V3 – Experimental & theoretical basics

KIT Faculty of Physics
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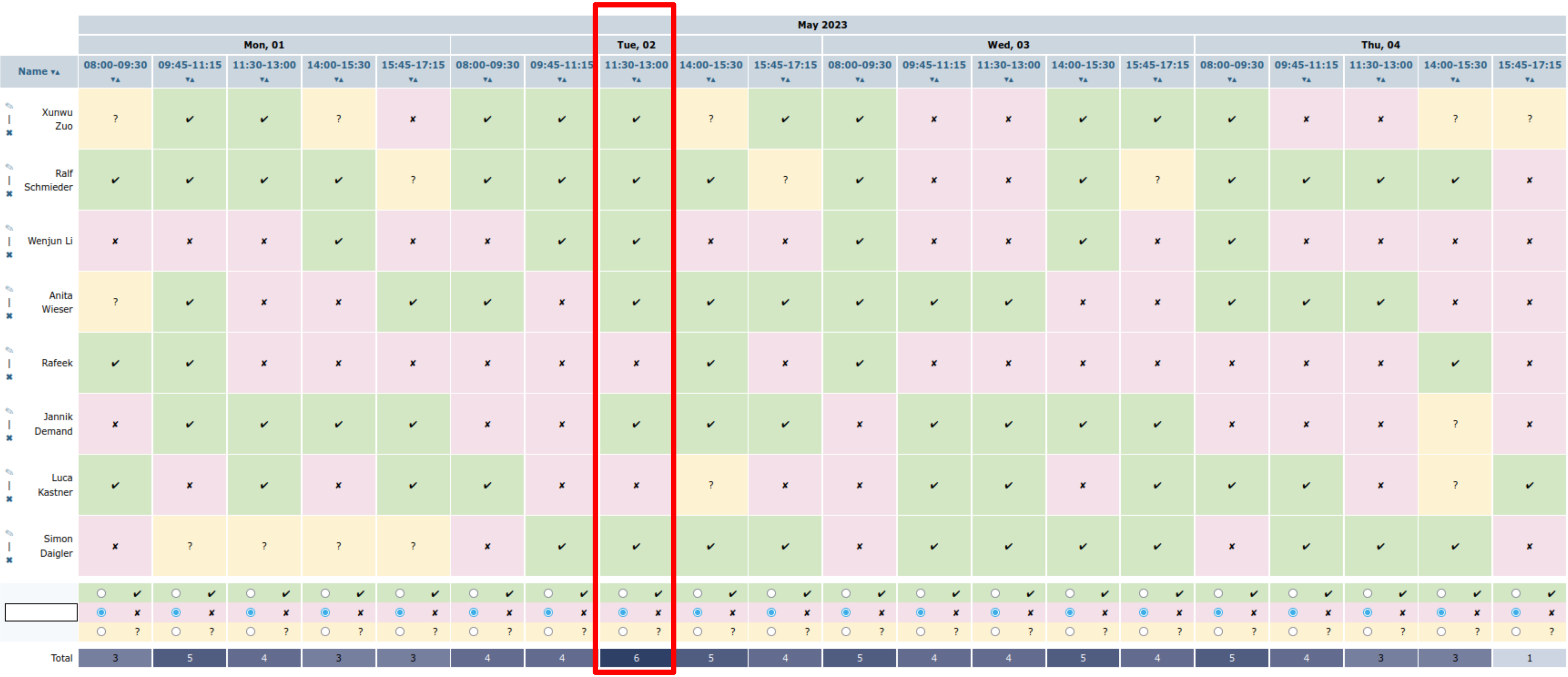
Exercises survey

