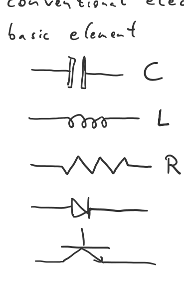
Vorlesung 4

Wednesday, 5. May 2021 13:52

2. Introduction to superconducting quantum circuits

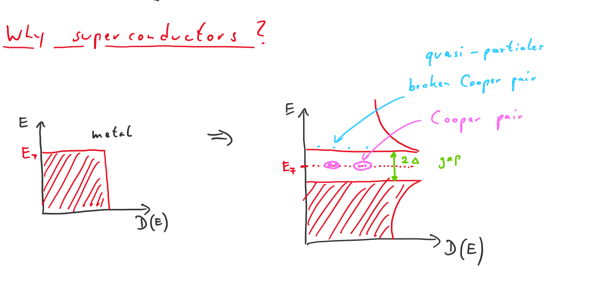
- conventional electronic circuits

basic element



properties: classical physics
no quantum mechanics
no quantization of field

Why superconductors?

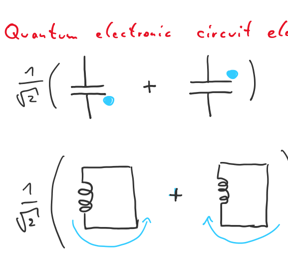


- single non-degenerate marcoscopic ground state
- no low-energy excitations
- $T < T_c$: vanishing internal electric resistance R_{int}

Nb : $2\Delta_s/h = 725\,GHz$, $T_c = 9.2\,K$

Al : $2\Delta_s/h = 100\,GHz$, $T_c = 1.2\,k$

Quantum electronic circuit elements



$\frac{1}{\sqrt{2}}(\ldots + \ldots)$ charge on a capacitor

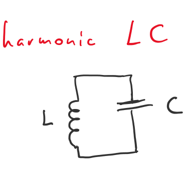
$\frac{1}{\sqrt{2}}(\ldots + \ldots)$ current or magnetic flux in a inductor

- charge Q
- flux ϕ

are conjugated variables

quantum uncertainty relation $\Delta\phi \Delta Q > h$

harmonic LC oscillator



$$\omega = \frac{1}{\sqrt{LC}} \sim 5\,GHz$$

typical inductor: $L = 1\,nH$

wire in vaccum $L \sim 1\,nH/mm$

typical capacitor: $C = 1\,pF$

size $10 \times 10\,\mu m^2$ and dielectric AlO_x $(\varepsilon = 10)$ of $10\,nm$ thickness.

classical physics → quantum mechanics

$$H = \frac{\phi^2}{2L} + \frac{Q^2}{2C} \qquad \hat{H} = \frac{\hat{\phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

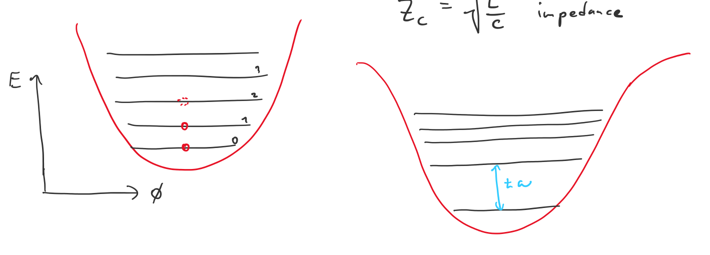
$$[\hat{\phi}, \hat{Q}] = i\hbar$$

$$\hat{Q} = \sqrt{\frac{\hbar}{2Z_c}}\,(\hat{a}^+ + \hat{a}) \qquad \hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{\phi} = i\sqrt{\frac{\hbar Z_c}{2}}\,(\hat{a}^+ - \hat{a}) \qquad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

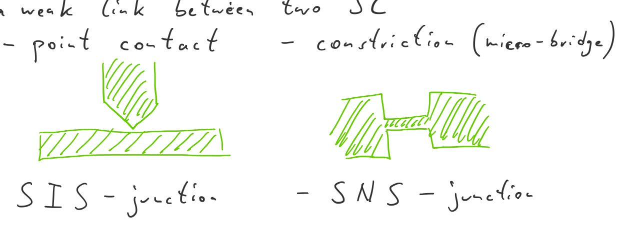
$$\hat{a}^+a\,|n\rangle = n\,|n\rangle$$

