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Seismic Modelling

Finite Element Method

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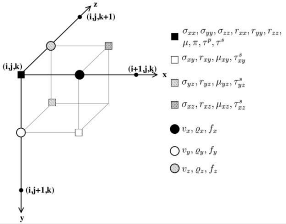
Wave equations

$$p_{ij} = \lambda \theta \delta_{ij} + 2\mu \epsilon_{ij}$$

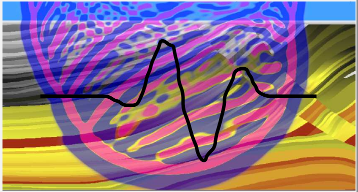
$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial p_{ij}}{\partial x_j} + f_i$$

Discretization



Solutions



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Multi-scale problems

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In some practical problems ”small-scale” features (structures much smaller than the seismic wavelength) can affect wave propagation. Those structures often exhibit strong contrasts in material properties.

- earth topography variations (mountains)
- boreholes, tunnels, TBM
- gas accumulations, fractures, voids, cracks
- earthquakes
- interaction of seismic waves with constructions
- problems in non-destructive material testing and medical imaging

To efficiently simulate ”small-scale” features a spatially variable discretization must be applied.

Modelling of planar interfaces

Grid-based methods suffer from artifacts produced by stair-case approximation.

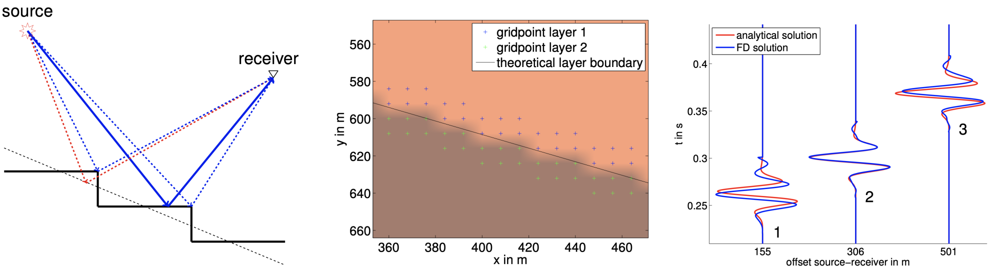

Figure 1: Left: Ray paths of exact (red) and FD solution (blue). Center: FD velocity model with stair-cases, Right: Seismograms of reflected P-wave at different offsets after dispersion correction (Habelitz 2016).

Finite Elements (FE)

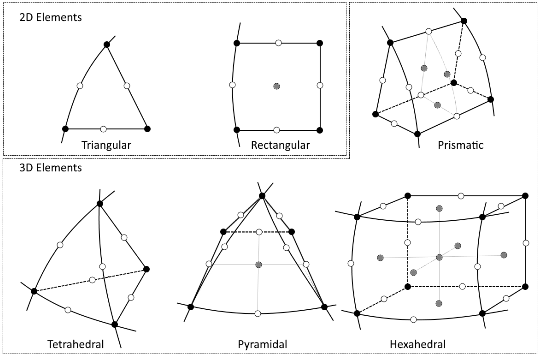

Figure 2: Typical Finite Elements Source:Comsol-Multiphysics.

- FE methods allow for the flexible space discretization by using finite elements.
- The weak formulation of the wave equation inside of each element is solved.
- The inner wavefield is approximated by basis functions.
- At the edges boundary conditions are applied.
- Geological interfaces must be aligned with the edges of the finite-elements. This may require sophisticated meshing.

Finite Elements (FE)

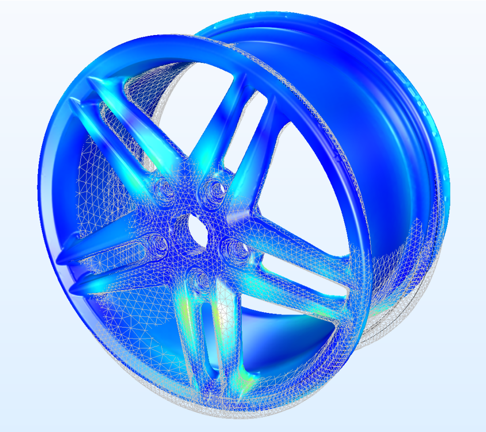

Figure 3: Example of Finite Element discretization: stresses and deformations of a wheel rim in a structural analysis. (Source:Comsol-Multiphysics).