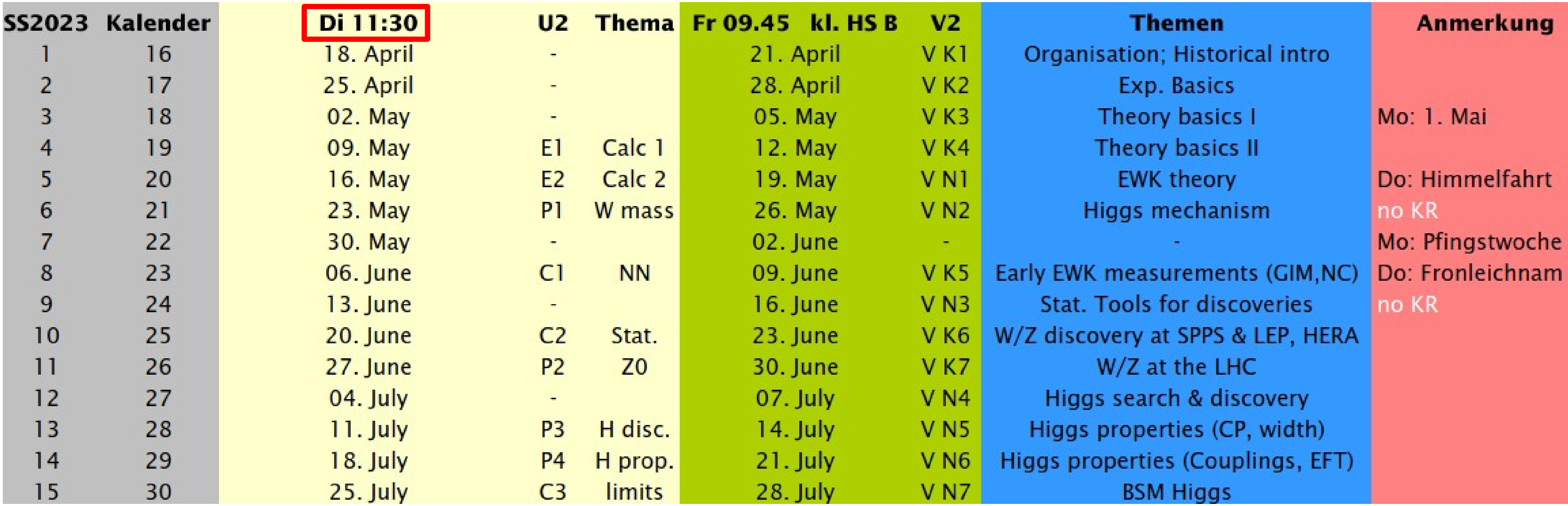
Day/time with largest availability



	May 2023																			
	Mon, 01					Tue, 02					Wed, 03					Thu, 04				
Name	08:00-09:30	09:45-11:15	11:30-13:00	14:00-15:30	15:45-17:15	08:00-09:30	09:45-11:15	11:30-13:00	14:00-15:30	15:45-17:15	08:00-09:30	09:45-11:15	11:30-13:00	14:00-15:30	15:45-17:15	08:00-09:30	09:45-11:15	11:30-13:00	14:00-15:30	15:45-17:15
Xunwu Zuo	?	✔	✔	?	✘	✔	✔	✔	?	✔	✔	✘	✘	✔	✔	✔	✘	✘	?	?
Ralf Schmieder	✔	✔	✔	✔	?	✔	✔	✔	✔	?	✔	✘	✘	✔	?	✔	✔	✔	✔	✘
Wenjun Li	✘	✘	✘	✔	✘	✘	✔	✔	✘	✘	✔	✘	✘	✔	✘	✔	✘	✘	✘	✘
Anita Wieser	?	✔	✘	✘	✔	✔	✘	✔	✔	✔	✔	✔	✔	✘	✘	✔	✔	✔	✘	✘
Rafeek	✔	✔	✘	✘	✘	✘	✘	✘	✔	✘	✔	✘	✘	✘	✘	✘	✘	✘	✔	✘
Jannik Demand	✘	✔	✔	✔	✔	✘	✘	✔	✔	✔	✘	✔	✔	✔	✔	✘	✘	✘	?	✘
Luca Kastner	✔	✘	✔	✘	✔	✔	✘	✘	?	✘	✘	✔	✔	✘	✔	✔	✔	✘	?	✔
Simon Daigler	✘	?	?	?	?	✘	✔	✔	✔	✔	✘	✔	✔	✔	✔	✘	✔	✔	✔	✘
	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?	○ ✔ ● ✘ ○ ?
Total	3	5	4	3	3	4	4	6	5	4	5	4	4	5	4	5	4	3	3	1

Preliminary schedule

Day/time with largest availability



SS2023	Kalender	Di 11:30	U2	Thema	Fr 09.45 kl. HS B	V2	Themen	Anmerkung
1	16	18. April	-		21. April	V K1	Organisation; Historical intro	
2	17	25. April	-		28. April	V K2	Exp. Basics	
3	18	02. May	-		05. May	V K3	Theory basics I	Mo: 1. Mai
4	19	09. May	E1	Calc 1	12. May	V K4	Theory basics II	
5	20	16. May	E2	Calc 2	19. May	V N1	EWK theory	Do: Himmelfahrt
6	21	23. May	P1	W mass	26. May	V N2	Higgs mechanism	no KR
7	22	30. May	-		02. June	-	-	Mo: Pfingstwoche
8	23	06. June	C1	NN	09. June	V K5	Early EWK measurements (GIM,NC)	Do: Fronleichnam
9	24	13. June	-		16. June	V N3	Stat. Tools for discoveries	no KR
10	25	20. June	C2	Stat.	23. June	V K6	W/Z discovery at SPPS & LEP, HERA	
11	26	27. June	P2	Z0	30. June	V K7	W/Z at the LHC	
12	27	04. July	-		07. July	V N4	Higgs search & discovery	
13	28	11. July	P3	H disc.	14. July	V N5	Higgs properties (CP, width)	
14	29	18. July	P4	H prop.	21. July	V N6	Higgs properties (Couplings, EFT)	
15	30	25. July	C3	limits	28. July	V N7	BSM Higgs	

The first exercise is therefore scheduled for next week Tuesday, 11:30;

Room to be announced.

Experimental basics cont.

Which collider?

- **Hadron collider (pp or p$\overline{\text{p}}$)**
  - Unknown initial state (partons), dense event environment
  - High energies for production of new particles but $\mathcal{O}(10^{-10})$ fraction of signal events over difficult backgrounds → discovery machine
- **Lepton collider (e^+e^-)**
  - Known initial state (leptons), clean reconstruction → precision meas.
  - Small total cross section, but process of interest with large fraction
  - Limited centre-of-mass energy