$$Z_c = \sqrt{\frac{L}{C}} \quad \text{impedance}$$

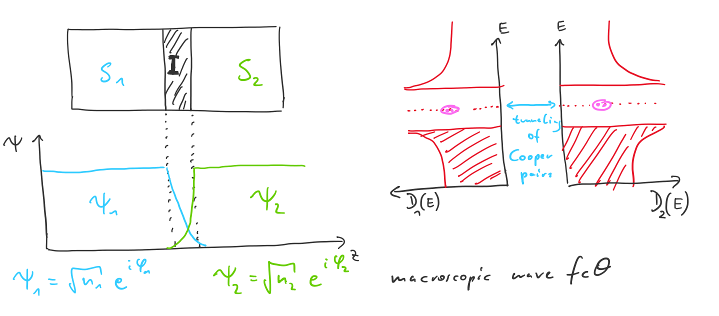


2.1 Josephson effect / junction

- a weak link between two SC
  - point contact
  - constriction (micro-bridge)



- SIS-junction
- SNS-junction



$\Psi_1 = \sqrt{n_1}\,e^{i\varphi_1}$ $\Psi_2 = \sqrt{n_2}\,e^{i\varphi_2}$ macroscopic wave fct

time-dependent Schrödinger eq.

$$\frac{\partial \Psi_{1,2}}{\partial t} = -\frac{i}{\hbar} E_{1,2}\Psi_{1,2}$$

weak coupling between both SC

$$\frac{\partial \Psi_{1,2}}{\partial t} = \frac{-i}{\hbar}\left(E_{1,2}\Psi_{1,2} + K\Psi_{2,1}\right)$$

K: coupling constant

separating the real and imaginary part gives the two Josephson equations

$$① \; I_J = I_c \sin\varphi$$

$$② \; \frac{d}{dt}\varphi = \frac{2\pi}{\phi_0} U$$

$E_1 - E_2 = 2eU$

$n_1 = n_2$

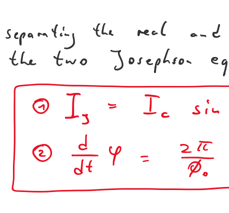
V : Volume

$I_c = \frac{4\pi K}{\phi_0} V n_1$

$\varphi = \varphi_2 - \varphi_1$

$\phi_0 = h/2e$

- for $\varphi = 0 \rightsquigarrow I_J = 0$



- for $U = const$ ⇒ phase difference increases linearly with time

$$\varphi_2 - \varphi_1 = \frac{2\pi}{\phi_0} U t + \varphi(t=0)$$

- substitution into the first J eq.

$$I_J = I_c \sin\left(\frac{2\pi}{\phi_0} U t + \varphi_0\right)$$
$$= I_c \sin(2\pi f t + \varphi_0) \qquad f = \frac{1}{\phi_0}U = \frac{2e}{h}U$$

$486{,}6\ MHz/\mu V$

Josephson energy:

$$E_{junction} = \int U I_J\,dt = \int \frac{\phi_0}{2\pi}\dot{\varphi}\, I_c \sin\varphi\,dt$$
$$= -\frac{I_c\phi_0}{2\pi}\cos\varphi$$
$$= -E_J \cos\varphi$$

$$E_J = \frac{I_c\phi_0}{2\pi}$$

classical inductance $U = -L\dot{I}$

(1) $I_J = I_c \sin\varphi \rightsquigarrow \dot{I}_J = I \cos\varphi\,\dot{\varphi}$

(2) $\frac{\partial\varphi}{\partial t} = \dot{\varphi} = \frac{2\pi}{\phi_0}U \rightsquigarrow U = \frac{\phi_0}{2\pi}\dot{\varphi} = \frac{\phi_0}{2\pi I\cos\varphi}\dot{I}$

⤳ non-linear inductance

Josephson energy E_J

$$E_J = \int U I_J\,dt = \int \frac{\phi_0}{2\pi}\dot{\varphi}\, I_c \sin\varphi\,dt = -\frac{I_c\phi_0}{2\pi}\cos\phi$$
$$=: -E_J\cos\phi$$

$$E_J = \frac{I_c\phi_0}{2\pi}$$

- non-linear inductance

(1) $I_J = I_c \sin\varphi \rightsquigarrow \dot{I}_J = I_c\cos\varphi\,\dot{\varphi}$

