Seismic Modelling

Velocity-stress Finite Difference Method

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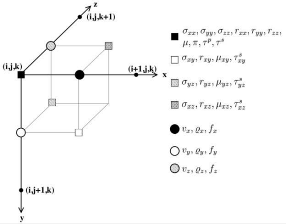
Wave equations

$$p_{ij} = \lambda \theta \delta_{ij} + 2\mu \epsilon_{ij}$$

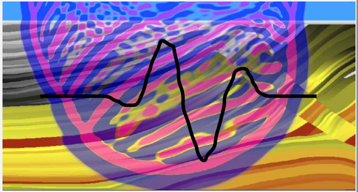
$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial p_{ij}}{\partial x_j} + f_i$$

Discretization



Solutions



Agenda

1D first order wave equation

In previous lectures we discretized the 1-D second order wave equation

$$\frac{\partial^2 p(x,t)}{\partial t^2} = c^2 \frac{\partial^2 p(x,t)}{\partial x^2} \tag{1}$$

Now we study the first order wave equation which is composed of the stress-strain relation

$$\frac{\partial p}{\partial t} = -\lambda \frac{\partial v}{\partial x} \tag{2}$$

and the equation of motion

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3}$$

with $p(x,t)$: pressure, $v(x,t)$: particle velocity, $\lambda(x)$: Lamé parameter, $\rho(x)$: mass density.
Note that $c^2(x) = \lambda(x)/\rho(x)$.

Discretization of 1D first order wave equation

We now discretize using full and staggered positions in space and time

$$x = jh \quad , \quad t = n\triangle t \tag{4}$$

$$\lambda(jh) := \lambda_j \quad , \quad \rho((j+1/2)h) := \rho_{j+1/2} \tag{5}$$

$$p(jh, (n+1/2)\triangle t) := p_j^{n+1/2} \quad , \quad v((j+1/2)h, n\triangle t) := v_{j+1/2}^n \tag{6}$$

A second order approximation O(2,2) of wave equation on a so-called staggered grid reads

$$\left.\frac{\partial p}{\partial t}\right|_j^n \approx \frac{p_j^{n+1/2} - p_j^{n-1/2}}{\triangle t} \quad = \quad -\lambda_j \frac{v_{j+1/2}^n - v_{j-1/2}^n}{h} \approx -\lambda_j \left.\frac{\partial v}{\partial x}\right|_j^n \tag{7}$$

$$\left.\frac{\partial v}{\partial t}\right|_{j+1/2}^{n-1/2} \approx \frac{v_{j+1/2}^n - v_{j+1/2}^{n-1}}{\triangle t} \quad = \quad -\frac{1}{\rho_{j+1/2}} \frac{p_{j+1}^{n-1/2} - p_j^{n-1/2}}{h} \approx -\frac{1}{\rho_{j+1/2}} \left.\frac{\partial p}{\partial x}\right|_{j+1/2}^{n-1/2} \tag{8}$$

1D Staggered grid velocity-stress formulation

The explicit FD time update O(2,2) then is

$$p_j^{n+1/2} = p_j^{n-1/2} - \frac{\lambda_j \triangle t}{h}\left(v_{j+1/2}^{n} - v_{j-1/2}^{n}\right)$$

$$v_{j+1/2}^{n} = v_{j+1/2}^{n-1} - \frac{\rho_{j+1/2}^{-1}\triangle t}{h}\left(p_{j+1}^{n-1/2} - p_j^{n-1/2}\right)$$

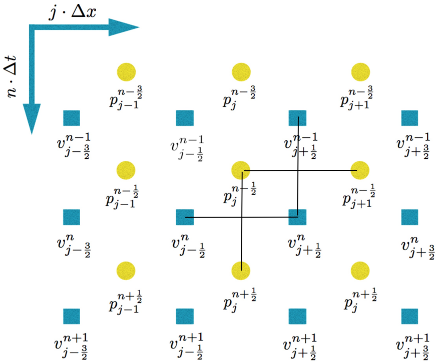


Figure 1: Staggered positions of pressure and particle velocities and FD stencils.

