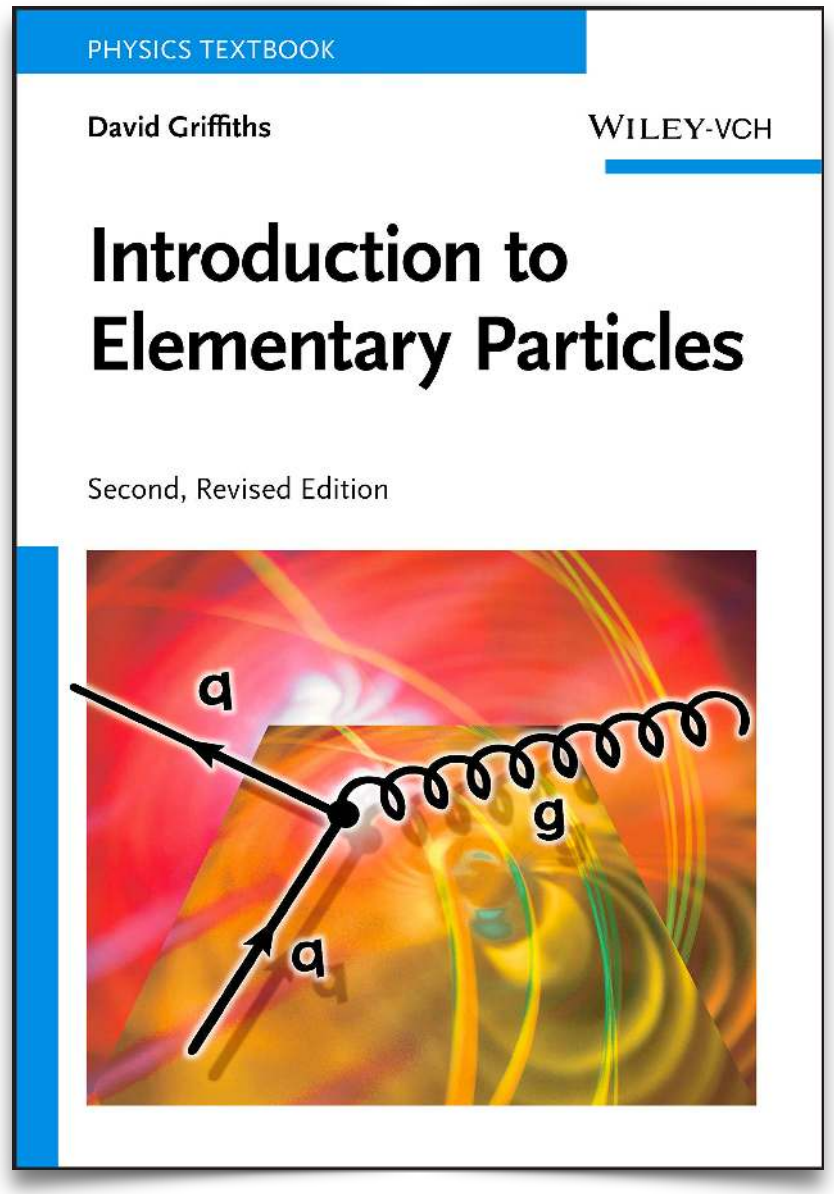
Literature

- **M. Thomson: Modern Particle Physics, Cambridge UP (2013)**
- **D. Griffith: Introduction to Elementary Particles, Wiley (2008)**
- A. Bettini: Introduction to Elementary Particle Physics, Cambridge UP (2008)
- C. Berger: Elementarteilchenphysik, Springer (2006)
- P. Schmüser: Feynman-Graphen und Eichtheorien für Experimentalphysiker, Springer (1995)