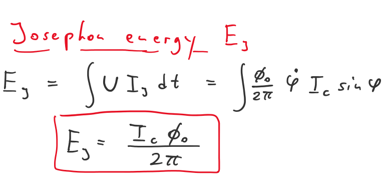
(2) $\frac{d\varphi}{dt} = \frac{2\pi}{\phi_0}U \rightsquigarrow U = \frac{\phi_0}{2\pi}\dot{\varphi} = \frac{\phi_0}{2\pi I_c\cos\varphi}\dot{I}_J$

classical inductance: $U = -L\dot{I}$

$$\rightsquigarrow L_J = -\frac{\phi_0}{2\pi I_c\cos\phi} \rightsquigarrow \text{non-linear inductance}$$

2.2 Resistively capacitively shunted junction model

RCSJ-Model



Kirchhoff's law

$$I = I_J + I_q + I_d$$

I_J : Josephson current

I_q : quasi-particle current (shunt current)

I_d : displacement current

- quasi particle current depends in a complicated way on the voltage U

⤳ approx: $I_q = \frac{U}{R}$

frequently, an artifical parallel resistance is attached „shunted JJ"

$$\Rightarrow I = I_c \sin\gamma + \frac{U}{R} + C\dot{U}$$

$$\Rightarrow I = I_c\sin\gamma + \frac{\phi_0}{2\pi R}\dot{\gamma} + \frac{C\phi_0}{2\pi}\ddot{\gamma} \quad (*)$$

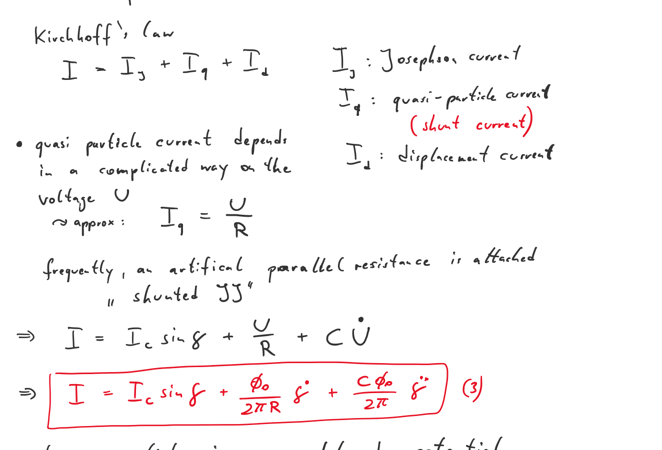
analogy: particle in a washboard potential

multiply (*) with $\frac{\phi_0}{2\pi}$, $E_J = \frac{I_c\phi_0}{2\pi}$ Josephson energy

$$C\left(\frac{\phi_0}{2\pi}\right)^2\ddot{\gamma} + \frac{1}{R}\left(\frac{\phi_0}{2\pi}\right)^2\dot{\gamma} - E_J\left(\frac{I}{I_c} - \sin\gamma\right) = 0$$

tilt position

$$U(\gamma) = -E_J\left(\cos\gamma + \gamma\,\frac{I}{I_c}\right)$$



- depending on the damping, the particle will move very non-uniformly
- at high current ⇒ particle very fast (486 MHz/μV)
- Josephson plasma oscillation at $I = 0$

$$\omega_{p0} = \sqrt{\frac{2\pi I_c}{\phi_0 C}}$$

$$\omega_p(I) = \omega_{p0}\left[1 - \left(\frac{I}{I_c}\right)^2\right]^{1/4}$$

Dimensionless unit: [illegible]

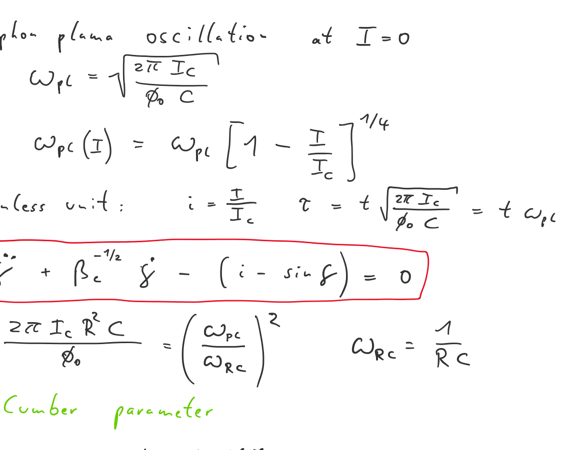
$$\ddot{\gamma} + \beta_c^{-1/2}\,\dot{\gamma} + (\sin\gamma - I/I_c) = 0$$

$$\beta_c = \frac{2\pi I_c R^2 C}{\phi_0} = \left(\frac{\omega_{p0}}{\omega_c}\right)^2 \qquad \omega_c = \frac{1}{RC}$$