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Weak formulation of wave equation

We start with the strong form of the 1-D wave equation for SH-waves (Igel 2016):

$$\rho(x)\partial_t^2 u(x,t) = \partial_x \mu(x) \partial_x u(x,t) + f(x) \tag{1}$$

where $u(x,t)$ = transverse displacement, $\mu(x)$ = shear modulus, $\rho(x)$=density, $f(x)$= external force. We omit the space and time dependencies:

$$\rho\partial_t^2 u = \partial_x \mu \partial_x u + f \tag{2}$$

We multiply equation 2 with an arbitrary, real, smooth, space-dependent test function $v = v(x)$ and integrate over a domain D (the volume of the element).

$$\int_D v\rho\partial_t^2 u dx - \int_D v\partial_x \mu \partial_x u dx = \int_D v f dx \tag{3}$$

Weak formulation of wave equation

We carry out an integration by parts:

$$-\int_D v\partial_x \mu \partial_x u dx = \int_D \mu \partial_x v \partial_x u dx + [v\mu\partial_x u]_{x_{min}}^{x_{max}} \tag{4}$$

where (x_{min}, x_{max}) define the edges of domain D. The last term on the RHS of equation 4 can be interpreted in terms of shear stress. According to Hooke's law (1-D SH waves)

$$\sigma_{xz} = \mu \partial_x u \tag{5}$$

This term vanishes if we assume a stress free surface at the edges of the elements: $\sigma_{xz} = 0$. Assuming that we have a free surface at the edges of our domain D we obtain

$$\int_D v\rho\partial_t^2 u dx + \int_D \mu \partial_x v \partial_x u dx = \int_D v f dx \tag{6}$$

which is the weak (integral) formulation of the 1D wave equation .

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Finite element formulation

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To enter the discrete world of Finite elements we perform two steps

1. We replace our exact solution u by an approximated solution $\bar{u}$.

$$u(x,t) \approx \bar{u}(x,t) = \sum_{i=1}^{N} u_i(t)\varphi_i(x) \tag{7}$$

   where $\varphi_i(x)$ are space dependent basis functions. Note that the coefficients $u_i(t)$ are time dependent only.

2. We choose our test functions as

$$v(x) := \varphi_j(x) \tag{8}$$

   This choice is known as the Galerkin principle.

Finite element formulation

With these two steps we obtain one equation for each of the j test functions $v(x) \rightarrow \varphi_j(x)$

$$\int_D \rho \partial_t^2 \bar{u} \varphi_j dx + \int_D \mu \partial_x \bar{u} \partial_x \varphi_j dx = \int_D \varphi_j f dx \tag{9}$$

By inserting equation 7 $\bar{u}(x,t) = \sum_{i=1}^{N} u_i(t)\varphi_i(x)$ we can turn the continuous weak form into a system of linear equations

$$\int_D \rho \partial_t^2 \left(\sum_{i=1}^{N} u_i(t) \varphi_i \right) \varphi_j dx + \int_D \mu \partial_x \left(\sum_{i=1}^{N} u_i(t) \varphi_i \right) \partial_x \varphi_j dx = \int_D \varphi_j f dx \tag{10}$$

Changing the order of integration and summation, we obtain a system of j linear equations

$$\sum_{i=1}^{N} \partial_t^2 u_i(t) \int_D \rho \varphi_i \varphi_j dx + \sum_{i=1}^{N} u_i(t) \int_D \mu \partial_x \varphi_i \partial_x \varphi_j dx = \int_D \varphi_j f dx \tag{11}$$