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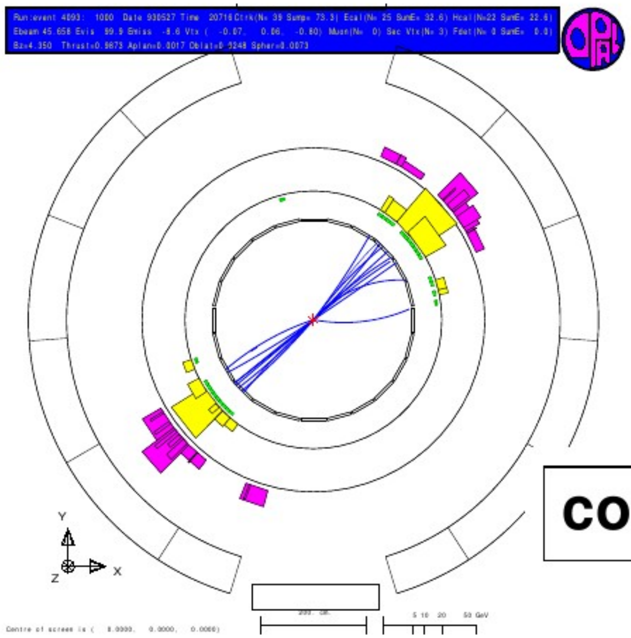
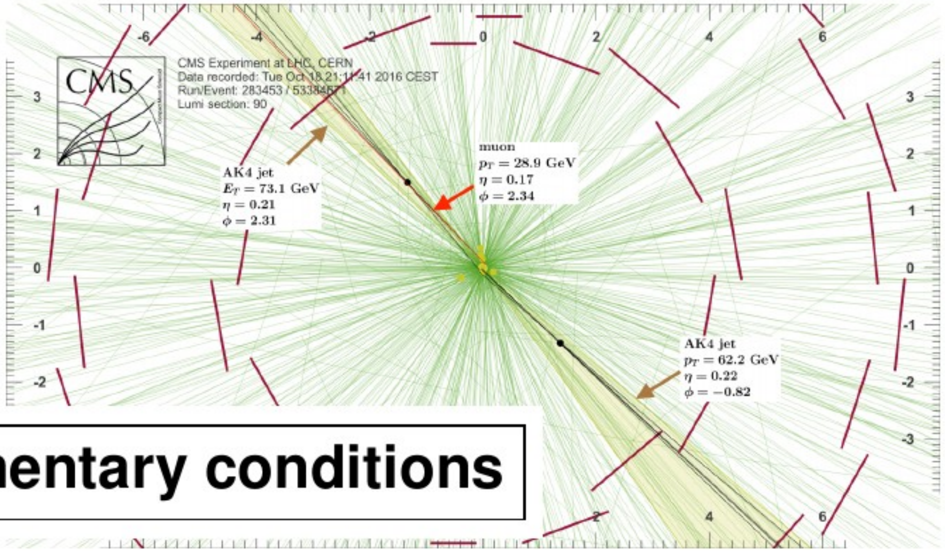
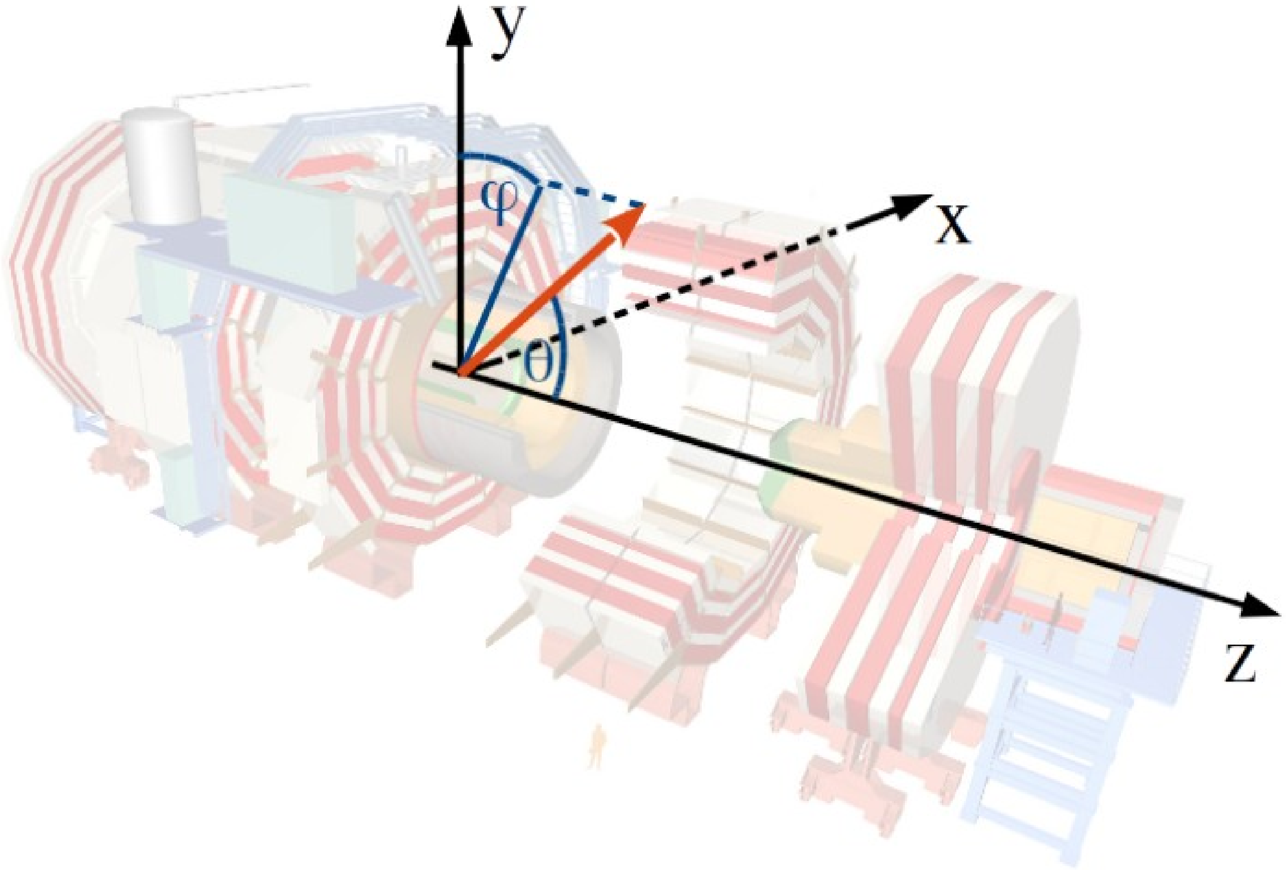




complementary conditions

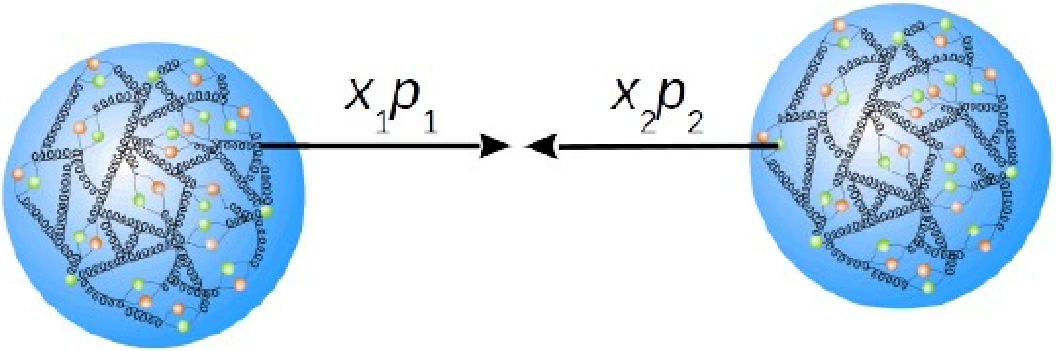
Kinematics at colliders

- Conventions for kinematic variables at collider detectors (motivated by cylindrical symmetry of detectors)
  - **Right-handed cylindrical coordinates** system
  - Azimuthal angle ϕ: angle to x axis in xy plane
  - Polar angle θ: angle to z axis (beam axis)



Transverse quantities

- Kinematics at **hadron colliders** (pp, $p\bar{p}$)
  - Collision of **partons with unknown fraction** x_i of total longitudinal momentum of proton (good approximation: all partons **massless** and **collinear** to beam)



$$\boxed{\hat{E}^2_{\text{cms}} = x_1 x_2 E^2_{\text{cms}}}$$

  - **Rest frame** of parton-parton collision **unknown**
    → parton centre-of-mass energy $\hat{E}_{\text{cms}}$ unknown

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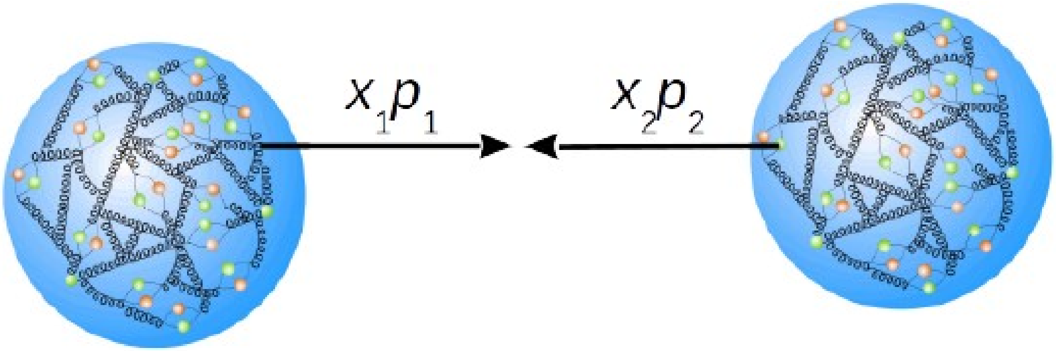