PDF uploaded to ILIAS



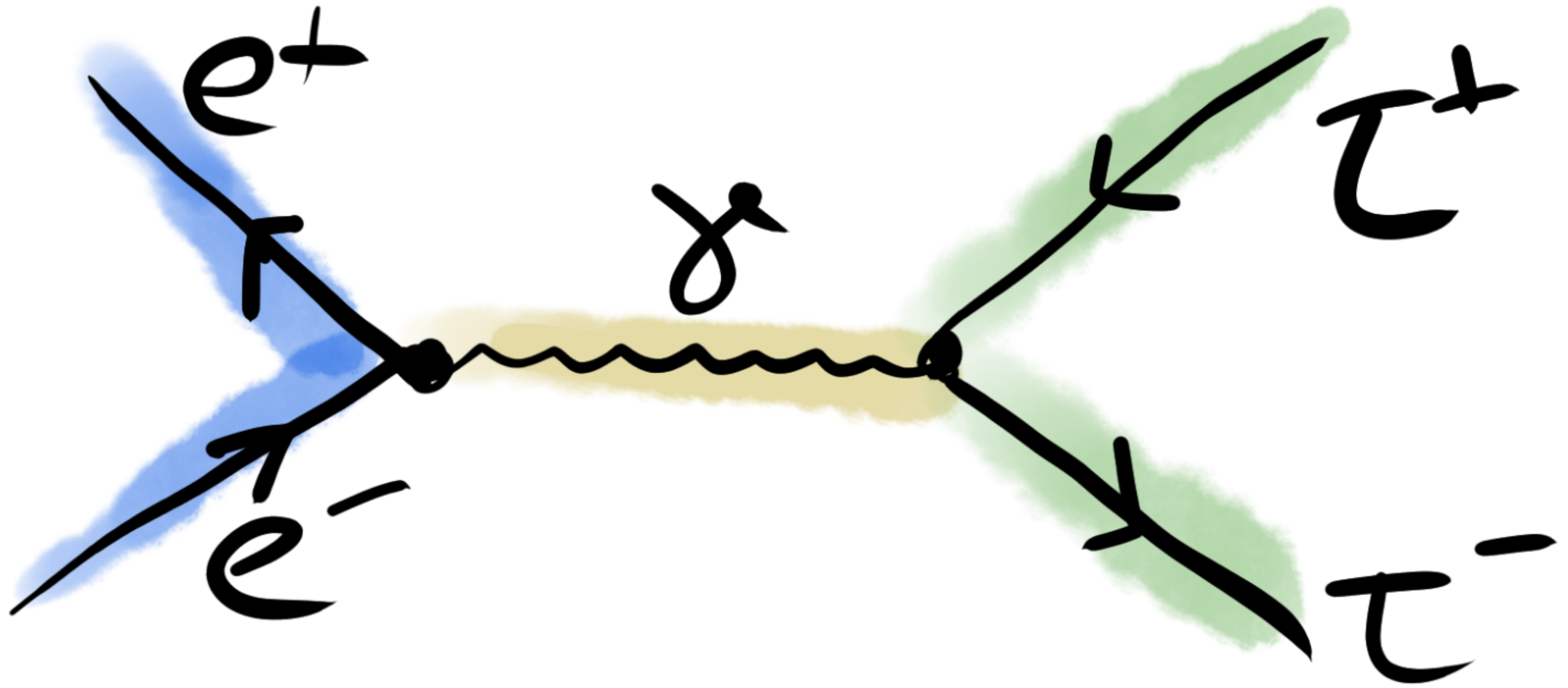
Please read the sections assigned at the end of each lecture



PINGO: Luminosity

From L01

- By 10/2023, the Belle II experiment at the e^+e^- KEKB collider in Japan had collected a dataset of about $0.92ab^{-1}$. The cross section to produce tau pairs, $\sigma(e^+e^- \to \tau^+\tau^-)$ is about 1nb. Approximately how many tau pairs had Belle II recorded at this time?* *assuming an efficiency of $\varepsilon=1$
    - about 1 Thousand (10^3)
    - about 1 Million (10^6)
    - about 1 Billion (10^9)
    - about 1 Trillion (10^{12})