Finite element formulation

We now introduce a matrix-vector notation

$$
\begin{aligned}
\text{displacement} \quad \boldsymbol{u}(t) &\rightarrow u_i(t) \\
\text{mass matrix} \quad \boldsymbol{M}(x) &\rightarrow M_{ij} = \int_D \rho \varphi_i \varphi_j dx \\
\text{stiffness matrix} \quad \boldsymbol{K}(x) &\rightarrow K_{ij} = \int_D \mu \partial_x \varphi_i \partial_x \varphi_j dx \\
\text{force} \quad \boldsymbol{f}(x) &\rightarrow f_j(x) = \int_D \varphi_j f dx
\end{aligned}
$$

Thus we can write the system of j equations as

$$\partial_t^2 \boldsymbol{u}\boldsymbol{M} + \boldsymbol{u}\boldsymbol{K} = \boldsymbol{f} \tag{12}$$

or with transposed system matrices as

$$\boldsymbol{M}^T \partial_t^2 \boldsymbol{u} + \boldsymbol{K}^T \boldsymbol{u} = \boldsymbol{f} \tag{13}$$

Finite element formulation

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By approximating the second order time derivative by a second order FD approximation

$$\mathbf{M}^T \left[\frac{\mathbf{u}(t+dt) - 2\mathbf{u}(t) + \mathbf{u}(t-dt)}{dt^2} \right] = \mathbf{f} - \mathbf{K}^T \mathbf{u} \tag{14}$$

we finally obtain the explicit time update FE equation

$$\boxed{\mathbf{u}(t+dt) = dt^2 (\mathbf{M}^T)^{-1} \left[\mathbf{f} - \mathbf{K}^T \mathbf{u} \right] + 2\mathbf{u}(t) - \mathbf{u}(t-dt)} \tag{15}$$

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Finite difference formulation

We compare the FE discretization of the weak formulation with the FD discretization of the strong formulation of the 1-D wave equation for SH-waves assuming $\mu = const$.

$$\rho(x)\partial_t^2 u(x,t) = \mu\partial_x^2 u(x,t) + f(x) \tag{16}$$

We apply second order FD-operators in both space and time O(2,2):

$$\rho(x)\left[\frac{u(x,t+dt) - 2u(x,t) + u(x,t-dt)}{dt^2}\right] = \mu\left[\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}\right] + f(x) \tag{17}$$

This gives the explicit update scheme

$$u(x,t+dt) = dt^2\rho(x)^{-1}\left[f(x) + \mu\left[\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}\right]\right] + 2u(x,t) - u(x,t-dt) \tag{18}$$

Comparison of FD and FE formulations

Finite elements

$$\boldsymbol{u}(t+dt) = dt^2 (\boldsymbol{M}^T)^{-1} \left[\boldsymbol{f} - \boldsymbol{K}^T \boldsymbol{u}\right] + 2\boldsymbol{u}(t) - \boldsymbol{u}(t-dt)$$

Finite differences

$$u(x,t+dt) = dt^2 \rho(x)^{-1} \left[f(x) + \mu \left[\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \right] \right] + 2u(x,t) - u(x,t-dt)$$

$$(\boldsymbol{M}^T)^{-1} \rightarrow \rho(x)^{-1} \quad \boldsymbol{K}^T \rightarrow -\mu \partial_x \partial_x$$

Finite elements

- global wavefields and operations
- $(\boldsymbol{M}^T)^{-1}$ costly to calculate
- $\boldsymbol{K}^T$ easy to calculate

Finite differences

- local wavefields and operations
- $\rho(x)^{-1}$ easy to calculate
- $-\mu \partial_x \partial_x$ easy to calculate

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System matrices

We now calculate the entries for the mass and stiffness matrix for a simple choice of basis function. The basis functions should be defined locally within each element. For this purpose we introduce a local coordinate system

$$\begin{aligned} \xi &= x - x_i \\ h_i &= x_{i+1} - x_i \end{aligned} \tag{19}$$

