Seismic Modelling

2D/3D acoustic Finite Difference Method

Thomas Bohlen, Geophysical Institute, KIT-Faculty of Physics

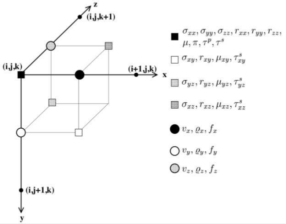
Wave equations

$$p_{ij} = \lambda\theta\delta_{ij} + 2\mu\epsilon_{ij}$$

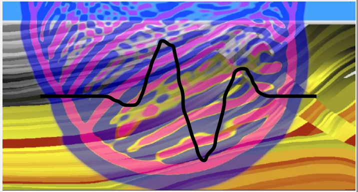
$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial p_{ij}}{\partial x_j} + f_i$$

Discretization



Solutions



Summary of last lecture (1)

We studied the 1-D second order wave equation

$$\frac{\partial^2 p(x,t)}{\partial t^2} = c^2 \frac{\partial^2 p(x,t)}{\partial x^2} \tag{1}$$

and derived the discrete form of accuracy O(2,2M):

$$\frac{1}{\triangle t^2}\left(p_j^{n+1} + p_j^{n-1} - 2p_j^n\right) = \frac{c^2}{h^2} \sum_{m=1}^{M} a_m \left(p_{j+m}^n + p_{j-m}^n - 2p_j^n\right) \tag{2}$$

We inserted a plane wave $p_j^n = p_0 \exp(i(kjh + \omega n \triangle t)$ and obtained the dispersion relation

$$\boxed{\frac{c_{fd}}{c} = \frac{\omega \triangle t}{khr} = \frac{2}{khr} \arcsin\left(r\sqrt{\sum_{m=1}^{M} a_m sin^2(\frac{mkh}{2})}\right)} \tag{3}$$

Summary of last lecture (1)

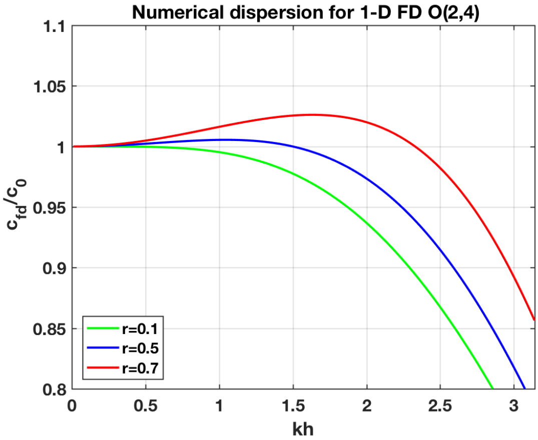


Figure 1: Numerical dispersion O(2,4) (M=2, equation 3)

- Strong effect of Courant number $r = \frac{c \triangle t}{h}$
- $c_{fd} / c > 1$ for large r
- $c_{fd} / c < 1$ for small r
- $\rightarrow$ effect of time discretization

This lecture (2)

1. Study the effect of time discretization in more detail
   a. Dispersion
   b. Stability
2. Extension to 2D and 3D acoustic FD modelling

Agenda

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Numerical dispersion due to time discretization

We want to analyze the dispersion caused by time discretization only. For this purpose we discretize w.r.t. time $t = n\triangle t$ only (x=continuous)

$$\frac{1}{\triangle t^2}\left(p^{n+1} - 2p^n + p^{n-1}\right) = c^2 \frac{\partial^2 p^n(x)}{\partial x^2} \tag{4}$$