PINGO: Luminosity

Solution

$1\,nb = 10^{-9}\,b$

$1\,pb = 10^{-12}\,b$

$1\,fb = 10^{-15}\,b$

$1\,ab = 10^{-18}\,b$

$$\mathcal{L} \simeq 1\,ab^{-1} = \frac{1}{1\,ab} = \frac{1}{10^{-18}\,b} = 10^{18}\,b^{-1}$$

$$\sigma(e^+e^- \to \mu^+\mu^-) \simeq 1\,nb = 10^{-9}\,b$$

$$\sigma = \frac{N_{sig} - N_{bkd}}{\epsilon \mathcal{L}}$$

$N \simeq \sigma \mathcal{L}$ (take $\epsilon = 1$)

$$= (10^{-9}\,b)(10^{18}\,b^{-1})$$

$$= 10^{9}$$

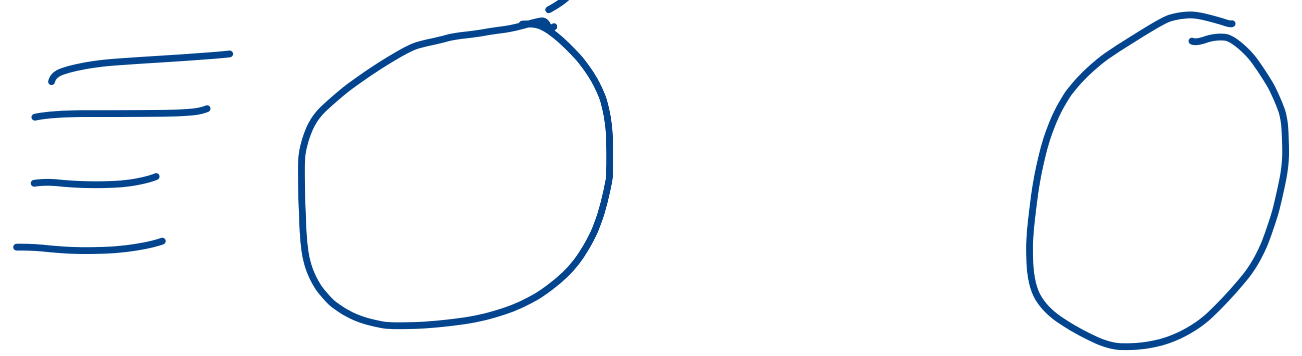
PINGO: Fixed target collision

- PINGO:
  - Session: Particle Physics 1 (WS 24/25).
  - Accession number: 559016
  - Link: https://pingo.coactum.de/events/559016

PINGO: Fixed target collision

- If LHC was a fixed target collider, what would the center of mass energy be if the proton beam has an energy of 7 TeV and the target is a proton at rest?
  - about 14 TeV
  - about 7 TeV
  - about 0.01 TeV

$P_1 = (E, \vec{p})$ $\qquad$ $P_2 = (m, 0)$



CM energy: $s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 p_2$

$= (E^2 - p^2) + m^2 + 2Em$

$\simeq 2m^2 + 2Em$

$\simeq 2Em$

[$m_{proton} = 938$ MeV small compared to E]

$= 2(7\,\text{TeV})(938\,\text{MeV})$

$= 2(7\,\text{TeV})(938 \times 10^{-6}\,\text{TeV})$

$= 0.01$ TeV

Other questions from L01?

Today

Classical mechanics

Quantum mechanics

Non-relativistic QM

Relativistic QM

Notation

- Vectors
  - 3-vectors: $x^a = \vec{x}, \quad a = (1,2,3)$
  - 4-vectors: $x^\mu = (t, \vec{x}), \quad \mu = (0,1,2,3)$
- **Contravariant** vector x^μ and the **covariant** vector x_μ are related by the metric tensor $g_{\mu\nu}$: $x_\mu = g_{\mu\nu} x^\nu$
  - $g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, i.e.: $x_0 \to x^0, x_1 \to -x^1, x_2 \to -x^2, x_3 \to -x^3$

Covariant vs. invariant

- ***Covariant* does not mean *invariant*!** *(2 different concepts)*
  - **Covariant** applies to an equation
  - **Invariant** applies to a quantity which has the same numerical value in any reference frame
    - $A^\mu A_\mu = b$ is a covariant equation
    - $A^\mu A_\mu$ is an invariant quantity

Covariant form is extremely useful since equation manifestly holds true in any reference frame (both sides get identical Lorentz boosts; one Λ^μ_ν per available index)

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Covariant?