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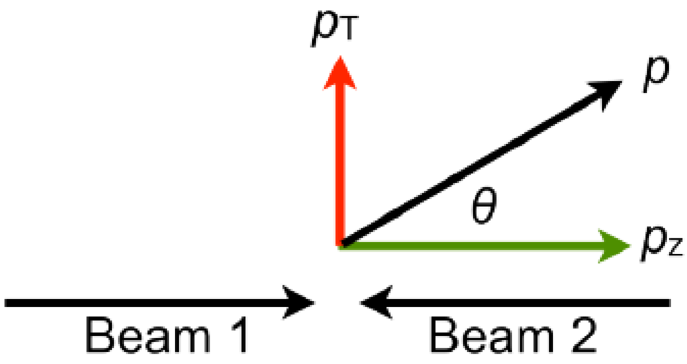
  - **Rest frame** of parton-parton collision **unknown**
    $\rightarrow$ parton centre-of-mass energy $\hat{E}_{\text{cms}}$ unknown

- **Transverse quantities** are **Lorentz invariant** under boosts **along beam direction**, e. g.

$$p_{\text{T}} = \sqrt{p_x^2 + p_y^2} = p \cdot \sin\theta$$



Pseudorapidity

- Rapidity y is relativistic measure of velocity in z direction

$$\boxed{y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right) = \tanh^{-1}\left(\frac{p_z}{E}\right)}$$

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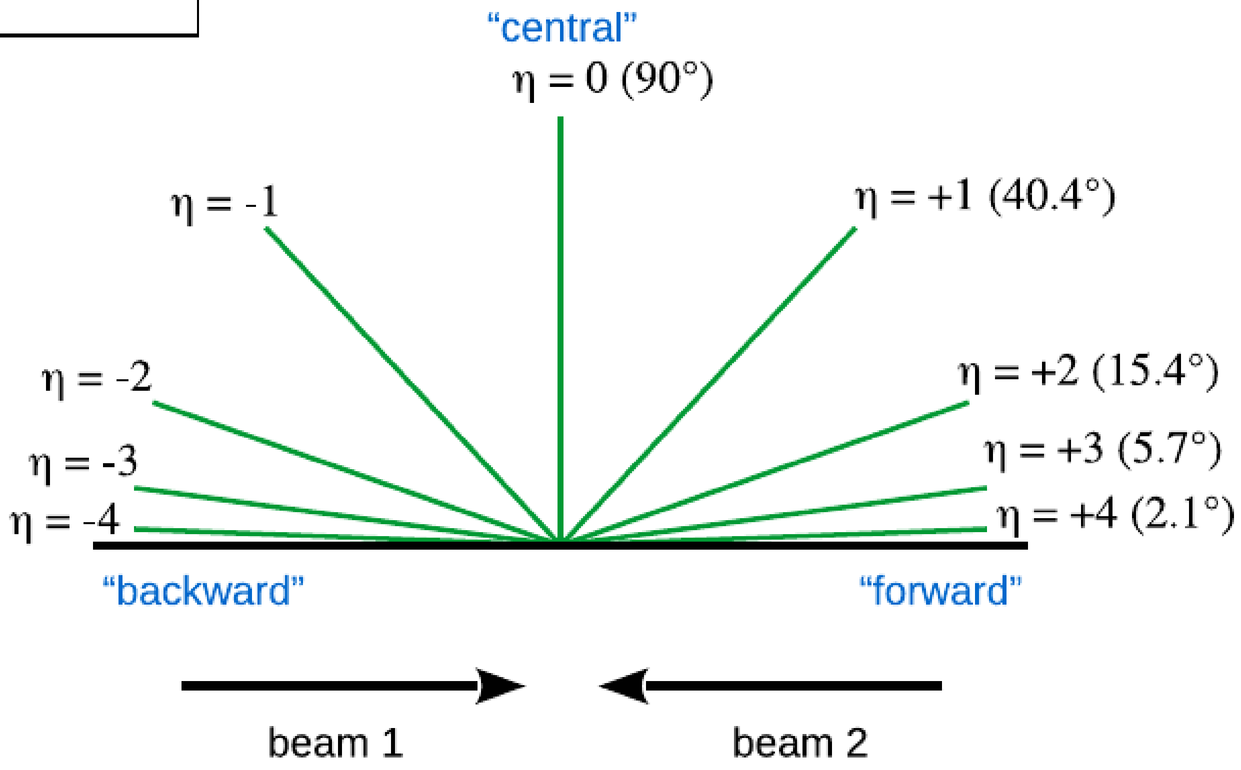
- Pseudorapidity

$$\boxed{\eta = -\ln\tan\left(\frac{\theta}{2}\right)}$$

  - Good **approximation of rapidity *y*** for momentum $\gg$ mass
  - **Easier to measure** than y: depends only on θ, not on mass