We insert a discrete plane wave

$$p^n(x) = p_0 \exp(i(kx + \omega n \triangle t))$$

and obtain with $z_t = \exp(i\triangle t\omega)$

$$\frac{(z_t^1 - 2z_t^0 + z_t^-1)}{\triangle t^2} = \frac{2(\cos(\omega\triangle t) - 2)}{\triangle t^2} = -c^2k^2 \overset{(c_{fd}=\omega/k)}{=} \frac{c^2\omega^2}{c_{fd}^2} \tag{5}$$

$$\frac{c_{fd}^2}{c^2} = \frac{-(\omega\triangle t)^2}{2(\cos(\omega\triangle t) - 1)} \overset{(\omega\triangle t = rkh)}{=} \frac{-(rkh)^2}{2(\cos(rkh) - 1)} \overset{Taylor}{\approx} 1 + \frac{(rkh)^2}{12} > 1 \tag{6}$$

Numerical dispersion due to time discretization

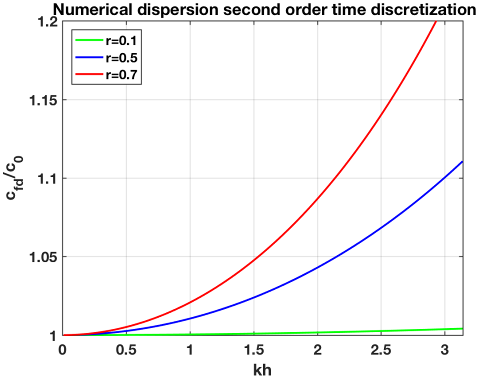


Figure 2: Numerical dispersion of 1D FD scheme due to second order time discretization only (equation 6).

- time dispersion leads to $c_{fd}/c > 1$
- strong dispersion for large r (coarse $\triangle t$)
- however, this can compensate for the opposite dispersion ($c_{fd}/c < 1$) intruced by space discretization !
- $\rightarrow$ find balance between time and space discretization and order of accuray $2M$ to minimize effective dispersion

Agenda

Von Neumann stability analysis

We first consider the 1D acoustic wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \tag{7}$$

We perform an analysis of the numerical stability of the explicit 1D acoustic FD integration

$$p_j^{n+1} = 2p_j^n - p_j^{n-1} + r^2 \left(a_0 p_j^n + \sum_{m=1}^{M} a_m \left(p_{j+m}^n + p_{j-m}^n \right) \right) \tag{8}$$

where

$$a_0 = -2 \sum_{m=1}^{M} a_m \tag{9}$$

and the Courant number is

$$r = \frac{c \triangle t}{h} \tag{10}$$

Von Neumann stability analysis

We first want to abbreviate the sum. By defining

$$a_{-m} = a_m \tag{11}$$

we can write equation 8 in a more condensed form

$$p_j^{n+1} = 2p_j^n - p_j^{n-1} + r^2 \sum_{m=-M}^{M} a_m p_{j+m}^n \quad \text{with} \quad a_0 = -2 \sum_{m=1}^{M} a_m \tag{12}$$

In order to separate the time and space discretization we make the following Ansatz

$$p_j^n = p^n e^{ikx} = p^n e^{ikjh} \tag{13}$$

Inserting this gives

$$p^{n+1} e^{ikjh} = -p^{n-1} e^{ikjh} + 2p^n e^{ikjh} + r^2 \sum_{m=-M}^{M} a_m p^n e^{ik(j+m)h} \tag{14}$$

Von Neumann stability analysis

Dividing by e^{ikjh} gives

$$p^{n+1} = -p^{n-1} + 2p^n + r^2 \sum_{m=-M}^{M} a_m p^n e^{ikmh} \tag{15}$$

We factor out p^n

$$p^{n+1} = -p^{n-1} + p^n \left[2 + r^2 \sum_{m=-M}^{M} a_m e^{ikmh} \right] = -p^{n-1} + 2Ap^n \tag{16}$$

with

$$A = 1 + \frac{r^2}{2} \sum_{m=-M}^{M} a_m e^{ikmh} \tag{17}$$

Von Neumann stability analysis

We define the time amplification factor

$$\gamma := \frac{p^{n+1}}{p^n} = \frac{p^n}{p^{n-1}} \tag{18}$$

The criterion for stability is

$$|\gamma| \leq 1 \quad f.a.n \in \mathbb{N}^+$$

This ensures that amplitudes do not increase at **every time step due to the discretization**.
By dividing equation 16 by p^{n-1} we get

$$\gamma^2 = -1 + 2A\gamma \quad \text{or} \quad \gamma^2 - 2A\gamma + 1 = 0 \tag{19}$$