$a^\mu b_\mu = D^{\alpha\beta} p_\alpha q_\beta$ ✓

$a_\mu b^\mu_\nu = d_\nu$ ✓

$a_\mu b^\mu_\gamma = d^\gamma$ ✗

Notation

energy-momentum: $p^\mu = (E, \vec{p})$

time-space: $x^\mu = (t, \vec{x})$

4-gradient: $\partial^\mu = \frac{\partial}{\partial_\mu} = \left(\frac{\partial}{\partial_t}, -\vec{\nabla}\right)$

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

D'Alembert operator $\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$

4-current $j^\mu = (\rho, \vec{j})$

electric scalar
magnetic vector

electromagnetic 4-potential: $A^\mu = (\phi, \vec{A})$

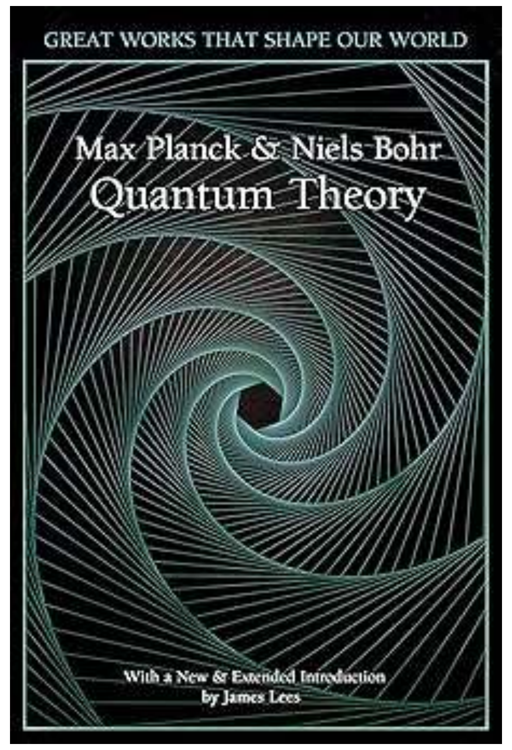
field-strength tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Classical mechanics → Quantum mechanics

"God does not play dice"
"The Lord is subtle but not malicious"
—Einstein

"Einstein, stop telling God what to do"
- Bohr

Suggested reading



Photograph by Paul Ehrenfest: https://history.aip.org/exhibits/einstein/ae63.htm

Schrödinger equation

- **Classical** energy-momentum of a free particle: $E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$
- **Quantize** by promoting E and P to operators: $E \rightarrow i\frac{\partial}{\partial t}$ and $p \rightarrow -i\vec{\nabla}$ and applying the resulting operator on a wave function $\psi(\vec{x}, t)$:

$$\boxed{-\frac{1}{2m}\nabla^2\psi = i\frac{\partial\psi}{\partial t}}$$

Recall: $|\psi(\vec{x}, t)|^2$ is the probability of finding the particle at $(\vec{x}, t)$

2^{nd} order derivative in space

1^{st} order derivative in time

→ *Different dependence on the time and space coordinates*

Schrödinger equation is not Lorentz-invariant

Non-relativistic QM → **Relativistic QM**

Klein-Gordon equation

Start over

- **Relativistic** energy-momentum of a free particle: $p^\mu p_\mu = E^2 - \vec{p}^2 = m^2$
- Canonical operator replacement (in 4-vector notation) $p_\mu = i\partial_\mu$ and application of the resulting operator on a wave function $\psi(\vec{x}, t)$:

$$\partial_\mu \partial^\mu \psi + m^2 \psi = 0$$

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + m^2 \psi = 0} \quad \leftrightarrow \left(\Box + m^2\right)\psi = 0$$

2^{nd} order derivative in time

2^{nd} order derivative in space

✔

Klein-Gordon equation is Lorentz-invariant

(Today identified as the correct solution for spin 0 particles)

Solutions to the K-G equation

- Plane wave solutions $\psi(\vec{x},t) = Ne^{i(\vec{p}\cdot\vec{x}-Et)}$
- $E = \pm\sqrt{p^2 + m^2}$
- Negative energy solutions **cannot be dismissed** in QM since all solutions are required to form a complete set of states.