Staggered grid FD approximation of the first order derivative

We perform a Taylor series expansion around $p_{j+1/2}$

$$p_{j+m} = p_{j+1/2} + p^{(1)}_{j+1/2}(m-1/2)h + \sum_{k=2}^{N} \frac{p^{(k)}_{j+1/2}(m-1/2)^k h^k}{k!}$$

We set $m = 1, ..., M > 0$ and subtract

$$p_{j+m} - p_{j-m} = 2p^{(1)}_{j+1/2}(m-1/2)h + 2\sum_{k=2}^{N} \frac{p^{(2k-1)}_{j+1/2}(m-1/2)^{2k-1} h^{2k-1}}{(2k-1)!}$$

to eliminate even derivatives. We move our desired derivative to the LHS:

$$(2m-1)p^{(1)}_{j+1/2} = \frac{1}{h}(p_{j+m} - p_{j-m}) - \frac{1}{h}\sum_{k=2}^{N} \frac{p^{(2k-1)}_{j+1/2}(2m-1)^{2k-1} h^{2k-1}}{(2k-1)!}$$

Staggered grid FD approximation of the first order derivative

We multiply for each m with β_m

$$\beta_m(2m-1)p^{(1)}_{j+1/2} = \frac{1}{h}\beta_m\left(p_{j+m} - p_{j-m}\right) - \frac{1}{h}\beta_m \sum_{k=2}^{N} \frac{p^{(2k-1)}_{j+1/2}(2m-1)^{2k-1}h^{2k-1}}{(2k-1)!}$$

We sum the equations for $m = 1, ..., M$:

$$\sum_{m=1}^{M} \beta_m(2m-1)p^{(1)}_{j+1/2} = \frac{1}{h}\sum_{m=1}^{M}\beta_m\left(p_{j+m} - p_{j-m}\right) - \frac{1}{h}\sum_{m=1}^{M}\beta_m \sum_{k=2}^{N} \frac{p^{(2k-1)}_{j+1/2}(2m-1)^{2k-1}h^{2k-1}}{(2k-1)!}$$

Staggered grid FD approximation of the first order derivative

If we choose β_m so that

$$\sum_{m=1}^{M} \beta_m(2m-1) = 1 \quad \text{and} \quad \sum_{m=1}^{M} \beta_m(2m-1)^{2k-1} = 0 \quad \text{with} \quad k = 2, ..., M \tag{9}$$

we obtain

$$p_{j+1/2}^{(1)} = \frac{1}{h} \sum_{m=1}^{M} \beta_m \left(p_{j+m} - p_{j-m}\right) - \frac{1}{h} \sum_{k=M+1}^{N} \frac{p_{j+1/2}^{(2k-1)} (2m-1)^{2k-1} h^{2k-1}}{(2k-1)!}$$

Staggered grid FD approximation of the first order derivative

The accuracy order is defined as smallest exponent of h in the error term

$$-\frac{1}{h}\sum_{k=M+1}^{N}\frac{p_{j+1/2}^{(2k-1)}(2m-1)^{2k-1}h^{2k-1}}{(2k-1)!}=\mathcal{O}(h^{2M})$$

which is $2(M+1)-1-1=2M$. So we finally obtain the following staggered FD aproximation for the first order derivative:

$$\boxed{p_{j+1/2}^{(1)}=\frac{1}{h}\sum_{m=1}^{M}\beta_m\left(p_{j+m}-p_{j-m}\right)+\mathcal{O}(h^{2M})} \tag{10}$$

FD coefficients for the first derivative

$2M$	β_1	β_2	β_3	β_4	β_5
2	1				
4	9/8	-1/24			
6	75/64	-25/389	3/640		
8	1225/1024	-245/3072	49/5120	-5/7168	
10	19845/16384	-735/8192	567/40960	-405/229376	35/294912

Table 1: FD-coefficients for the staggered approximation of the first derivative (equation 10) calculated via equation 9. The accuracy order is $2M$.

FD update O(2,2M)

The explicit FD time update O(2,2M) for the 1D wave equation then becomes

$$
\begin{aligned}
p_j^{n+1/2} &= p_j^{n-1/2} - \frac{\lambda_j \triangle t}{h} \sum_{m=1}^{M} \beta_m \left(v_{j+m-1/2}^{n} - v_{j-m+1/2}^{n} \right) \\
v_{j+1/2}^{n} &= v_{j+1/2}^{n-1} - \frac{\rho_{j+1/2}^{-1} \triangle t}{h} \sum_{m=1}^{M} \beta_m \left(p_{j+m}^{n-1/2} - p_{j-m+1}^{n-1/2} \right)
\end{aligned}
$$

Dispersion analysis

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We consider the equations

$$\frac{\partial p}{\partial t} = -\lambda \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{11}$$