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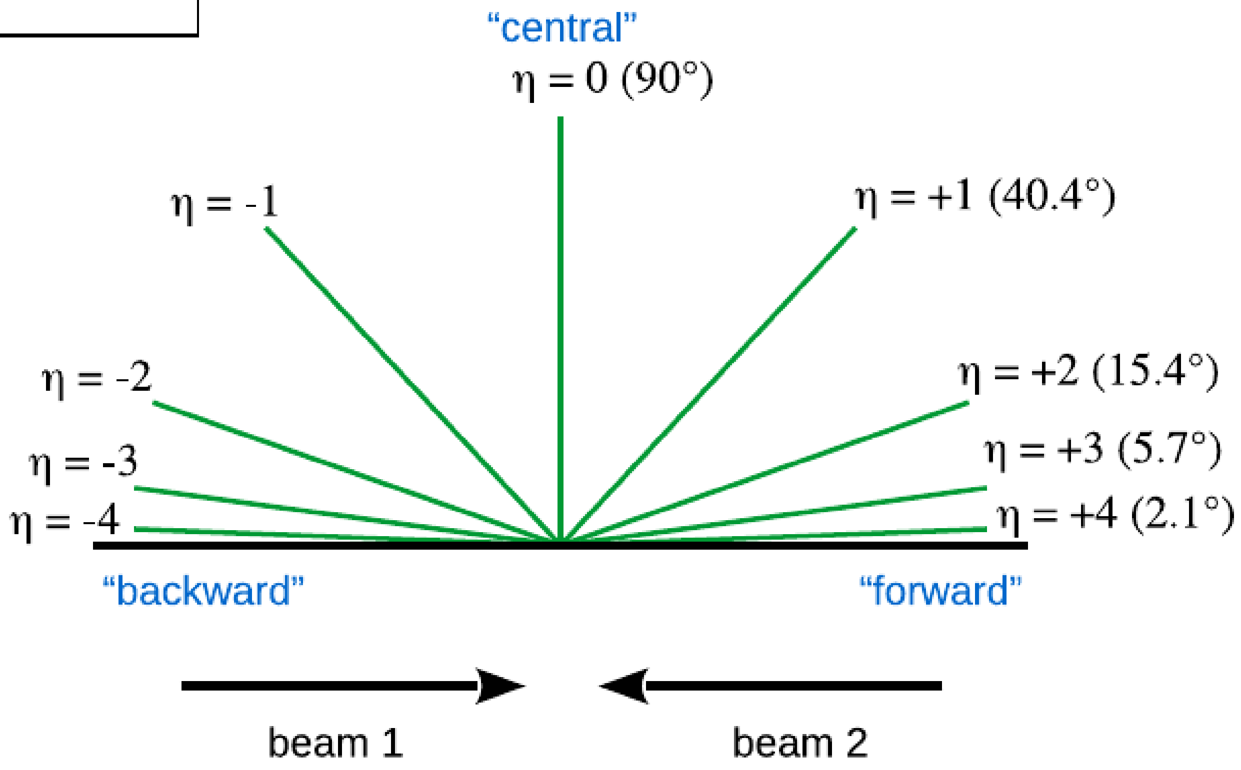

Rapidity **differences** are Lorentz invariant under boosts in z direction

Missing transverse momentum

- Missing transverse momentum, often “missing transverse energy” (“missing E_T”, $\not{E}_\mathrm{T}$, MET)
- **Indirect** detection of weakly interacting neutral particles, e. g. neutrinos
- Concept: colliding **partons without significant transverse momentum** $\rightarrow$ sum of transverse momenta of final-state particles is 0

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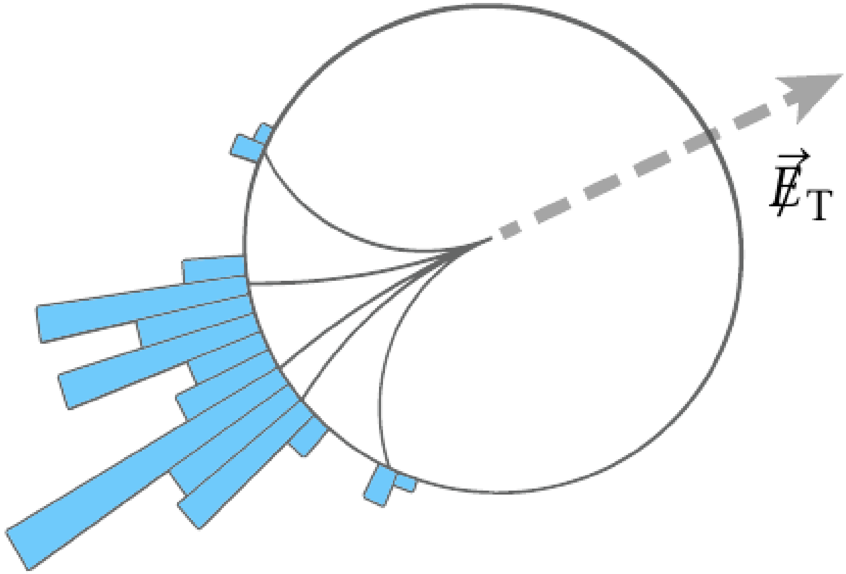
- Concept: colliding **partons without significant transverse momentum** → sum of transverse momenta of final-state particles is 0
  - **Measurement of imbalance** in transverse energy or momentum sum, e. g. based on energy deposits in calorimeter cells

$$\vec{\not{E}}_\mathrm{T} = - \sum_{\text{calo cells}} E_i \sin\theta_i \begin{pmatrix} \cos\phi_i \\ \sin\phi_i \\ 0 \end{pmatrix}$$



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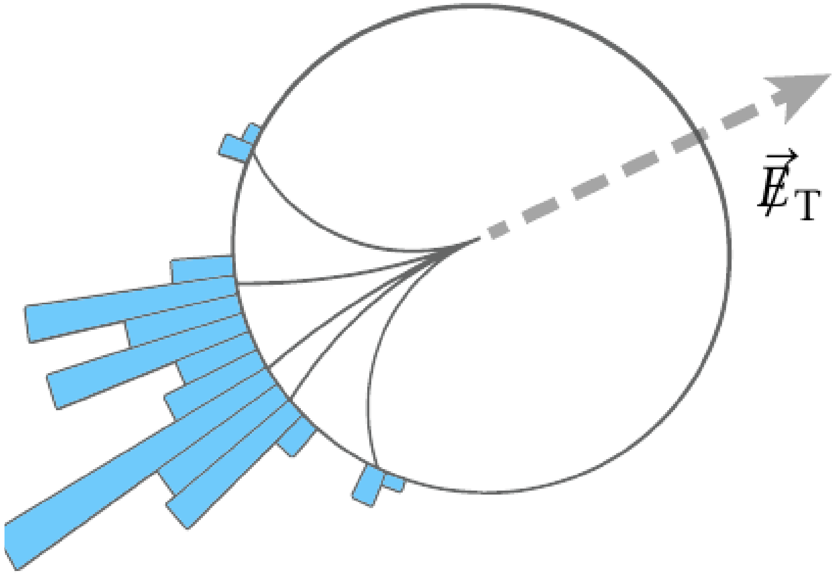

- Experimental **challenge**: many sources of ‘fake’ $\not{E}_\mathrm{T}$, e. g. muons, non-instrumented regions, or noisy detectors

Theoretical basics

Notations (Reminder)

- **Vectors** 3-vector: $x^i = \vec{x}$ $(i = 1, 2, 3)$

  4-vector: $x^\mu = (t, \vec{x})$ $(\mu = 0, 1, 2, 3)$

- Contravariant x^μ and covariant x_μ representation connected via metric tensor:

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad x_\mu = g_{\mu\nu} x^\nu \equiv \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu$$

- **Einstein convention:** **Summation over identical indices is implied!**
  - **Lowercase Roman indices:** **a, b, c, … = 1, 2, 3**
  - **Lowercase Greek indices:** **μ, v, ρ, … = 0, 1, 2, 3**
  - **Uppercase Roman indices:** **A, B, C, … = 1, 2, … , 8**