Two problems with the K-G equation

- Negative energy solutions of the K-G equation have unphysical negative probability densities.
- The K-G equation does not account for the spin degree of freedom.
  - (Recall that earlier we said that the K-G equation is identified as the solution for spin 0 particles only)

Probability density & probability current

- Recall the continuity equation for the conservation of QM probability:

$$\boxed{\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.}$$

Probability density: $\rho(\mathbf{x}, t) = \psi^*(\mathbf{x}, t)\,\psi(\mathbf{x}, t).$

Probability flux density (aka curent): $\mathbf{j}(\mathbf{x}, t)$

Assuming the particle does not decay or interact, its associated total probability will be constant

- In non-relativistic QM:

$$(1) \quad i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

$$(2) \quad -i\frac{\partial \psi^*}{\partial t} = -\frac{1}{2m}\nabla^2\psi^*$$

$$\psi^* \cdot (1) - \psi \cdot (2)$$

$$-\frac{1}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right)$$

$$-\frac{1}{2m}\nabla\cdot\underbrace{\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)}_{j = \frac{1}{2im}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)} = i\frac{\partial}{\partial t}\underbrace{\left(\psi^*\psi\right)}_{\rho = \psi^*\psi}$$

Now see what happens with the K-G equation

$$\boxed{\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - m^2\psi}$$

Same procedure as with the Schr. eqn. in the last slide:

$$\underbrace{\nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*)}_{j = -i(\psi^*\nabla\psi - \psi\nabla\psi^*)} = \frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)$$

Problem!

Dirac equation

Start over

- Proposed solution: try an equation **linear** in derivatives

$$\hat{E}\psi = (\overrightarrow{\alpha} \cdot \hat{p} + \beta m)\psi$$

- To be physically satisfactory, it must:

1. Give the correct energy momentum relation $E^2 = p^2 + m^2$ for a free particle (i.e., satisfy the K-G equation)

2. Allow a probabilistic interpretation of the wave function $\psi(\vec{x}, t)$

3. Be Lorentz covariant (i.e., true in any reference frame)

Condition 1: Satisfy $E^2 = p^2 + m^2$

- Start with the Dirac equation: $\hat{E}\psi = (\vec{\alpha} \cdot \hat{p} + \beta m)\psi$
- Write in terms of the operators: $i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi$
- Square it:

$$\frac{\partial^2 \psi}{\partial t^2} = \alpha_x^2 \frac{\partial^2 \psi}{\partial x^2} + \alpha_y^2 \frac{\partial^2 \psi}{\partial y^2} + \alpha_z^2 \frac{\partial^2 \psi}{\partial z^2} - \beta^2 m^2 \psi$$

$$+ (\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2 \psi}{\partial x \partial y} + (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2 \psi}{\partial y \partial z} + (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2 \psi}{\partial z \partial x}$$

$$+ i(\alpha_x\beta + \beta\alpha_x)m\frac{\partial \psi}{\partial x} + i(\alpha_y\beta + \beta\alpha_y)m\frac{\partial \psi}{\partial y} + i(\alpha_z\beta + \beta\alpha_z)m\frac{\partial \psi}{\partial z}.$$

This part needs to survive so as to reduce to the K-G equation

Condition 1: Satisfy $E^2 = p^2 + m^2$

- Need the following:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I,$$

$$\alpha_j \beta + \beta \alpha_j = 0,$$

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad (j \neq k),$$

Not possible if the α_i and β are normal numbers

$$\frac{\partial^2 \psi}{\partial t^2} = \alpha_x^2 \frac{\partial^2 \psi}{\partial x^2} + \alpha_y^2 \frac{\partial^2 \psi}{\partial y^2} + \alpha_z^2 \frac{\partial^2 \psi}{\partial z^2} - \beta^2 m^2 \psi$$

This part needs to survive so as to reduce to the K-G equation

$$+ (\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2 \psi}{\partial x \partial y} + (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2 \psi}{\partial y \partial z} + (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2 \psi}{\partial z \partial x}$$

$$+ i(\alpha_x\beta + \beta\alpha_x)m\frac{\partial \psi}{\partial x} + i(\alpha_y\beta + \beta\alpha_y)m\frac{\partial \psi}{\partial y} + i(\alpha_z\beta + \beta\alpha_z)m\frac{\partial \psi}{\partial z}.$$