We now perform the dispersion analysis for the staggered approximation O(2,2M)

$$\frac{p_j^{n+1/2} - p_j^{n-1/2}}{\triangle t} = -\frac{\lambda_j}{h}\sum_{m=1}^{M}\beta_m\left(v^n_{j+m-1/2} - v^n_{j-m+1/2}\right) \tag{12}$$

$$\frac{v^n_{j+1/2} - v^{n-1}_{j+1/2}}{\triangle t} = -\frac{\rho^{-1}_{j+1/2}}{h}\sum_{m=1}^{M}\beta_m\left(p^{n-1/2}_{j+m} - p^{n-1/2}_{j-m+1}\right) \tag{13}$$

Dispersion analysis

Ansatz

$$p = p_0 \exp i(kx + \omega t), \quad v = v_0 \exp i(kx + \omega t) \tag{14}$$

We insert eq. 14 into eq. 11 and obtain

$$i\omega p_0 = \lambda i k v_0 \rightarrow \frac{p_0}{v_0} = -\frac{\lambda k}{\omega} = -\frac{\lambda}{c} \tag{15}$$

The discrete form of eq. 14 is

$$p_j^n = p_0 \exp i(kjh + \omega n \triangle t) \quad v_j^n = v_0 \exp i(kjh + \omega n \triangle t) \tag{16}$$

We insert this into eq. 12

$$p_0 \left(\exp(i\omega \triangle t/2) - \exp(-i\omega \triangle t/2)\right) = \frac{v_0 \lambda \triangle t}{h} \sum_{m=1}^{M} \beta_m \left(\exp(ikh(m - \frac{1}{2})) - \exp(-ikh(m - \frac{1}{2}\right.$$

Dispersion analysis

We apply

$$\exp(iu) - \exp(-iu) = -2i\sin(u)$$

and get

$$-p_0 2i\sin(\frac{\omega\triangle t}{2}) = 2i\frac{v_0\lambda\triangle t}{h}\sum_{m=1}^{M}\beta_m\sin(kh(m-\frac{1}{2}))$$

With eq. 15 we get

$$\sin(\frac{\omega\triangle t}{2}) = r\sum_{m=1}^{M}\beta_m\sin(kh(m-\frac{1}{2})) \tag{17}$$

with the Courant number

$$r = \frac{c\triangle t}{h} \tag{18}$$

Dispersion analysis

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With this we obtain the ratio between numerical velocity and model velocity for the 1D staggered FD scheme approximating the first order wave equation

$$\frac{c_{fd}}{c} = \frac{\omega \triangle t}{khr} = \frac{2}{khr} arcsin \left[r \left(\sum_{m=1}^{M} \beta_m \sin(kh(m-1/2)) \right) \right] \tag{19}$$

This differs from the dispersion relation for the FD scheme approximating the second order wave equation

$$\frac{c_{fd}}{c} = \frac{\omega \triangle t}{khr} = \frac{2}{khr} \arcsin \left(r \sqrt{\sum_{m=1}^{M} a_m sin^2 (\frac{mkh}{2})} \right) \tag{20}$$

Dispersion analysis

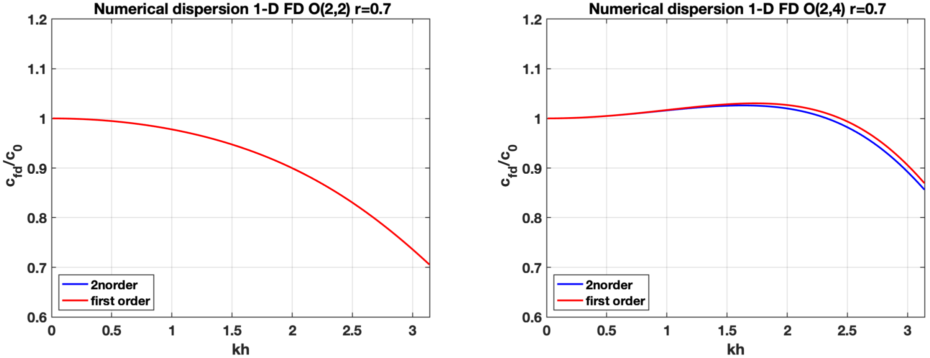

Figure 2: Numerical dispersion for staggered and non-staggered 1D FD schemes for 2M=2 (left) and 2M=4 (right) (calculated with eq. 19 and 20).

