Karlsruhe Institute of Technology

Seismic Modelling

Wave equations and Greens functions

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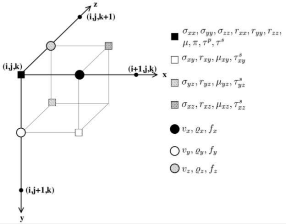
Wave equations

$$p_{ij} = \lambda \theta \delta_{ij} + 2\mu \epsilon_{ij}$$

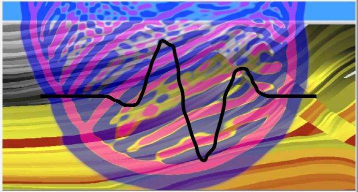
$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial p_{ij}}{\partial x_j} + f_i$$

Discretization



Solutions



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1. Introduction
2. Derivation of the wave equation for acoustic media
3. Greens functions
4. Analytical solutions
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Introduction

In this lecture ...

- ... we derive the acoustic wave equation which will be solved numerically by different modelling methods,
- ... we define the solutions of the acoustic wave equation in terms of Greens functions (GF) (impulse responses),
- ... we compare the GF for homogeneous acoustic media in 1D, 2D, and 3D in the frequency domain,
- ... we show how analytical solutions can be calculated,
- ... compare seismograms for 1D, 2D, 3D media.

In summary we gain a better understanding of wave propagation in homogeneous media and the significant differences resulting from 1D, 2D, and 3D assumptions.
We learn how to calculate analytical solutions to validate numerical methods.

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Stress-strain relation and equation of motions

For the derivation of wave equations we take the stress-strain constitutive relation

$$p_{ij} = \lambda\theta\delta_{ij} + 2\mu\epsilon_{ij} \tag{1}$$

where p_{ij} are the elements of the stress tensor $\mathbf{p}$, λ and μ Lamé parameters, $\theta = \mathrm{div}(\vec{u})$ the cubic dilation and ϵ_{ij} the elements of the deformation tensor. With the displacement vector $\vec{u}$, the deformation tensor is defined as

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

We also take the equation of motion, which is given by

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial p_{ij}}{\partial x_j} + f_i \tag{3}$$

with the mass density ρ and $\vec{f}$ containing external forces.

Approximations in fluids

For the acoustic case (wave propagation in fluids) we can make two simplifications:

1. The shear modulus μ becomes zero $\Rightarrow \mu = 0$
2. We have hydrostatic pressure only and shear stresses vanish, which simplifies the stress tensor to

$$\mathbf{p} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

Modified stress-strain relation and equation of motion

With those two simplifications, the stress-strain relation (1) reduces to

$$-p = \lambda \, \mathrm{div}(\vec{u}) \tag{4}$$

and the equation of motion (3) simplifies to

$$\rho \frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial p}{\partial x_i} \quad (i = 1, 2, 3), \tag{5}$$

assuming that we have no external forces.

Velocity-stress first order acoustic wave equations

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Differentiation of equations (4) with respect to the time t and introducing particle velocities $v_i = \frac{\partial u_i}{\partial t}$ lead to the equations

$$-\frac{\partial p}{\partial t} = \lambda \operatorname{div}(\vec{v}) \tag{6}$$

$$\rho \frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial x_i} \tag{7}$$

This is the so-called velocity-stress formulation of the acoustic wave equation. Since equation (7) consists of three equations, we now have a coupled system of 4 first-order differential equations.

Second order acoustic wave equation

In order to obtain only one second-order differential equation, we differentiate once more equation (6) with respect to time, leading to

$$-\frac{\partial^2 p}{\partial t^2} = \lambda\, \mathrm{div}\left(\frac{\partial}{\partial t}\vec{v}\right) = \lambda\left(\frac{\partial}{\partial x_1}\frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x_2}\frac{\partial v_2}{\partial t} + \frac{\partial}{\partial x_3}\frac{\partial v_3}{\partial t}\right) \tag{8}$$

and then insert equation 7:

$$-\frac{\partial^2 p}{\partial t^2} = -\lambda\left(\frac{\partial}{\partial x_1}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_3}\right)\right) \tag{9}$$

$$\Leftrightarrow \quad \frac{\partial^2 p}{\partial t^2} = \lambda \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right) \tag{10}$$

Equation (10) is the second-order acoustic wave equation.