Notations

- **4-vectors:**
  - Time-space: $x^\mu = (t, \vec{x})$
  - 4-momentum: $p^\mu = (E, \vec{p})$
  - 4-current: $j^\mu = (\rho, \vec{j})$
  - electromagnetic 4-potential: $A^\mu = (\phi, \vec{A})$
  - 4-gradient: contravariant $\partial^\mu = \dfrac{\partial}{\partial_\mu} = \left(\dfrac{\partial}{\partial_t}, -\vec{\nabla}\right)$

    covariant $\partial_\mu = \dfrac{\partial}{\partial^\mu} = \left(\dfrac{\partial}{\partial^t}, \vec{\nabla}\right)$

- **Laplace & d'Alembert operators:** $\Delta = \nabla^2 \qquad \Box = \partial^\mu \partial_\mu = \dfrac{\partial^2}{\partial_t^2} - \nabla^2$

- **Field-strength tensor:** $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Notations

- **4-vectors:**
  - **Time-space:** $x^\mu = (t, \vec{x})$ $[x^\mu] = \text{GeV}^{-1}$
  - **4-momentum:** $p^\mu = (E, \vec{p})$ $[p^\mu] = \text{GeV}$
  - **4-current:** $j^\mu = (\rho, \vec{j})$
  - **electromagnetic 4-potential:** $A^\mu = (\phi, \vec{A})$

Remark: Units

  - **4-gradient: contravariant** $\partial^\mu = \frac{\partial}{\partial_\mu} = \left(\frac{\partial}{\partial_t}, -\vec{\nabla}\right)$ $[\partial^\mu] = \text{GeV}$

    covariant $\partial_\mu = \frac{\partial}{\partial^\mu} = \left(\frac{\partial}{\partial^t}, \vec{\nabla}\right)$

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Quantum mechanics

Classical: Energy-momentum of free particle: $E = \frac{p^2}{2m}$

Canonical operator replacement: $\boxed{\hat{E} \rightarrow i\frac{\partial}{\partial t} \qquad \hat{\vec{p}} \rightarrow -i\vec{\nabla}}$

→ Schrödinger equation (free particle):

$$\hat{H}\phi = \frac{\hat{\vec{p}}^2}{2m}\phi = -\frac{\Delta}{2m}\phi = \hat{E}\phi = i\frac{\partial}{\partial t}\phi$$

2nd order derivative in space

1st order derivative in time

Observation: Cannot be Lorentz invariant → no solution for relativistic QM

Relativistic extension

Recall 3-dim. scalar product: **invariant under rotations in 3-dim. space**

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x^i y_i = x^i y_i = |\vec{x}| \cdot |\vec{y}| \cdot \cos(\angle[\vec{x}, \vec{y}])$$

"4-dim." scalar product: **invariant under Lorentz transformations**

$$p^\mu p_\mu = E^2 - p^2 = m^2$$

→ Relativistic: Energy-momentum of free particle: $E^2 = p^2 + m^2$

Operator replacement → $-\partial^2 t \phi = (-\Delta + m^2)\phi$

$$\Longleftrightarrow \ (\partial^2 t - \Delta + m^2)\phi = (\Box + m^2)\phi = 0$$

Klein-Gordon Equation

Klein-Gordon Equation

Spin-0 particle wave eq.: $(\Box + m^2)\phi = (\partial_\mu \partial^\mu + m^2)\phi = 0$

Relativistic, i.e. Lorentz-invariant? **OK**

Free wave solutions: $\phi(\vec{x}, t) = Ne^{\pm i(p^\mu x_\mu)}$

But:

Possibility of negative energy $E = \pm\sqrt{(p^2 + m^2)}$

Probability current $\partial^\mu j_\mu = 0$

has non-positive definite probability density ρ = j^0,
e.g. for plane wave solution with E < 0: $\rho = 2\left|N\right|^2 E < 0$

Meaning for single-particle states … ???

Dirac's Approach 1927

Find representation of energy-momentum conservation that is linear in both, space and time derivatives!

Ansatz: $\hat{H} = \vec{\alpha}\hat{\vec{p}} + \beta m$ **with** $\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$$\rightarrow i\partial t\,\psi = \hat{H}\psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)\psi$$

What are these α and β? They cannot be simple numbers … !

Re-apply both operators and require Klein-Gordon eq. to be fulfilled:

$$\begin{aligned}(i\partial t)^2\,\psi &= \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)^2\psi \\ &= \left(-\sum_{i,j=1}^{3}\underbrace{\left(\frac{\alpha_i\alpha_j+\alpha_j\alpha_i}{2}\right)}\partial_i\partial_j - im\sum_{i=1}^{3}\underbrace{\left(\alpha_i\beta+\beta\alpha_i\right)}\partial_i + \underbrace{(\beta m)^2}\right)\psi \\ &\equiv \left(-\vec{\nabla}^2 + m^2\right)\psi\end{aligned}$$

Anti-commutator relations $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ $\{\alpha_i, \beta\} = 0$ $\beta^2 = 1$

Dirac: Properties of α_i and β

- **Operators α_i and β can be expressed by matrices:**
  - **must be hermitian since $\hat{H}$ should have real eigenvalues**
  - **must be traceless**

$$Tr(\alpha_i) = Tr(\alpha_i \underset{\mathbb{I}}{\beta\beta}) = Tr(\beta\alpha_i\beta) = -Tr(\beta\beta\alpha_i) = -Tr(\alpha_i) = 0$$

unity matrix **cyclic permutation** **anti-commutation**

  - **must have at least four dimensions**
    - $\alpha_i^2 = \mathbb{I}$ **,** $\beta^2 = \mathbb{I} \rightarrow$ **has only eigenvalues ±1**
    - **Dimension must be even to obtain zero trace**
    - **In two dimensions only 3 independent traceless matrices: Pauli matrices σ_i, fourth matrix $\mathbb{I}$ is not traceless**
    - **four dimensions required**

Representations of α_i and β

- **α_i and β matrices:**

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

($\sigma_i (i = 1, 2, 3)$ are the Pauli Matrices)

- **γ^μ matrices:** $\gamma^0 \equiv \beta \quad \gamma^i \equiv \beta\alpha_i$ $\Rightarrow$ $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

compact notation of algebra

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

- Basis of 4x4 matrices
- Orthonormal
- Traceless (apart from $\mathbb{I}_4$)

$\mathbb{I}_4$	1 matrix
γ^μ	4 matrices
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	6 matrices
$\gamma^\mu\gamma^5$	4 matrices
$\gamma^5 \equiv \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$	1 matrix

γ matrices

- Important tool for relativistic formulation of Dirac equation
  (NB: γ^μ is ***not a 4-vector*** but the same in each coordinate system)

$$\boxed{\begin{aligned} &\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) \\ &\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \\ &(\gamma^\mu)^\dagger \equiv (\gamma^\mu)^{T*} = \begin{cases} \gamma^0 \text{ for } \mu = 0 \\ -\gamma^\mu \text{ for } \mu = 1,2,3 \end{cases} \end{aligned}}$$

- γ matrices in common *chiral* representation (4×4 matrices!)