Stability analysis

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A von Neumann stability analysis gives the following upper limit for the Courant number. We omit the details of the derivation.

$$r = \frac{c\triangle t}{h} \leq \frac{1}{\sum_{m=1}^{M} |\beta_m|} \tag{21}$$

(When can identify this already by inspecting the dispersion relation 17 when we insert $\sin(kh(m - 1/2)) = 1$ and $\sin(\omega\triangle t/2) = 1$).

Stability analysis

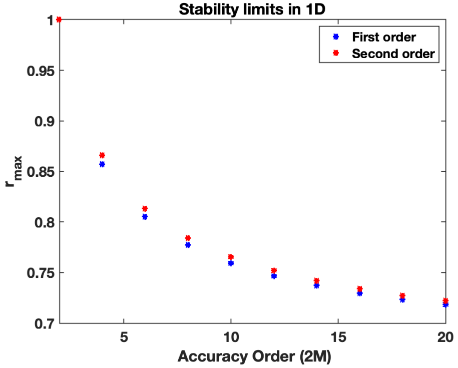

Figure 3: Maximum Courant numbers for staggered and full grid FD modelling in 1D.

1. Order (staggered grid)

$$r_{max} = \frac{1}{\sum_{m=1}^{M} |\beta_m|}$$

2. Order (full grid)

$$r_{max} = \frac{2}{\sqrt{\sum_{m=1}^{M} 2|a_m| + |a_0|}}, \quad a_0 = -2\sum_{m=1}^{M} a_m$$

Agenda

2D elastic wave equation

Stress-strain relation

$$\frac{\partial \sigma_{xx}}{\partial t} = \Pi \frac{\partial v_x}{\partial x} + (\Pi - 2\mu) \frac{\partial v_y}{\partial y} \qquad (22)$$

$$\frac{\partial \sigma_{yy}}{\partial t} = \Pi \frac{\partial v_y}{\partial y} + (\Pi - 2\mu) \frac{\partial v_x}{\partial x}$$

$$\frac{\partial \sigma_{xy}}{\partial t} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

Equation of motion

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x \qquad (23)$$

$$\rho \frac{\partial v_y}{\partial t} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y$$

$\Pi := \lambda + 2\mu$ P-wave modulus; $\alpha^2 = \Pi/\rho$: P-wave velocity; $\alpha^2 = \mu/\rho$: SV-wave velocity.

Standard Staggered grid

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Stress-strain relation

$$\begin{aligned}
\frac{\sigma_{xx}^{n+}[i,j]-\sigma_{xx}^{n-}[i,j]}{\triangle t} &= \Pi[i,j]\frac{v_x^n[i+,j]-v_x^n[i-,j]}{h}+\lambda[i,j]\frac{v_y^n[i,j+]-v_x^n[i,j-]}{h} \qquad (24)\\
\frac{\sigma_{yy}^{n+}[i,j]-\sigma_{yy}^{n-}[i,j]}{\triangle t} &= \Pi[i,j]\frac{v_y^n[i,j+]-v_y^n[i,j-]}{h}+\lambda[i,j]\frac{v_x^n[i+,j]-v_x^n[i-,j]}{h}\\
\frac{\sigma_{xy}^{n+}[i+,j+]-\sigma_{xy}^{n-}[i+,j+]}{\triangle t} &= \mu[i+,j+]\left(\frac{v_x^n[i+,j+1]-v_x^n[i+,j]}{h}+\frac{v_y^n[i+1,j+]-v_x^n[i,j+]}{h}\right)
\end{aligned}$$

Equation of motion

$$\begin{aligned}
\frac{v_x^n[i+,j]-v_x^{n-1}[i+,j]}{\triangle t} &= \rho^{-1}[i+,j]\left(\frac{\sigma_{xx}^{n-}[i+1,j]-\sigma_{xx}^{n-}[i,j]}{h}+\frac{\sigma_{xy}^{n-}[i+,j+]-\sigma_{xy}^{n-}[i+,j-]}{h}\right) \qquad (25)\\
\frac{v_y^n[i,j+]-v_y^{n-1}[i,j+]}{\triangle t} &= \rho^{-1}[i,j+]\left(\frac{\sigma_{xy}^{n-}[i+,j+]-\sigma_{xy}^{n-}[i-,j+]}{h}+\frac{\sigma_{yy}^{n-}[i,j+1]-\sigma_{yy}^{n-}[i,j]}{h}\right)
\end{aligned}$$