Second order acoustic wave equation for constant density

Assuming a constant mass density ρ, we obtain the widely used form of the acoustic wave equation

$$\frac{\partial^2 p}{\partial t^2} = \frac{\lambda}{\rho}\nabla^2 p = c^2\,\nabla^2 p \tag{11}$$

with the velocity $c = \sqrt{\frac{\lambda}{\rho}}$ and $\rho = const$. This can also be written as

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{1}{\partial t^2}\right) p = 0} \tag{12}$$

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Green's functions for the acoustic wave equation

Our goal is to find analytical solutions for the inhomogenous acoustic wave equation

$$\left[\Delta - \frac{1}{c^2}\frac{1}{\partial t^2}\right] p(x, x_s, t) = -4\pi f(x_s, t) \tag{13}$$

The Green's function $G(x, x_s, t)$ is the solution of

$$\left[\Delta - \frac{1}{c^2}\frac{1}{\partial t^2}\right] G(x, x_s, t) = -4\pi\delta(x - x_s)\delta t \tag{14}$$

with the source location x_s. If we know $G(x, x_s, t)$ we can construct the solution to any other source time function (RHS) $f(x_s, t)$ by convolution

$$p(x, x_s, t) = \int G(x, x_s, t) f(x_s, t - \tau) d\tau = G(x, x_s, t) * f(x_s, t) \tag{15}$$

The Greens function can be interpreted as the impulse response of the wave equation.

Green's functions for the acoustic wave equation

In the following we will discuss the Green's functions for homogenous media $c = const$ in the frequency domain. We therefore first apply a Fourier transformation (FT) of equation 14 w.r.t. time. Our definition of the FT is

$$\hat{G}(x, x_s, \omega) = \int_{-\infty}^{\infty} G(x, x_s, t) e^{-i\omega t} dt \tag{16}$$

$$G(x, x_s, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(x, x_s, \omega) e^{i\omega t} d\omega \tag{17}$$

Green's functions for the acoustic wave equation

We make use of

$$-\omega^2 \hat{G}(x, x_s, \omega) = \int_{-\infty}^{\infty} \frac{\partial^2 G(x, x_s, t)}{\partial t^2} e^{i\omega t} dt \quad (18)$$

$$1 = e^{i\omega 0} = \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} dt \quad (19)$$

By FT we obtain the Helmholtz equation

$$\left[k^2 + \Delta\right] \hat{G}(x, x_s, \omega) = -4\pi\delta(x - x_s) \quad (20)$$

with the wave number $k = \frac{\omega}{c}$

Green's functions for acoustic wave equation

Assuming a constant velocity c, one can derive the following solutions of the Helmholtz equation in the far field. For details we refer to Morse & Freshbach (1953):

$$1\text{D}: \Delta = \left[\frac{\partial^2}{\partial x^2}\right], \qquad \hat{G}^{1D}(x, x_s, \omega) = \frac{2\pi i}{k} e^{-ikr} \tag{21}$$

$$2\text{D}: \Delta = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right], \qquad \hat{G}^{2D}(x, x_s, \omega) = \sqrt{\frac{2\pi}{kr}} e^{-ikr} e^{-i\pi/4} \tag{22}$$

$$3\text{D}: \Delta = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right], \qquad \hat{G}^{3D}(x, x_s, \omega) = \frac{e^{-ikr}}{r} \tag{23}$$

with $r = ||x - x_s||$.

2D and 3D Geometrical Spreading

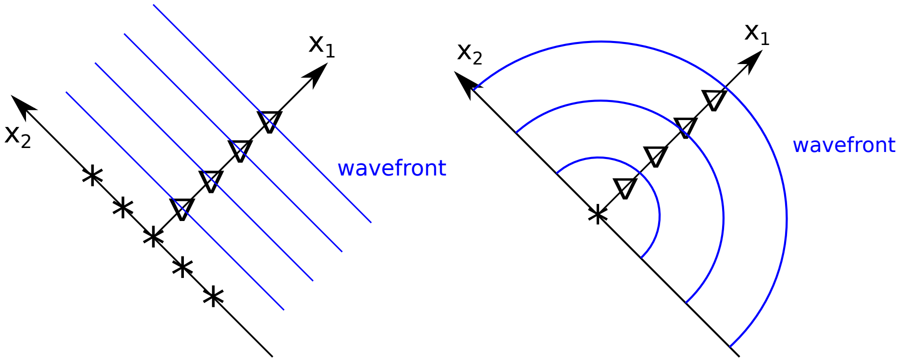

Figure 1: Waves propagating in 2D and 3D. Left: Line Source (2D), Right: Point Source (3D).