The element size h_i may vary. The element i is located in the intervall $x \in [x_i, x_{i+1}]$

Local linear basis functions

In the local coordinate system simple linear basis functions can be defined as

$$\varphi_i(\xi) = \begin{cases} \frac{\xi}{h_{i-1}} + 1 & -h_{i-1} \leq \xi \leq 0 \\ 1 - \frac{\xi}{h_i} & 0 \leq \xi \leq h_i \\ 0 & \text{elsewhere} \end{cases} \quad (20)$$

The linear basis functions are defined over two elements. The derivatives are

$$\partial_\xi \varphi_i(\xi) = \begin{cases} \frac{1}{h_{i-1}} & -h_{i-1} \leq \xi \leq 0 \\ -\frac{-1}{h_i} & 0 \leq \xi \leq h_i \\ 0 & \text{elsewhere} \end{cases} \quad (21)$$

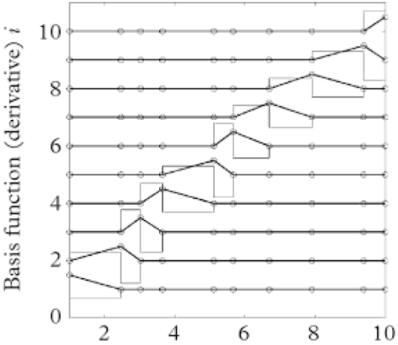

Figure 4: Local basis function and first derivatives as defined in equations 20 and 21 (Igel 2016).

The mass matrix

The mass matrix has been defined as

$$M_{ij} = \int_D \rho \varphi_i \varphi_j dx = \int_{D_\xi} \rho \varphi_i \varphi_j d\xi \tag{22}$$

For the diagonal elements we obtain

$$M_{ii} = \rho_{i-1} \int_{-h_{i-1}}^{0} \left(\frac{\xi}{h_{i-1}} + 1 \right)^2 d\xi + \rho_i \int_0^{h_i} \left(1 - \frac{\xi}{h_i} \right)^2 d\xi = \frac{1}{3} (\rho_{i-1} h_{i-1} + \rho_i h_i) \tag{23}$$

For the off-diagonal elements the basis functions overlap only in one element, for example

$$\begin{aligned} M_{i,i-1} &= \rho_{i-1} \int_{-h_{i-1}}^{0} \left(\frac{\xi}{h_{i-1}} + 1 \right) \frac{-\xi}{h_{i-1}} d\xi = \frac{1}{6} \rho_{i-1} h_{i-1} \\ M_{i,i+1} &= \rho_{i-1} \int_0^{h_i} \left(1 - \frac{\xi}{h_i} \right) \frac{\xi}{h_i} d\xi = \frac{1}{6} \rho_{i-1} h_{i-1} \end{aligned} \tag{24}$$

The mass matrix

The banded nature of the mass matrix, assumimg constant element size h and density ρ reads

$$
\boldsymbol{M} = \frac{\rho h}{6} \begin{pmatrix} \cdots & & & & & & & \\ & 1 & 4 & 1 & & & & \\ & & 1 & 4 & 1 & & & \\ & & & 1 & 4 & 1 & & \\ & & & & 1 & 4 & 1 & \\ & & & & & & & \cdots \end{pmatrix} \tag{25}
$$

The stiffness matrix

The stiffness matrix has been defined as

$$K_{ij} = \int_D \mu \partial_x \varphi_i \partial_x \varphi_j dx = \int_{D_\xi} \mu \partial_\xi \varphi_i \partial_\xi \varphi_j d\xi \tag{26}$$

For the diagonal elements we obtain assuming constant shear modulus μ in each element

$$K_{ii} = \mu_{i-1} \int_{-h_{i-1}}^{0} \left(\frac{1}{h_{i-1}}\right)^2 d\xi + \mu_i \int_0^{h_i} \left(-\frac{1}{h_i}\right)^2 d\xi = \frac{\mu_{i-1}}{h_{i-1}} + \frac{\mu_i}{h_i} \tag{27}$$