Condition 1: Satisfy $E^2 = p^2 + m^2$

- Need the following:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I,$$
$$\alpha_j \beta + \beta \alpha_j = 0,$$
$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad (j \neq k),$$

Not possible if the α_i and β are normal numbers

- The simplest objects that can satisfy these relations are anti-commuting $\{a, b\} = ab + ba$ matrices with properties:
  - Trace $= 0$
  - Eigenvalues $= \pm 1$
  - Even dimension
  - Hermitian (since $\hat{H}_D$ must be hermitian to have real eigenvalues)

Condition 1: Satisfy $E^2 = p^2 + m^2$

- *$\alpha_x, \alpha_y, \alpha_z$ and β are 4 mutually anticommuting Hermitian matrices of even dimension and trace 0.*

- The lowest-D object that can represent $\alpha_x, \alpha_y, \alpha_z$ and β are 4×4 matrices. *Do you see why?*

- The Dirac hamiltonian $\hat{H}_D = \overrightarrow{\alpha} \cdot \hat{p} + \beta m$ is a 4×4 ***matrix of operators*** *that* ***must*** *act on a 4-component wave function*:

Dirac spinor $$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Condition 1: Satisfy $E^2 = p^2 + m^2$

- The algebra is fully defined by these relations

$$\begin{aligned} \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 &= I, \\ \alpha_j \beta + \beta \alpha_j &= 0, \\ \alpha_j \alpha_k + \alpha_k \alpha_j &= 0 \qquad (j \neq k), \end{aligned}$$

- Use the *Pauli-Dirac representation*, based on the Pauli spin matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Condition 2: Probabilistic interpretation

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

- Follow the same procedure as that used for the Schrödinger and Klein-Gordon equations (S19-20), but note that ψ^* must be replaced by the Hermitian conjugate $\psi^\dagger = (\psi^*)^T$, *since the wavefunctions are now 4-component Dirac spinors.*

(1) Dirac equation $$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + m\beta\psi = +i\frac{\partial \psi}{\partial t},$$

(2) Hermitian conjugate $$+i\frac{\partial \psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial \psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial \psi^\dagger}{\partial z}\alpha_z^\dagger + m\psi^\dagger\beta^\dagger = -i\frac{\partial \psi^\dagger}{\partial t}.$$

$$\psi^\dagger \times (1) - \psi \times (2) \Rightarrow \nabla \cdot (\psi^\dagger \boldsymbol{\alpha}\psi) + \frac{\partial(\psi^\dagger\psi)}{\partial t} = 0,$$

Identify probability density and probability current

$$\rho = \psi^\dagger\psi \quad \text{and} \quad \mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi.$$

All solutions of the Dirac equation have +probability density ✓

Relation to spin

Read at home: Thomson Section 4.4

- The Dirac equation provides a natural description of spin-half particles.

Spin emerges as a direct consequence of requiring the wavefunction to satisfy the Dirac equation

QM basics: Time dependence of an observable corresponding to an operator $\hat{O}$ is given by

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle = i\langle\psi|[\hat{H},\hat{O}]|\psi\rangle \quad \Longrightarrow$$

If the operator for an observable commutes with the H, it is a constant of the motion

$$[\hat{H}_D, \hat{\mathbf{L}}] = -i\boldsymbol{\alpha} \times \hat{\mathbf{p}}. \quad ✗$$

$$[\hat{H}_D, \hat{\mathbf{S}}] = i\,\boldsymbol{\alpha} \times \hat{\mathbf{p}}. \quad ✗$$

$$\left[\hat{H}_D, \hat{\mathbf{J}}\right] \equiv \left[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}\right] = 0. \quad ✓$$

Condition 3: Must be Lorentz covariant

- Start with the Dirac equation: $\hat{E}\psi = (\vec{\alpha} \cdot \hat{p} + \beta m)\psi$
- Quantize it: $$i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi$$
- Multiply by β:

$$i\beta\alpha_x\frac{\partial\psi}{\partial x} + i\beta\alpha_y\frac{\partial\psi}{\partial y} + i\beta\alpha_z\frac{\partial\psi}{\partial z} + i\beta\frac{\partial\psi}{\partial t} - \beta^2 m\psi = 0$$