Correction filter

With these Greens functions we can formulate a geometrical spreading correction filter (Forbriger et al. 2014).

$$F(r,\omega) = \frac{\hat{G}^{2D}}{\hat{G}^{3D}} = \frac{\hat{G}^{1D}}{\hat{G}^{2D}} = \sqrt{\frac{2\pi r}{k}} e^{-i\pi/4} = \sqrt{\frac{2\pi rc}{\omega}} e^{-i\pi/4} \tag{24}$$

The factor $e^{i\pi/4}$ produces a phase shift of $\frac{\pi}{4}$, and the first term corresponds to a half integration (low-pss filter). With $k = \frac{\omega}{c}$, this can be written as

$$F(r,\omega) = \sqrt{2rc}\sqrt{\frac{\pi}{\omega}}\, e^{-i\pi/4} = \sqrt{2rc}\,\mathrm{FT}\{\sqrt{t^{-1}}\} = F_{amp}\,\mathrm{FT}\{\sqrt{t^{-1}}\}, \tag{25}$$

where $\mathrm{FT}\{\sqrt{t^{-1}}\}$ is the Fourier transform of the function $\sqrt{t^{-1}}$ (Forbriger et al. 2014). This filter is independent of offset r. The other term $F_{amp} = \sqrt{2rc}$ is an offset (r) or travel time ($t = rc$) dependent amplitude correction.

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Calculation of analytical solutions

We now can compute analytical solutions of the acoustic wave equation in 1D, 2D, 3D

$$\left[\Delta - \frac{1}{c^2}\frac{1}{\partial t^2}\right] p(x, x_s, t) = -4\pi f(x_s, t), \quad x_s = 0 \tag{26}$$

at distance $x = R$ via the convolution

$$p^{1D,2D,3D}(R, 0, t) = \int G^{1D,2D,3D}(R, 0, t) f(t - \tau) d\tau \tag{27}$$

or in the frequency domain via

$$\hat{p}^{1D,2D,3D}(R, 0, \omega) = \hat{G}^{1D,2D,3D}(R, 0, \omega)\hat{f}(\omega) \tag{28}$$

with $\hat{G}^{1D,2D,3D}(R, 0, \omega)$ as defined in equations 21, 22, 23, respectively.

Source signal

We assume an homogeneous acoustic medium with $c = 500$ m/s and a shifted Ricker signal as source wavelet with a center frequency of $f_c = 50$ Hz located at $x = 0$

$$f(x_s = 0, t) = (1 - 4\tau^2)e^{-2\tau^2}, \tag{29}$$
$$\tau = \pi(t - t_d)f_c,$$
$$t_d = 1/f_c$$

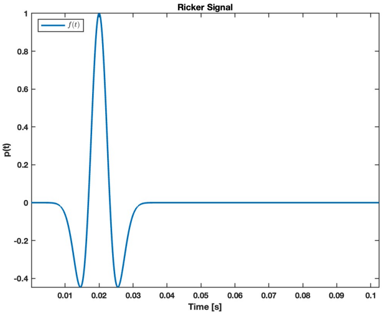

Figure 2: Ricker signal

Amplitude spectrum of source signal

$$\hat{f}(0,\omega) = \int_{-\infty}^{\infty} f(0,t) e^{i\omega t} dt$$

$$|\hat{f}(0,\omega)| = \sqrt{\Re(\hat{f}(0,\omega))^2 + \Im(\hat{f}(0,\omega))^2}$$