The solutions are

$$\gamma_{1,2} = A \pm \sqrt{A^2 - 1} \tag{20}$$

Von Neumann stability analysis

The criterion for stability $|\gamma| \leq 1$ is fulfilled if

$$-1 \leq A \leq 1 \tag{21}$$

Consequently

$$-1 \leq 1 + \frac{r^2}{2} \sum_{m=-M}^{M} a_m e^{ikmh} \leq 1 \tag{22}$$

We subtract 1 and multiply with 2

$$-4 \leq r^2 \sum_{m=-M}^{M} a_m e^{ikmh} \leq 0 \tag{23}$$

Von Neumann stability analysis

The term

$$
\begin{aligned}
\sum_{m=-M}^{M} a_m e^{ikmh} &= \sum_{m=1}^{M} a_m\left(e^{ikmh} - e^{-ikmh}\right) + a_0 \qquad (24)\\
&= \sum_{m=1}^{M} a_m\left(e^{ikmh} - e^{-ikmh} - 2\right)\\
&= -\sum_{m=1}^{M} a_m 2\left(\cos(kmh) - 1\right)\\
&= -\sum_{m=1}^{M} a_m 4\left(\sin^2(kmh/2)\right) < 0
\end{aligned}
$$

Von Neumann stability analysis

Therefore, the requirement for stability is

$$r^2 \left| \sum_{m=-M}^{M} a_m e^{ikmh} \right| \leq 4 \tag{25}$$

An upper limit for the sum is

$$\left| \sum_{m=-M}^{M} a_m e^{ikmh} \right| \leq \sum_{m=-M}^{M} |a_m| \tag{26}$$

This gives

$$r^2 \sum_{m=-M}^{M} |a_m| \leq 4 \quad \text{or} \quad r \leq \frac{2}{\sqrt{\sum_{m=-M}^{M} |a_m|}} := r_{max} \tag{27}$$

This is the upper limit of the Courant number to ensure numerical stability for the 1D acoustic case.

Von Neumann stability analysis

1D explicit FD modelling with equation 8 is stable if

$$r = \frac{c\triangle t}{h} \leq r_{max} \tag{28}$$

where

$$r_{max} = \frac{2}{\sqrt{\sum\limits_{m=1}^{M} 2|a_m| + |a_0|}}, \quad a_0 = -2\sum_{m=1}^{M} a_m \tag{29}$$

FD coefficients and maximum Courant numbers r_{max}

M	$a_0 = 2\sum_{m=1}^{M} a_m$	a_1	a_2	a_3	a_4	a_5	accuracy order (2M)	r_{max}
1	2	1					2	1
2	5/2	4/3	-1/12				4	$\frac{\sqrt{3}}{2}$
3	49/18	3/2	-3/20	1/90			6	$\frac{\sqrt{765}}{34}$
4	205/72	8/5	-1/5	8/315	-1/560		8	$\frac{\sqrt{630}}{32}$
5	5269/1800	5/3	-5/21	5/126	-5/1008	1/3150	10	$\frac{\sqrt{150}}{16}$

Table 1: FD-coefficients and maximum Courant numbers in 1D.

Stability limits for 1D FD modelling

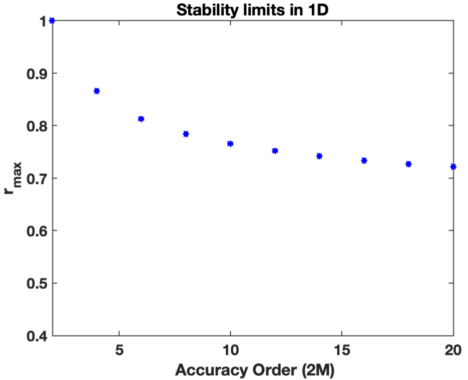


Figure 3: Maximum Courant numbers for 1D FD modelling calculated with equation 29.