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}$$

with Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad [\sigma_a, \sigma_b] = 2i\epsilon_{abc}$$

- Special combination: $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ with $\{\gamma^5, \gamma^0\} = 0$, $(\gamma^5)^2 = 1$
- Feynman dagger ('d slash'): $\boxed{\partial\!\!\!/ = \gamma^\mu \partial_\mu}$

Dirac equation: Solution

- **Final formulation:** $$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

- **Solutions:**

$$\psi_+(\vec{x}) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

$$\mathrm{E}(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$$ (free wave)

$$\psi_-(\vec{x}) = v(\vec{p})e^{-i(\vec{p}\vec{x}-Et)}$$

(at rest)

These are Dirac spinors

$$e^{\mp imt} \cdot \boxed{\begin{array}{ll} u_\uparrow(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & u_\downarrow(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ v_\downarrow(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} & v_\uparrow(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{array}}$$

+m solution

-m solution

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(at rest)

Lorentz transformation $\Lambda : (m,0,0,0) \rightarrow (E, p_x, p_y, p_z)$

$$e^{\mp imt} \cdot \left. \begin{array}{l} u_\uparrow(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u_\downarrow(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ v_\downarrow(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad v_\uparrow(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{array} \right\}$$

$$u_\uparrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}E+m\\0\\p_z\\p_x+ip_y\end{pmatrix} \quad u_\downarrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}0\\E+m\\p_x-ip_y\\-p_z\end{pmatrix}$$

$$v_\downarrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}p_z\\p_x+ip_y\\E+m\\0\end{pmatrix} \quad v_\uparrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}p_x-ip_y\\-p_z\\0\\E+m\end{pmatrix}$$

Transformation behaviour

- **Classification of physical objects according to transformation behaviour**
- **With Lorentz-Transformation Λ:**

Größe	Beispiel	Tranformation		
Skalar	m	$\Lambda : m$	$\to m'$	$= m$
Vektor	p^μ	$\Lambda : p^\mu$	$\to p'^\mu$	$= \Lambda^\mu_\nu p^\nu$
Tensor	$T^{\mu\nu}$	$\Lambda : T^{\mu\nu}$	$\to T'^{\mu\nu}$	$= \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$
Vektorfeld	$A^\alpha(x^\mu)$	$\Lambda : A^\alpha(x^\mu)$	$\to A'^\alpha(x'^\alpha)$	$= \Lambda^\alpha_\beta A^\beta(\Lambda^\mu_\nu x^\nu)$
Tensorfeld	$F^{\alpha\beta}(x^\mu)$	$\Lambda : F^{\alpha\beta}(x^\mu)$	$\to F'^{\alpha\beta}(x^\mu)$	$= \Lambda^\alpha_\sigma \Lambda^\beta_\rho F^{\sigma\rho}(\Lambda^\mu_\nu x^\nu)$
Spinor	$\psi^\alpha(x^\mu)$	$\Lambda : \psi^\alpha(x^\mu)$	$\to \psi'^\alpha(x'^\mu)$	$= S^\alpha_\beta \psi^\beta(\Lambda^\mu_\nu x^\nu)$

Dirac equation: Solution

- Dirac equation in covariant notation

$$\boxed{(i\partial\!\!\!/ - m) = 0}$$

  - Hermitian conjugate spinor: $\psi^\dagger = \psi^{T*} \rightarrow$ adjoint spinor $\overline{\psi} = \psi^\dagger\gamma^0$
  - Adjoint Dirac equation: $i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0$ (apply $|\{\}^\dagger\gamma^0$ to Dirac eq.)
- ***Solution*** of Dirac equation for ***free particles***
  - Ansatz: ***plane wave*** $\psi(x) = \chi(p)\exp[\mp ipx]$ (later: interaction as small perturbation) $\rightarrow$ Dirac equation in momentum space

positive energy $(p\!\!\!/ - m)u(p) = 0$

negative energy $(p\!\!\!/ + m)v(p) = 0$

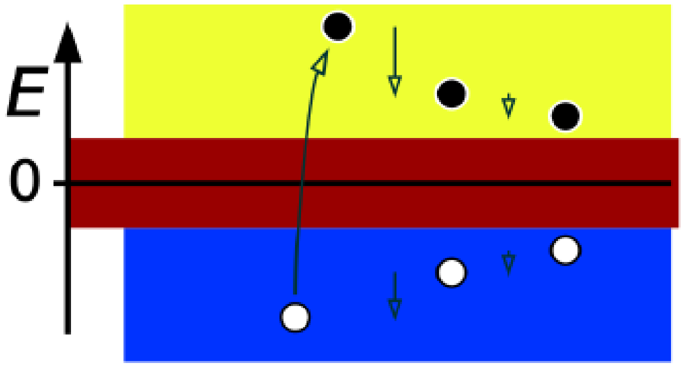
with $p\!\!\!/ = \gamma^\mu p_\mu$

  - Solutions with ***positive energy***: $\psi_{1,2} = u_{1,2}(p)\exp[-ipx]$
    - Particle with charge q and momentum $\vec{p}$
    - Spin $\vec{s}$ parallel (‘up’, u_1) and anti-parallel (‘down’, u_2)
  - Solutions with ***negative energy***: $\psi_{3,4} = v_{2,1}(p)\exp[ipx]$
    - Particle with charge q and momentum $-\vec{p}$
    - Spin $-\vec{s}$ parallel (‘up’, v_2) and anti-parallel (‘down’, v_1)

Dirac equation: Interpretation

- Interpretation as ***single-particle wave function***
  - *Problem*: negative-energy states → electrons can emit infinitely much energy via photons
- Solution: ***Dirac sea*** model
  - Ground state ('vacuum'): all negative-energy states filled with electrons, following Pauli principle
    - No transitions from positive-energy state to negative-energy state
    - But the other way round: electron can be elevated from negative to positive-energy state

Dirac Sea



wikimedia.org

  - Hole in Dirac sea: ***anti particle*** with $-q$ but momentum $+\vec{p}$ and spin $+\vec{s}$
  - *Problems*: infinitely much charge, not applicable to bosons
- Solution (Feynman, Stückelberg): ***multi-particle system***
  - Requires ***quantised fields*** ('2. quantisation')
  - Particle with ***negative energy backward in time*** = anti-particle with ***positive energy forward in time***