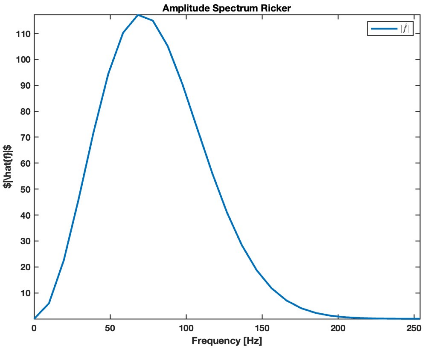

Figure 3: Amplitude spectrum of Ricker

Phase spectrum of source signal

$$\hat{f}(0,\omega) = \int_{-\infty}^{\infty} f(0,t)e^{-i\omega t}dt$$

$$\phi(0,\omega) = \arctan\left(\frac{\Im(\hat{f}(0,\omega))}{\Re(\hat{f}(0,\omega))}\right)$$

$$\phi(0,\omega) = -\omega * t_d$$

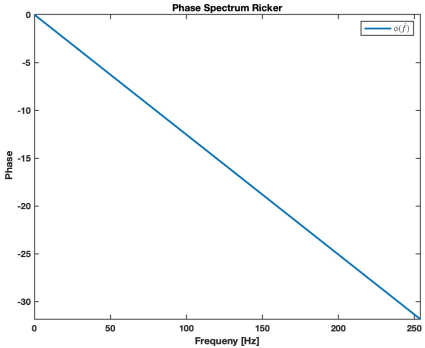

Figure 4: Phase spectrum of Ricker

Analytical solution: 3D

$$\hat{p}^{3D}(R,0,\omega) = \hat{G}^{3D}(R,0,\omega)\hat{f}(0,\omega)$$

$$\boxed{\hat{G}^{3D} = \frac{e^{ikr}}{R}}$$

$$k = \frac{\omega}{c}, c = 500m/s, R = 20m$$

$$p^{3D}(R,0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{p}^{3D}(R,0,\omega)e^{i\omega t}d\omega$$

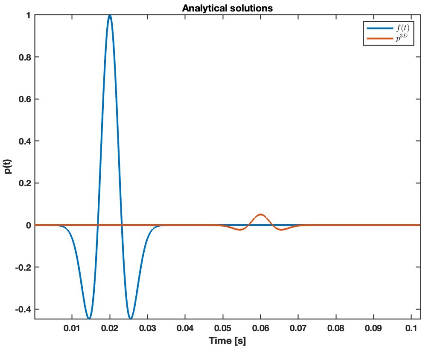

Figure 5: Source signal and analytical solution: 3D

Analytical solution: 3D, 2D

$$\hat{p}^{2D}(R,0,\omega) = \hat{G}^{2D}(R,0,\omega)\hat{f}(0,\omega)$$

$$\boxed{\hat{G}^{2D} = \sqrt{\frac{2\pi c}{\omega R}}e^{-ikR}e^{-i\pi/4}}$$

$$p^{2D}(R,0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{p}^{2D}(R,0,\omega)e^{i\omega t}d\omega$$

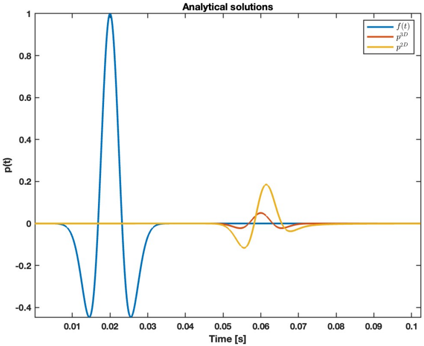

Figure 6: Source signal and analytical solutions: 3D, 2D

Analytical solution: 3D, 2D, 1D

$$\hat{p}^{1D}(R,0,\omega) = \hat{G}^{1D}(R,0,\omega)\hat{f}(0,\omega)$$

$$\boxed{\hat{G}^{1D} = \frac{2\pi i c}{\omega} e^{-ikR}}$$

$$p^{1D}(R,0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{p}^{1D}(R,0,\omega)e^{i\omega t}d\omega$$