- r_{max} decreases with increasing M

Numerical verification of stability criteria

We perform 1D FD-simulations with O(2,4). The maximum Courant number is

$$r_c = \frac{c \triangle t}{h} \leq \frac{\sqrt{3}}{2} = 0.8660254 \tag{30}$$

We choose a homogeneous medium with $c = 500$ m/s. $\triangle t = 1.732$ ms and $h = 1m$. In our first simulation we have

$$r_1 = 0.866 < r_c \tag{31}$$

The simulation shoud be stable.

Stability for acoustic FD modelling

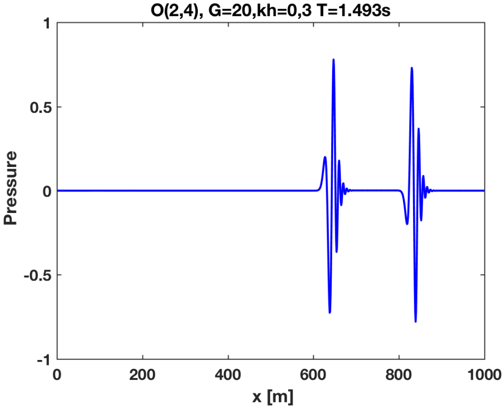


Figure 4: **Movie** of snapshots (1DFD, O(2,4)), $r_1 = 0.866 < r_c = 0.8660254$.

Numerical verification of stability criteria

Now we increase the velocity at the single gridpoint located at $x = 600$m to $c = 500.05$ m/s. The velocity remains at $c = 500$ m/s elsewhere. We thus now have

$$\begin{aligned} r_2 &= 0.86608 > r_c = 0.8660254 \quad at \quad x = 600m \\ r_1 &= 0.866 < r_c = 0.8660254 \quad \text{elsewhere} \end{aligned} \tag{32}$$

We violate the criterion at one grid point slightly.

Stability for acoustic FD modelling

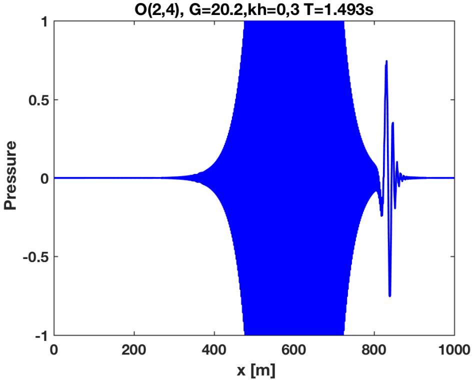


Figure 5: **Movie** of snapshots (1DFD, O(2,4)) where $r > r_c$ at one single grid point located at $x = 600m$.

- amplitides "explode" immediately
- the stability criterion must be fulfilled strictly

Agenda

1. The Finite-Difference Method

1.3 1D Dispersion analysis (time discretization)

1.4 1D Stability analysis

1.5 Extension to 2D and 3D acoustic FD modelling

2D/3D acoustic FD modelling

We show the 3D case. In 3D (D=3) we have the scalar wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right) \tag{33}$$

We discretize $x = ih, y = jh, z = kh$

$$\begin{aligned} p_{ijk}^{n+1} &= 2p_{ijk}^{n} - p_{ijk}^{n-1} \\ &+ r^2 \sum_{m=-M}^{M} a_m p_{i+m,j,k}^{n} \\ &+ r^2 \sum_{m=-M}^{M} a_m p_{i,j+m,k}^{n} \\ &+ r^2 \sum_{m=-M}^{M} a_m p_{i,j,k+m}^{n} \end{aligned} \tag{34}$$