For the off-diagonal elements

$$\begin{aligned} K_{i,i-1} &= \mu_{i-1} \int_{-h_{i-1}}^{0} \left(\frac{1}{h_{i-1}}\right) \frac{-1}{h_{i-1}} d\xi = -\frac{\mu_{i-1}}{h_{i-1}} \\ K_{i,i+1} &= \mu_{i-1} \int_0^{h_i} \left(\frac{-1}{h_i}\right) \frac{1}{h_i} d\xi = -\frac{\mu_i}{h_i} \end{aligned} \tag{28}$$

The stiffness matrix

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For the stiffness matrix we obtain in a similar way

$$\mathbf{K} = \frac{\mu}{h} \begin{pmatrix} \cdots & & & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & & & \cdots \end{pmatrix} \tag{29}$$

Which corresponds to second order Finite Difference operator matrix.

System matrices

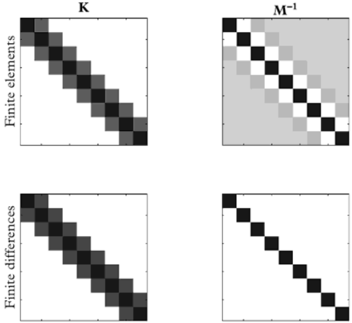

Figure 5: System matrices for the FE and FD method (Igel 2016).

Finite elements

$$\mathbf{u}(t+dt) = dt^2(\mathbf{M}^T)^{-1}\left[\mathbf{f} - \mathbf{K}^T\mathbf{u}\right] + 2\mathbf{u}(t) - \mathbf{u}(t-dt)$$

Finite differences

$$\begin{aligned} u(x,t+dt) \quad = \quad & dt^2\rho(x)^{-1}\left(f(x)\right. \\ & \left. +\mu\left[\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}\right]\right) \\ & +2u(x,t) - u(x,t-dt) \end{aligned}$$

$$(\mathbf{M}^T)^{-1} \rightarrow \rho(x)^{-1} \quad \mathbf{K}^T \rightarrow -\mu\partial_x\partial_x$$

Summary: Finite element method

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- The FE method allows to simulate multi-scale problems by flexible discretization in space. Furthermore, the edges of the FE can be aligned with interfaces to avoid stair-case artifacts.
- The FE calculates local approximations inside of each element. This requires the calculation of system matrices in the initialization phase.
- The FE method is a series expansion method. The continuous solution is replaced by a finite sum over basis functions.
- The stress-free surface is implicitly solved. This is an advantage compared to the FD method when free surface topography needs to be considered explicitly.
- The time update can be performed by a conventional FD-scheme.
- For seismological applications the inversion of the mass matrix is expensive. The spectral element methods solves this by a special choice of basis functions and integration scheme.

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Igel, H. (2016), *Computational Seismology: A Practical Introduction*, 1. edn, Oxford University Press. **URL:** *https://global.oup.com/academic/product/computational-seismology-9780198717409?cc=de&lang=en&*

Questions

1. What are the main advantages of the FE method ? What are potential shortcomings ?
2. Why is the calculation of $(M^T)^{-1}$ costly ?
3. Is the weak formulation fully equivalent to the strong formulation of the wave equation ? Why do we solve the weak formulation with FE methods and not the strong formulation ?
4. Why can the FE method be considered as a series expansion method ?
5. What are the boundary conditions between adjacent elements ?
6. How can we deal with strong surface topography in the FE method and the FD method ?
7. Let us compare the FE and FD method w.r.t. the following aspects: accuracy, computational efficiency, stability, numerical dispersion, implementation.

Questions

8 Let us consider the following scenarios. Would you apply the FD or the FE method ? wave propagation in (a) mountains, (b) marine environments, (c) bore holes, (e) subduction zones, (f) global earth, (g) shallow sediments, (h) strongly heterogenous media.