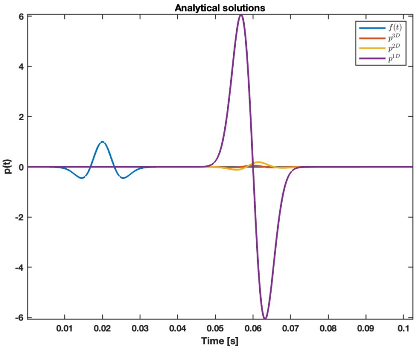

Figure 7: Source signal and analytical solution: 3D, 2D, 1D

Amplitude spectra of Greens functions

$$\hat{G}^{3D} = \frac{e^{-ikr}}{R}$$

$$\hat{G}^{2D} = \sqrt{\frac{2\pi c}{\omega R}} e^{-ikR} e^{-i\pi/4}$$

$$\hat{G}^{1D} = \frac{2\pi i c}{\omega} e^{-ikR}$$

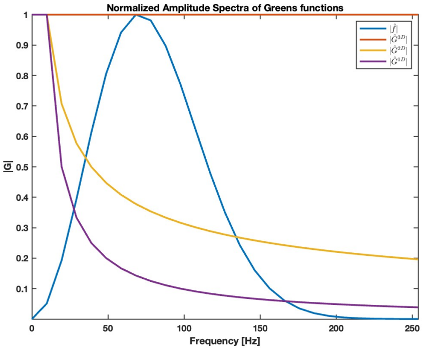

Figure 8: Amplitude spectra of Greens functions

Analytical solution: amplitude and phase spectra

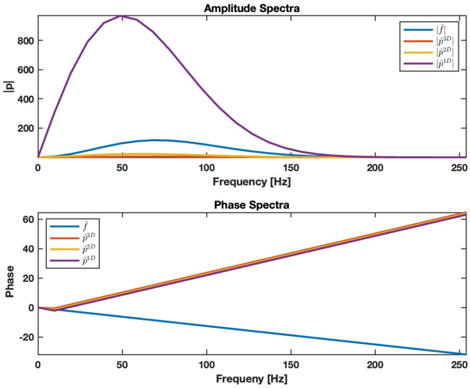

Figure 9: Amplitude and phase spectra of analytical solutions

Analytical solutions: summary

1. 1D: **full integration , no amplitude decay, no phase shift**
2. 2D: **half integration, amplitude decay with** $1/\sqrt{R}$, **phase shift of** $-\pi/4$
3. 3D: **no integration, amplitude decay with** $1/R$, **no phase shift**

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Summary

- We derived the classical acoustic wave equation and discussed the Greens-functions which are solutions of the Helmholtz equation

$$\left[k^2 + \Delta\right] \hat{G}(x, x_s, \omega) = -4\pi\delta(x - x_s)$$

- The solutions are:

$$\hat{G}^{1D} = \frac{2\pi i c}{\omega} e^{-ikr}, \quad \hat{G}^{2D} = \sqrt{\frac{2\pi c}{\omega r}} e^{-ikr} e^{-i\pi/4}, \quad \hat{G}^{3D} = \frac{e^{-ikr}}{r}$$

- 3D: source signal remains unchanged and amplitudes decrease with $1/r$.
- 2D: source signal is half integrated and amplitudes decrease with $1/\sqrt{r}$
- 1D: source signal is fully integrated and amplitudes do not change with r.

References

Forbriger, T., Groos, L. & Schäfer, M. (2014), ‘Line-source simulation for shallow-seismic data. Part 1: theoretical background.’, *Geophysical Journal International* **198**(3), 1387–1404.

Morse, P. & Freshbach, H. (1953), *Methods of theoretical physics*, McGraw-Hill.

Questions

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1. Please give a few examples of media in which the acoustic wave equations should be sufficient to describe wave propagation.
2. Is the approximation of homogeneous mass density ($\rho =$const) crucial ?
3. Why do we consider solutions of 1D and 2D wave equation at all ?
4. Let us summarize the main differences between wave propagation in 1D, 2D, and 3D homogeneous acoustic media w.r.t. to signal shape and amplitude decay.
5. Is it (always) possible to calculate the 3D solution from a given 2D solution (or vice versa) ? What are the assumptions ? In which applications in applied seismics or seismology might such a spreading correction be required ?
6. Can we calculate analytical solutions for non-homogenous media ?
7. Do S-waves in homogeneous elastic media behave similarly ?