Dispersion analysis for 2D acoustic FD modelling

We insert a harmonic plane wave of the form

$$p = p_o \exp\left(i\left(wt - kx\sin(\theta) - kz\cos(\theta)\right)\right) \tag{35}$$

where θ denotes the angle between the vertical axis z and the direction of propagation. We introduce the following abbreviations $z_t = e^{i\omega\triangle t}$, $z_x = e^{ikh\sin(\theta)}$, $z_z = e^{ikh\cos(\theta)}$. This gives

$$z_t^1 + z_t^{-1} - 2 = r^2\left(\sum_{m=1}^{M} a_m\left(z_x^m + z_x^{-m} - 2\right) + \sum_{m=1}^{M} a_m\left(z_z^m + z_z^{-m} - 2\right)\right) \tag{36}$$

We make use of

$$z^m + z^{-m} - 2 = e^{(ikmh)} + e^{(-ikmh)} - 2 = 2\left(\cos(mkh) - 1\right)\right) = -4\sin^2(mkh/2) \tag{37}$$

Dispersion analysis for 2D acoustic FD modelling

Therefore

$$\sin^2\left(\frac{\omega\triangle t}{2}\right) = r^2 \sum_{m=1}^{M} a_m \left[\sin^2\left(\frac{mkh\sin(\theta)}{2}\right) + \sin^2\left(\frac{mkh\cos(\theta)}{2}\right)\right] \tag{38}$$

The dispersion for the 2D acoustic O(2,2M) FD integration thus is

$$\boxed{\frac{c_{fd}}{c} = \frac{\omega\triangle t}{khr} = \frac{2}{khr}\arcsin\left[r\sqrt{\sum_{m=1}^{M} a_m \left[\sin^2\left(\frac{mkh\sin(\theta)}{2}\right) + \sin^2\left(\frac{mkh\cos(\theta)}{2}\right)\right]}\right]} \tag{39}$$

Dispersion for 2D acoustic FD modelling O(2,2)

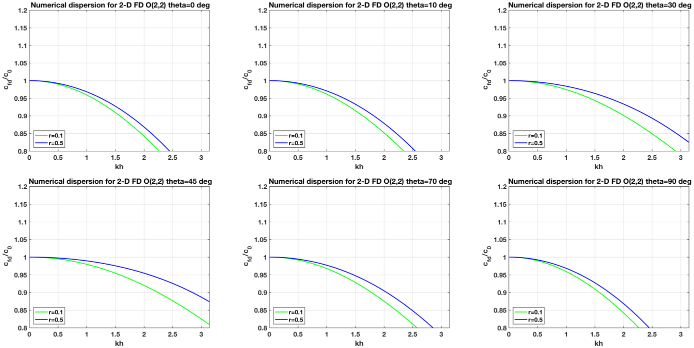


Figure 6: Numerical dispersion of 2D acoustic FD modelling O(2,2) (equation 39).
Click on first frame to play

Dispersion for 2D acoustic FD modelling O(2,2)

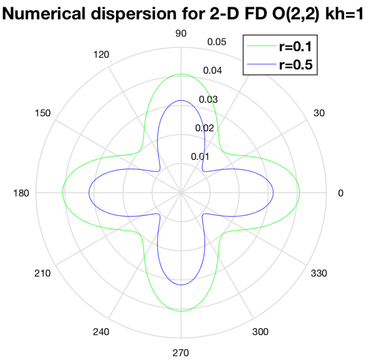


Figure 7: Plot of $|c_{fd}/c - 1|$ of equation 39 using kh=1 and 2M=2.

