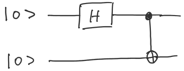
Vorlesung 3bis

Tuesday, 7. May 2019 21:43

Entanglement

two qubit register $|b_1 b_2\rangle$



CNOT : $|x, y\rangle \rightarrow |x, y \oplus x\rangle$

$$|00\rangle \xrightarrow{H \otimes I_2} \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Let's measure the first bit

$\sim$ prob. $\frac{1}{2}$ for $|0\rangle$ and $|1\rangle$

if we measure $|0\rangle \sim$ state is $|00\rangle$

$|1\rangle \sim$ $|11\rangle$

$\Rightarrow$ we measure always the same states for the two qubits

$\Rightarrow$ this is also true when we separate the two qubits of the entangled state

Alice and Bob

- the order of measuring the two qubits is not important
- afterwards, when Alice and Bob compare their results, they find identical results

4 Bell - states

$$\phi^+ = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\phi^- = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$\psi^+ = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$\psi^- = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$\Rightarrow$ orthonormal basis

$\Rightarrow$ qubits of ϕ^+ or ψ^+ are known as EPR - paires (Einstein, Podolsky, Rosen)

Are $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ or $H_2|11\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$ entangled? ⇒ No because product states

$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$H_2|11\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Def: A state $|\phi\rangle$ is not entangled when it is composed of a product state of individual qubit states

Quantum teleportation

Problem: Alice wants to tell Bob a Qubit state
She has to measure it and distroys it!
Making a copy does not work neither
(no-cloning theorem)
The qubit $|x\rangle$ has the state

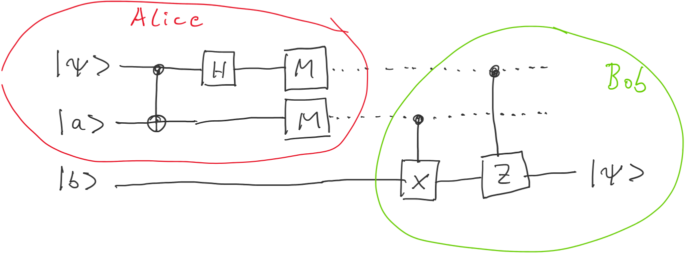
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Solution: Alice and Bob share an entangled EPR-pair: $|ab\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

goes to Alice

goes to Bob

Alice entangles her qubit $|x\rangle$ with half of the EPR-pair. Then she measures it (and distroys them)
⇒ Bob's half changes to Alice's qubit
using two classical information, he can get the qubit state $|\psi\rangle$



Algorithm:

0. $|ab\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

1. Alice applies the CNOT-gate

   $|x\rangle|a\rangle \leftarrow |x\rangle|a \oplus x\rangle$

2. Alice applies Hadamard gate

   $|x\rangle \leftarrow H|x\rangle$

3. Alice measures her qubits $|x\rangle$ and $|a\rangle$ and passes her result to Bob

4. For $a = 1$, Bob applies a bitflip operation X

   $|b\rangle \leftarrow X|b\rangle$

5. For $x = 1$, Bob applies a phasefip operation Z

   $|b\rangle \leftarrow Z|b\rangle$

Analysis

Let's consider the quantum register

$$|x\rangle|a\rangle|b\rangle$$

$|x\rangle$ is in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

therefore the quantum register is initially in

$$|\phi_0\rangle = (\alpha|0\rangle + \beta|1\rangle)\cdot\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|100\rangle + |111\rangle)$$

First step of Alice : CNOT

$$|\phi_1\rangle = \frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|110\rangle + |101\rangle)$$

Second step : Hadamard

$$|\phi_2\rangle = \frac{\alpha}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)(|00\rangle + |11\rangle) +$$

$$+\frac{\beta}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)(|10\rangle + |01\rangle)$$

$$= \frac{\alpha}{2}(|000\rangle + |011\rangle + |100\rangle + |111\rangle)$$

$$+\frac{\beta}{2}(|010\rangle + |001\rangle - |110\rangle - |101\rangle)$$

separation in respect to Alice

$$|\phi_2\rangle = \frac{1}{2}\Big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) +$$

$$|01\rangle(\beta|0\rangle + \alpha|1\rangle) +$$

$$|10\rangle(\alpha|0\rangle - \beta|1\rangle) +$$

$$-|11\rangle(\beta|0\rangle - \alpha|1\rangle)\Big)$$

Third step : Alice gets with propability $\frac{1}{4}$ the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

after the measurement Bob's half EPR-pair becomes a state that is related to Alice's qubit $|x\rangle$

Fourth and Fives step:

Bob corrects his qubit $|b\rangle$ to become $|\psi\rangle$

No-Cloning-Theorem

Can we copy a quantum state $\alpha|0\rangle+\beta|1\rangle$?

Consider two qubits:

$$|a\rangle = \alpha|0\rangle+\beta|1\rangle$$
$$|b\rangle = |0\rangle$$

We want to go from $(\alpha|0\rangle+\beta|1\rangle)|0\rangle$

to $(\alpha|0\rangle+\beta|1\rangle)(\alpha|0\rangle+\beta|1\rangle)$

$$\Rightarrow \alpha^2|00\rangle+\alpha\beta|01\rangle+\alpha\beta|10\rangle+\beta^2|11\rangle \quad (*)$$

Classically, this is no problem.

For ex.: CNOT: $|x\rangle|y\rangle \longrightarrow |x\rangle|y\oplus x\rangle$

$$|00\rangle \rightsquigarrow |00\rangle$$
$$|10\rangle \rightsquigarrow |11\rangle$$

CNOT is unitary. So, why not for quantum?