Discrete symmetries

- Transformation of spinors under ***discrete symmetry operations C, P, T*** important in particle physics
  - ***Charge conjugation C***: particle → anti-particle

$$\psi(x) \to \psi'(x) = i\gamma^2\psi^*(x)$$

  - ***Parity P***: mirroring at origin $x = (t, \vec{x}) \to x' = (t, -\vec{x})$

$$\psi(x) \to \psi'(x') = i\gamma^0\psi(x)$$

  - ***Time reversal T***: $x = (t, \vec{x}) \to x' = (-t, \vec{x})$

$$\psi(x) \to \psi'(x') = i\gamma^1\gamma^3\psi(x)$$

- ***CPT theorem*** (Pauli, Lüders 1957):

Every locally Lorentz-invariant quantum-field theory is invariant under CPT symmetry

Observables from spinors

- Physical observables: ***bilinear forms of spinors***
- Classification by ***transformation behaviour*** under *C* and *P* transformations

Bilinear Form		***C***	***P***	***T***
$\overline{\psi}\psi$	scalar	+	+	+
$\overline{\psi}\gamma^5\psi$	pseudo-scalar	+	-	-
$\overline{\psi}\gamma^\mu\psi$	vector	-	γ^0: +, γ^i: -	γ^0: +, γ^i: -
$\overline{\psi}\gamma^\mu\gamma^5\psi$	axial-vector	+	γ^0: -, γ^i: +	γ^0: +, γ^i: -
$\overline{\psi}\Sigma^{\mu\nu}\psi$	tensor ($\Sigma^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu\gamma^\nu]$)	-	σ^{0j}: -, σ^{ij}: + σ^{0j}: -, σ^{ij}: +	σ^{0j}: +, σ^{ij}: + σ^{0j}: +, σ^{ij}: +

Helicity and chirality

- ***Helicity*** $\boldsymbol{\lambda}$ ('direction of rotation')
  - Projection of spin onto unit vector in direction of momentum

$$\boxed{\lambda = \vec{\Sigma} \cdot \frac{\vec{p}}{|\vec{p}|}} \quad \text{with spin operator } \vec{\Sigma},\ \Sigma_i = \tfrac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

  - Commutes with Hamilton operator: λ is ***'good' quantum number***
  - ***Not Lorentz-invariant***: for massive particles, there is always a reference frame in which momentum but not spin is in opposite direction
- ***Chirality*** ('handedness')
  - Important particle property in electroweak interaction
  - ***Eigenvalue of*** $\boldsymbol{\gamma^5}$ ***operator*** (+: right handed, -: left handed)
  - u, v are chirality eigenstates: $\gamma^5 u = \pm u$, $\gamma^5 v = \pm v$
  - Any spinor can decomposed into left- and right-handed components

$$\boxed{\psi = (\psi_R + \psi_L) \quad \text{with} \quad \psi_{R/L} = P_{R/L}\psi,\ P_{R/L} = \tfrac{1}{2}\left(1 \pm \gamma^5\right)}$$

  - For ***massless*** particles: chirality = helicity

Spin and helicity

The spin and helicity operators for the Dirac equation are:

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \qquad h \equiv \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2\,|\vec{p}|}\begin{pmatrix} \vec{\sigma}\vec{p} & 0 \\ 0 & \vec{\sigma}\vec{p} \end{pmatrix}$$

In general Dirac spinors are not eigen states of the spin or helicity operators, but they are eigen states of S$_z$. For particles moving in z direction:

$$h_z \equiv \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\underbrace{\frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}}_{h_z} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix}}_{u_\downarrow(p_z)} = -\frac{1}{2}\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix}}_{u_\downarrow(p_z)}$$

Analogously one finds:

$$h_z\, u_\uparrow(p_z) = +\frac{1}{2}u_\uparrow(p_z)$$

$$h_z\, u_\downarrow(p_z) = -\frac{1}{2}u_\downarrow(p_z)$$

$$h_z\, v_\uparrow(p_z) = -\frac{1}{2}v_\uparrow(p_z)$$

$$h_z\, v_\downarrow(p_z) = +\frac{1}{2}v_\downarrow(p_z)$$

A spinor with spin against movement direction has negative helicity

Chirality

**Formally, chirality (handedness) is defined through the operator γ^5.
It can be shown that kinetic terms of the Lagrangian are covariant under γ^5.
With its help one can define projection operators for the chirality:**

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2},$$

$$\left.\begin{array}{ll} \psi_L = P_L\psi & \text{lefthanded} \\ \psi_R = P_R\psi & \text{righthanded} \end{array}\right\} \text{component of } \psi$$

The following relations can easily be demonstrated:

$$\underbrace{P_L^2 = P_L, \quad P_R^2 = P_R,}_{\text{projection operators}} \quad \underbrace{P_L\,P_R = P_R\,P_L = 0,}_{\text{projections perpendicular to each other}} \quad \psi_L + \psi_R = \psi.$$

projection operators

**projections perpendicular
to each other**

Chirality

$$P_L u_\downarrow(p_z) = \frac{1-\gamma^5}{2} u_\downarrow(p_z) = \frac{1}{2}\sqrt{E+m} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$= \frac{1}{2}\sqrt{E+m} \left(1 + \frac{p_z}{E+m}\right) \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$P_R u_\downarrow(p_z) = \frac{1+\gamma^5}{2} u_\downarrow(p_z) = \frac{1}{2}\sqrt{E+m} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$= \frac{1}{2}\sqrt{E+m} \underbrace{\left(1 - \frac{p_z}{E+m}\right)}_{\approx (1-\beta)} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Righthanded components are suppressed for particles of negative helicity: $(1-\beta)$

Summary

- **Relativistic quantum mechanics incorporates relativistic energy-momentum relation:**

$$E^2 = p^2 + m^2 \qquad \textbf{or} \qquad p^\mu p_\mu = E^2 - p^2 = m^2$$

- **Most important equations of motion:**
  - **Spin-0 particles (scalars): Klein-Gordon** $\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0$
  - **Spin-1/2 particles: Dirac** $\left(i\gamma^\mu \partial_\mu - m\right) \psi = 0$
  - **Spin-1 particles (vectors): Proca** $\left(\partial_\nu \partial^\nu + m^2\right) A^\mu = 0$

    for completeness

- **From canonical operator replacement:**

$$\boxed{\hat{E} \rightarrow i\frac{\partial}{\partial t} \qquad \hat{\vec{p}} \rightarrow -i\vec{\nabla}} \qquad \textbf{or} \qquad \boxed{\hat{p^\mu} \rightarrow i\partial^\mu}$$