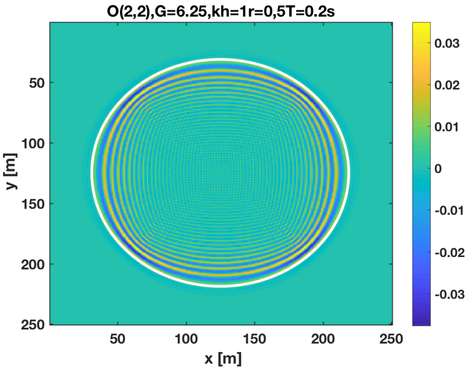
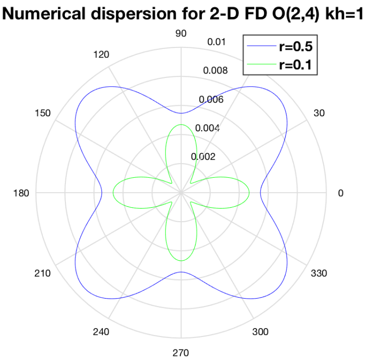


Figure 8: Snapshot of acoustic O(2,2) FD modelling ($kh = 1$, $r = 0.5$)

Dispersion for 2D acoustic FD modelling O(2,4)



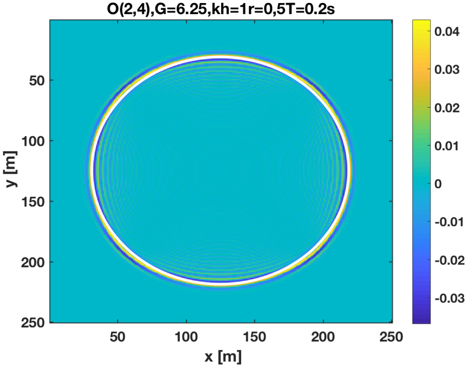
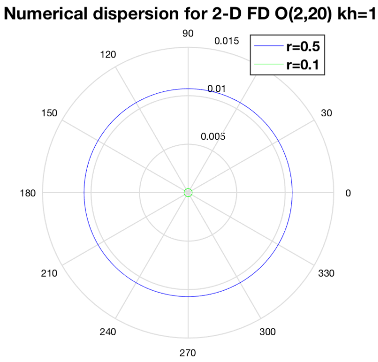


Figure 9: Plot of $|c_{fd}/c - 1|$ of equation 39 using kh=1 and 2M=4.

Figure 10: Snapshot of acoustic O(2,4) FD modelling ($kh = 1$, $r = 0.5$)

Dispersion for 2D acoustic FD modelling O(2,20)



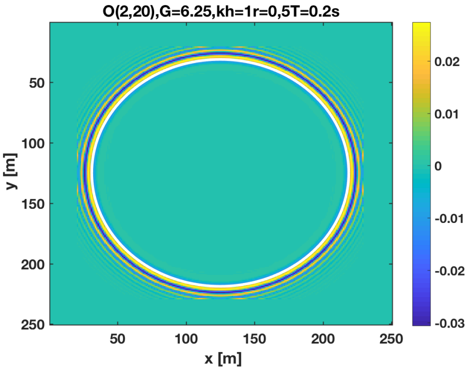


Figure 11: Plot of $|c_{fd} / c - 1|$ of equation 39 using kh=1 and 2M=20.

Figure 12: Snapshot of acoustic O(2,20) FD modelling ($kh = 1$, $r = 0.5$)

Dispersion for 2D acoustic FD modelling O(2,20)

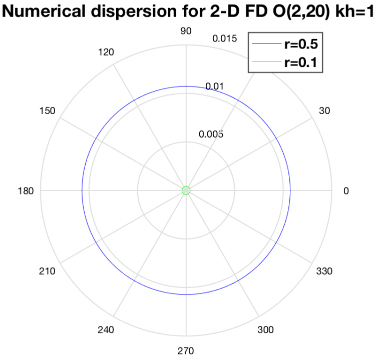


Figure 13: Plot of $|c_{fd}/c - 1|$ of equation 39 using kh=1 and 2M=20.