Let's try

$$\text{CNOT}\,(\alpha|0\rangle+\beta|1\rangle)|0\rangle \longrightarrow \underbrace{\alpha|00\rangle+\beta|11\rangle}_{\text{entangled state}}$$

different from (*) apart from $\alpha\beta=0$

which is the classical bit

No-Cloning Theorem: for any quantum state $|\psi\rangle\otimes|s\rangle$ there is no unitary transformation that would lead to $|\psi\rangle\otimes|\psi\rangle$

Proof: suppose there would be a transformation

$$|\psi\rangle\otimes|s\rangle \to |\psi\rangle\otimes|\psi\rangle$$

and $|\phi\rangle\otimes|s\rangle \to |\phi\rangle\otimes|\phi\rangle$

the scalar product should be identical because a unitary transformation conserves the angles: $\langle Uv|Uv\rangle = \langle v|v\rangle$

$$\langle\psi\otimes s|\phi\otimes s\rangle = \langle\psi\otimes\psi|\phi\otimes\phi\rangle$$

for vectors of the same length

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}=\begin{pmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{pmatrix}$$

$$\langle\psi|\phi\rangle\underbrace{\langle s|s\rangle}_{=1} = \langle\psi|\phi\rangle\langle\psi|\phi\rangle$$

$$\Rightarrow \langle\psi|\phi\rangle = \langle\psi|\phi\rangle^2$$

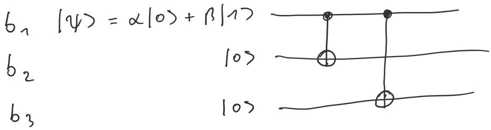
$\Rightarrow$ only for 0 or 1

$\rightsquigarrow$ $\phi=\psi$ or $\phi\perp\psi$ (orthogonal)

$\Rightarrow$ two states can only be copied when they are orthogonal (similar to classical bits)

Error correction

- we want to protect the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ against a bitflip
- the no-cloning-theorem forbits copying the state into save keeping bits
- we can extend the state



register

$$|b_1\rangle|b_2\rangle|b_3\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle = \alpha|000\rangle + \beta|100\rangle$$

appying the two CNOT gates gives

$$\longrightarrow \alpha|000\rangle + \beta|111\rangle$$ (entangled state)

possible bitflip errors

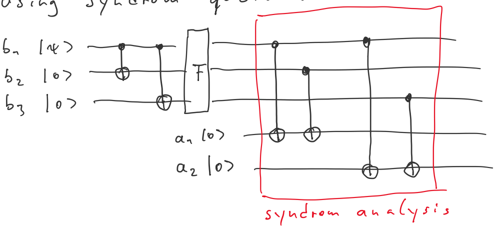
bitflip	register state
none	$\alpha\|000\rangle + \beta\|111\rangle$
b_1	$\alpha\|100\rangle + \beta\|011\rangle$
b_2	$\alpha\|010\rangle + \beta\|101\rangle$
b_3	$\alpha\|001\rangle + \beta\|110\rangle$

we cannot measure the qubits for error correction without distroying the state

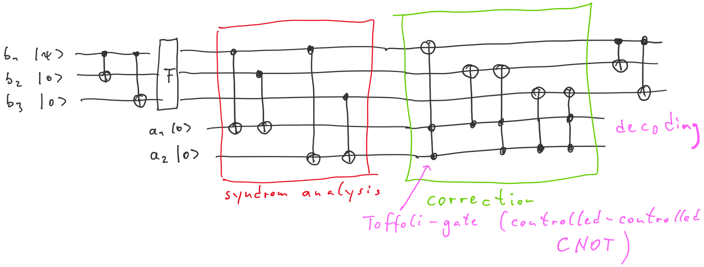
analysis

bitflip	property
none	$b_1 = b_2 = b_3$
b_1	$b_1 \neq b_2 = b_3$
b_2	$b_2 \neq b_1 = b_3$
b_3	$b_3 \neq b_1 = b_2$

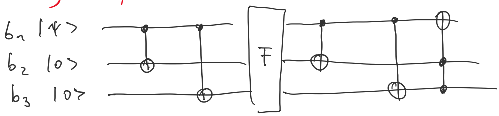
We have to compare the states without measuring in order to perform the correction using syndrom-qubits :



$a_1 a_2$	bitflip	property
0 0	none	$b_1 = b_2 = b_3$
1 0	b_2	$b_2 \neq b_1 = b_3$
0 1	b_3	$b_3 \neq b_1 = b_2$
1 1	b_1	$b_1 \neq b_2 = b_3$



Correcting only the Data bit



2^n-dimensional Hadamard transformation

$$H_n = \bigotimes_{i=1}^{n} H$$

$$H_2 = H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} H & H \\ H & H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

H_n applied on a n-quantum register means to apply H on each qubit

Ex:

$H \otimes H = H_2$ applies H on $|x\rangle$ and $|y\rangle$

$H \otimes I_2$ applies H on $|x\rangle$

$I_2 \otimes H$ applies H on $|y\rangle$

$$H \otimes I_2 = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$I_2 \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Di Vincenzo's criteria (1995)

1) Scalable physical system with well characterized qubits
2) Ability to initialize the quantum register to a simple state
3) Long relevant decoherence times (longer than gate times)
4) A "universal" set of quantum gates
5) A qubit-specific measurement capability

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6) The ability to inter-convert stationary and flying qubits
7) The ability to faithfully trasmit flying qubits between specified locations