Standard staggered grid

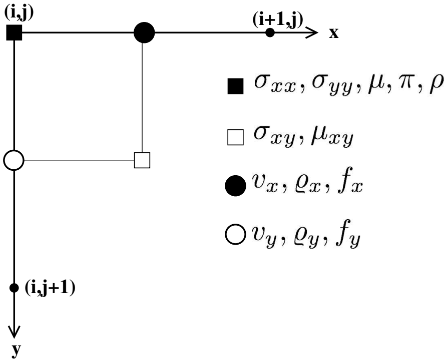

Figure 4: 2D elastic Standard Staggered Grid (SSG).

$$\begin{aligned}
\rho_x[i+,j] &= \frac{\rho[i+1,j]+\rho[i,j]}{2} \\
\rho_y[i,j+] &= \frac{\rho[i,j+1]+\rho[i,j]}{2} \\
\mu_{xy}^{-1}[i+,j+] &= \frac{1}{4}\Big(\mu^{-1}[i,j+1]+\mu^{-1}[i,j]+ \\
&\quad +\mu^{-1}[i+1,j]+\mu^{-1}[i+1,j+1]\Big)
\end{aligned} \tag{26}$$

Point sources

An explosive sources at grid point $[i_s, j_s]$ is generated by adding the source signal $S(n\triangle t) = S^n$ to the diagonal components of the stress tensor:

$$\begin{aligned} \sigma_{xx}^{n+}[i_s, j_s] &= \sigma_{xx}^{n+}[i_s, j_s] + S^{n+} \\ \sigma_{yy}^{n+}[i_s, j_s] &= \sigma_{yy}^{n+}[i_s, j_s] + S^{n+} \end{aligned} \tag{27}$$

A directional force sources is produced by adding the source signal $S(n\triangle t) = S^n$ to the particle velocities:

$$\begin{aligned} v_x^n[i_s, j_s] &= v_x^n[i_s, j_s] + S^n \quad \text{horizontal force} \\ v_y^n[i_s, j_s] &= v_y^n[i_s, j_s] + S^n \quad \text{vertical force} \end{aligned} \tag{28}$$

Movies

- Explosion
- Vertical Force

Free surface boundary condition - image method

The free surface is defined by the boundary condition

$$\sigma_{xy} = \sigma_{yy} = 0 \quad \text{at} \quad y = 0. \tag{29}$$

If we define the depth of the planar free surface to be in $y = 1 \cdot h$ ($j = 1$). then this can be realized by the so-called image method:

$$\begin{aligned} \sigma_{yy}[i, 1] &= 0 \\ \sigma_{xy}[i+, 1+] &= -\sigma_{xy}[i+, 0+] \end{aligned} \tag{30}$$

Vertical point force at free surface of half space.

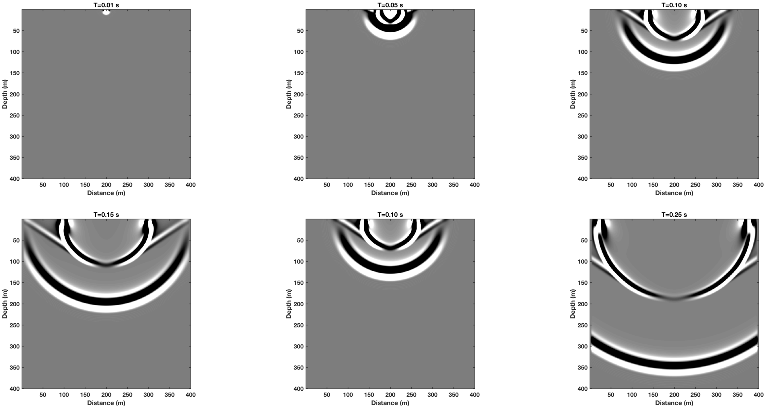

Figure 5: A point force at the free surface radiates Rayleigh waves, P- and SV-waves. Plots show snapshots of vertical particle velocity v_y. The free surface is realized by the image method. Click on first frame to play ($f_c = 30Hz$ $V_p = 1500m/s$, $V_s = 800m/s$, $\varrho = 2000kg/m^3$, $\lambda_p = 50m$, $\lambda_s = 23m$) .

Domain decomposition

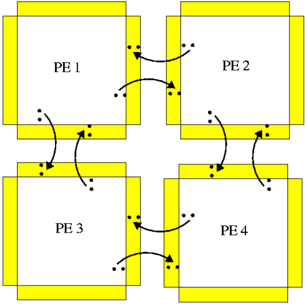

Figure 6: Decomposition of the global grid into subgrids each computed by a different processing element (PE).