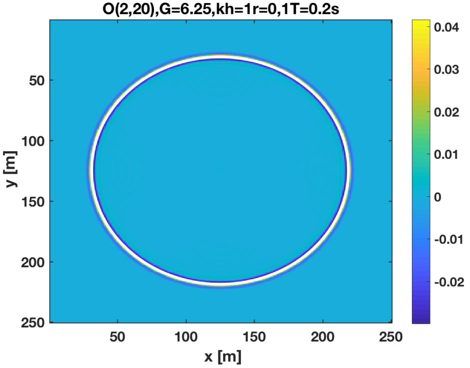


Figure 14: Snapshot of acoustic O(2,20) FD modelling ($kh = 1$, $r = 0.1$)

Stability for 2D and 3D acoustic FD modelling

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If we perform the same Von Neumann stability analysis for the 2D (D=2) and 3D (D=3) discrete wave equations 34 the criterion for stability is simply

$$r^2 \sum_{m=-M}^{M} D|a_m| \leq 4 \tag{40}$$

We therefore get the factor $\sqrt{D}$ in the final stability condition

$$\boxed{r = \frac{c\triangle t}{h} \leq \frac{2}{\sqrt{D}\sqrt{\sum\limits_{m=-M}^{M} |a_m|}} = \frac{2}{\sqrt{D}\sqrt{\sum\limits_{m=1}^{M} 2|a_m| + |a_0|}}} \tag{41}$$

Equation 41 defines the upper limit of the Courant number to ensure numerical stability of O(2,2M) acoustic FD modelling in D=1,2,3 dimensions.

Stability limits for 1D FD modelling

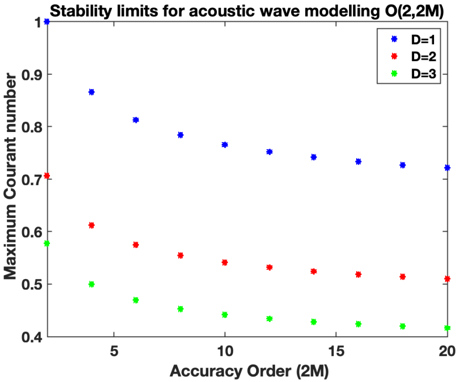

Figure 15: Maximum Courant numbers for acoustic O(2,2M) FD modelling calculated with equation 41.

- r_{max} decreases with increasing M and dimension D

Summary of lecture

- We studied the numerical dispersion caused by time discretization: $c_{fd}\,/\,c > 1$
- Dispersion caused by space and time discretization can counterbalance.
- Von Neumann stability analysis: $r < r_{max}$. Strict stability condition must be met.
- Extension of the analysis of diserion and stability to 2D and 3D is straightforward.
  - Numerical dispersion $c_{fd}\,/\,c = f(r, kh, 2M, \theta)$
  - In higher dimensions $D = 2, 3$ the stability limit is reduced by a factor $(1\,/\,\sqrt{D})$

Questions

1. How can we distinguish between the numerical dispersion produced by a too coarse time discretization from the numerical dispersion produced by a too coarse space discretization ?
2. Do the opposite effects of numerical dispersion caused by time and space discretization generally partly compensate each other and is this generally advantageous ? Ich which case can we observe a perfect cancellation ?
3. Do you know numerical methods that allow to increase the accuracy order of the time discretization beyond second order ?
4. Please recap the principle assumption and central idea of the Von Neumann stability analysis.
5. Can we in most cases expect a stronger or weaker numerical error for waves that propagate along the cartesian directions ?

Questions

6 Let us consider a complex velocity model with a minimum velocity of 200 m/s in the shallow part and 2000 m/s in the deeper part. The main frequency of the seismic source shall be 50 Hz. We want to simulate waves with a 2D O(2,2) acoustic FD scheme. How should we choose the space interval h and time step $\triangle t$ to ensure a stable and sufficiently accurate simulation ?

7 Can we predict the dispersion error in FD modeling analytically ? If yes, how could we correct the synthetic seismograms to obtain (more) accurate solutions ?