- Define $\gamma^\mu \equiv (\beta, \beta\alpha_i)$, and use $\beta^2 = I$:

$\gamma^0 := \beta,\ \gamma^1 := \beta\alpha_x,\ \gamma^2 := \beta\alpha_y,\ \gamma^3 := \beta\alpha_z$

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = 0$$

Covariant form of the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0,$$

Recall:

$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$;

Condition 3: Must be Lorentz covariant

- Straightforward to obtain the properties of the γ matrices from those of the α and β matrices
  - $(\gamma^0)^2 = 1$
  - $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$
  - $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ for $\mu \neq \nu$

$$\left.\begin{array}{l} (\gamma^0)^2 = 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \text{ for } \mu \neq \nu \end{array}\right\} \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

γ Matrices (Key points):

- $\gamma^\mu = \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3\right)$ is **not a four-vector**, but the same in each coordinate system
- Usually use the Dirac-Pauli (or chiral) representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ and } \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}$$

with Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \left[\sigma_a, \sigma_b\right] = 2i\epsilon_{a,b,c}$

- Special combination $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ with $\{\gamma^5, \gamma^0\} = 0$ and $\left(\gamma^5\right)^2 = 1$

Covariant current

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

- Probability density: $\rho = \psi^\dagger \psi$
- Probability current: $\vec{j} = \psi^\dagger \vec{\alpha} \psi$

Derived on S27

Can be written compactly: $\boxed{j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi}$

$\star$ j^μ is a 4 vector (See Appendix B3 in Thomson for proof)

Verify: $j^0 = \psi^\dagger \gamma^0 \gamma^0 \psi \Leftarrow (\gamma^0)^2 = 1$

$= \psi^\dagger \psi$ ✓

$\Rightarrow$ Further simplify

$j^\mu = \bar{\psi} \gamma^\mu \psi$

$(\bar{\psi} = \psi^\dagger \gamma^0)$

$j^1 = \psi^\dagger \gamma^0 \gamma^1 \psi$

$= \psi^\dagger \alpha_1 \psi$ ✓ $\Leftarrow$ $\left\{ \begin{array}{l} \gamma^\mu = (\beta, \beta\alpha_i) \\ \gamma^0\gamma^\mu = (\gamma^0\beta, \gamma^0\beta\alpha_i) \\ \gamma^0\gamma^i = \gamma^0\beta\alpha_i \\ \quad = \beta\beta\alpha_i \\ \quad = \alpha_i \end{array} \right.$

Adjoint spinor

in the Dirac-Pauli representation of the γ-matrices

$$\bar{\psi} = \psi^\dagger \gamma^0$$

$$= (\psi^*)^T \gamma^0$$

$$= (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & & 0 & \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

PINGO: Dirac equation

- PINGO:
  - Session: Particle Physics 1 (WS 24/25).
  - Accession number: 559016
  - Link: https://pingo.coactum.de/events/559016

PINGO: Dirac equation

- Which particles can be described by the Dirac equation?
  - Higgs Boson(s)
  - Charged leptons
  - Photons
  - Quarks
  - Neutrinos

Summary

- Relativistic quantum mechanics incorporates relativistic energy- momentum relation: $p^\mu p_\mu = E^2 - \vec{p}^2 = m^2$
- Canonical operator replacement $p_\mu = i\partial_\mu$ ($E \rightarrow i\frac{\partial}{\partial t}$ and $p \rightarrow -i\vec{\nabla}$)
- Klein-Gordon equation of motion $\left(\partial^\mu\partial_\mu + m^2\right)\phi = 0$ for spin-0 particles (scalars)
- Dirac equation $\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$: equation of motion for relativistic spin-1/2 particles
  - *Next time we'll see that the 4D spinor ψ simultaneously describes particles and anti-particles with spin up/down*

Reading assignment

- Modern particle physics (Mark Thomson)
  - Chap. 2
    - 2.3.1-2.3.3
  - Chap. 4
    - 4.1-4.5.1

What questions do you have?