- Domain decomposition using MPI
- Very efficient for HPC
- Width of padding layer is *M* for exchange of wavefield

Agenda

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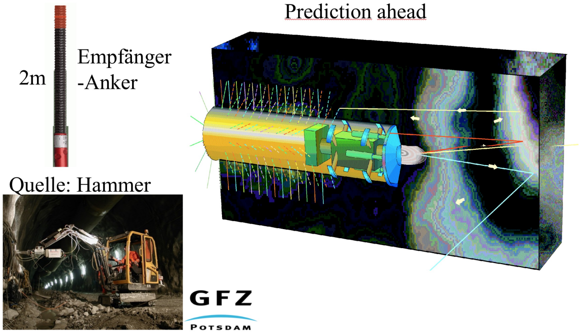
Tunnel seismic prediction



- Seismic waves must be excited and recorded behind the TBM
- Source: hammer (point force)
- Receiver: geophones (particle velocities)
- Goal: look ahead into the drilling direction

Figure 7: Tunnel seismic prediction

3D elastic FD modelling

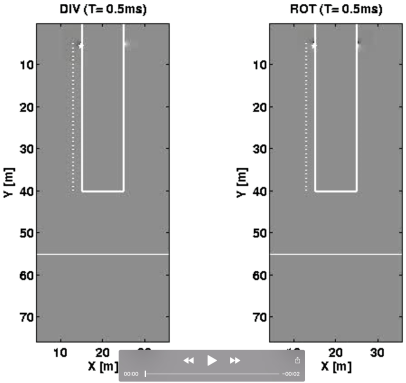


Figure 8: Snapshots of $\nabla \cdot \vec{v}$ (compressional part) and $(\nabla \times \vec{v})_y$ (shear part).

1. Rayleigh wave propagates to the front face
2. conversion into S-wave
3. S-S reflection at interface
4. conversion back into Rayleigh wave

RSSR has highest amplitudes and contains information about reflectors ahead of the tunnel.

Tunnel seismic prediction

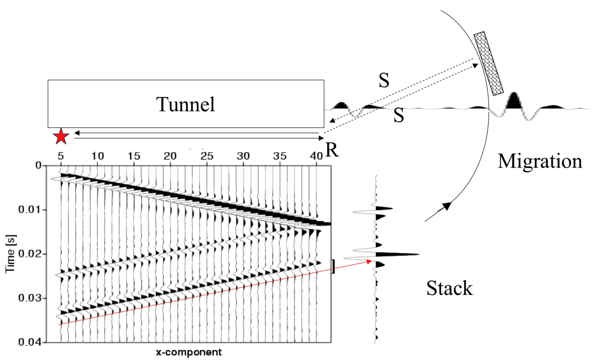


Figure 9: Migration of RSSR waves

- Summation of RSSR waves recorded along the side walls
- Back-projection (migration) to reflector location
- Image of S-S reflections ahead of tunnel

Summary of lecture

- We studied the FD approximation of the first order wave equations on staggered grids.
- The numerical properties (dispersion and stabilty) are almost identical to the FD discretization of second order wave equation on regular (non-staggered) grids.
- The staggered grid approximation is widely applied to the first order elastic wave equation because of the existence of a unique staggered distribution of wavefield parameters.
- Sources and boundary conditions aligned with the grid can be realized efficiently in elastic FD modelling.
- Elastic wave simulation in complex environments (tunnels) can be helpful to understand wave propgation effects and develop new ideas for seismic imaging.

Questions

1. What are the main differences between acoustic and elastic wave propagation in general ?
2. What are the challenges of elastic versus acoustic FD modelling ?
3. What are the advantages and disadvantages of the discretization of the first order wave equation on staggered grids compared to the discretization of the second order wave equation on non-staggered (full) grids ? Let us discuss the following aspects: (1) wave equation formulation, (2) dispersion, (3) stability, (4) source and receiver locations, (5) realization of explicit boundary conditions, (6) material parameter averaging, (7) number of floating point operations per grid point.
4. How can the boundary conditions at ordinary geological discontinuities (continuity of stress and displacement) in the FD method be fulfilled ? Do we have to consider them explicitly ? How can we realize the boundary condition of the (topographic) free surface ?

Questions

5 Why do we have to average the material parameters on a staggered grid (slide 22) ?

6 Let us consider the domain decomposition illustrated on slide 27. How would you generally choose the size of the individual subgrids and the width of the padding layer ?

7 Let us consider the example of tunnel seismic prediction. What are the the advantages and disadvantages of RSSR waves compared to conventional P-wave reflection seismics ?