Mc Cumber parameter

$\beta_c \gg 1$: underdamped JJ

$\beta_c \ll 1$: overdamped JJ



- for $\beta_c = 0$ $U = 0$ for $|I| < I_c$

$$U = R_n\sqrt{I^2 - I_c^2} \quad \text{for } |I| > I_c$$

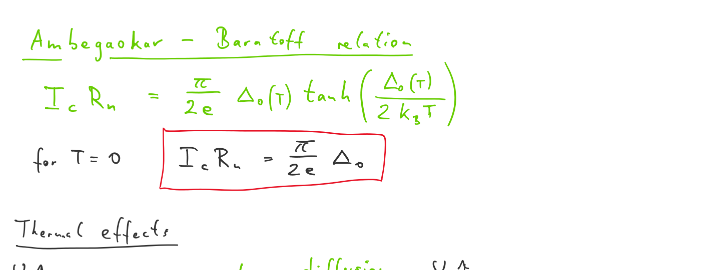
Ambegaokar - Baratoff relation

$$I_c R_n = \frac{\pi}{2e}\Delta_0(T)\tanh\left(\frac{\Delta_0(T)}{2k_BT}\right)$$

for $T = 0$

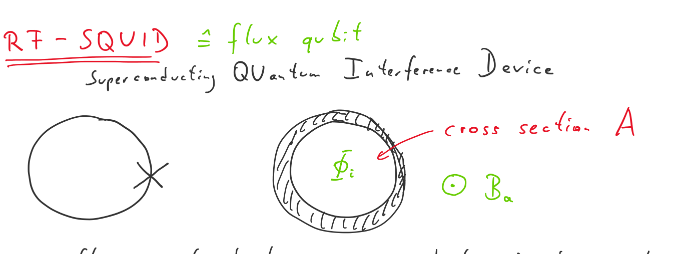
$$I_c R_n = \frac{\pi}{2e}\Delta_0$$

Thermal effects



2.3 RF-SQUID ≙ flux qubit

Superconducting QUantum Interference Device



- mag. flux enclosed by a superconducting ring is quantized

$$\Phi_i = n\phi_0 \quad , n = 0, 1, 2 \ldots$$

- flux generated by the ring

$$\phi_s = L I_s$$

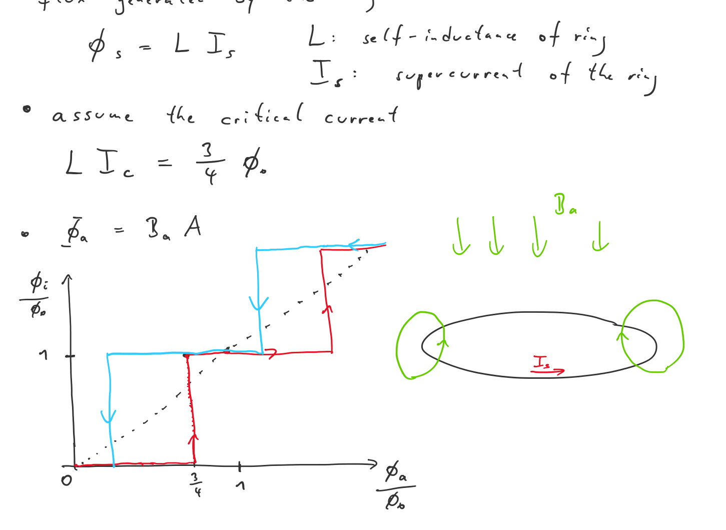
L: self-inductance of ring

I_s : supercurrent of the ring

- assume the critical current

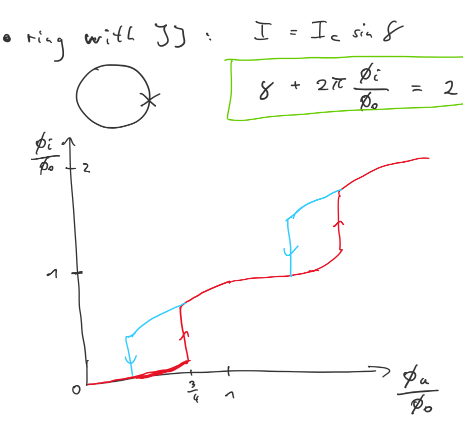
$$L I_c = \frac{3}{4}\phi_0$$

- $\Phi_a = B_a A$



- ring with JJ: $I = I_c \sin\gamma$

$$\gamma + 2\pi\frac{\phi_i}{\phi_0} = 2\pi n$$

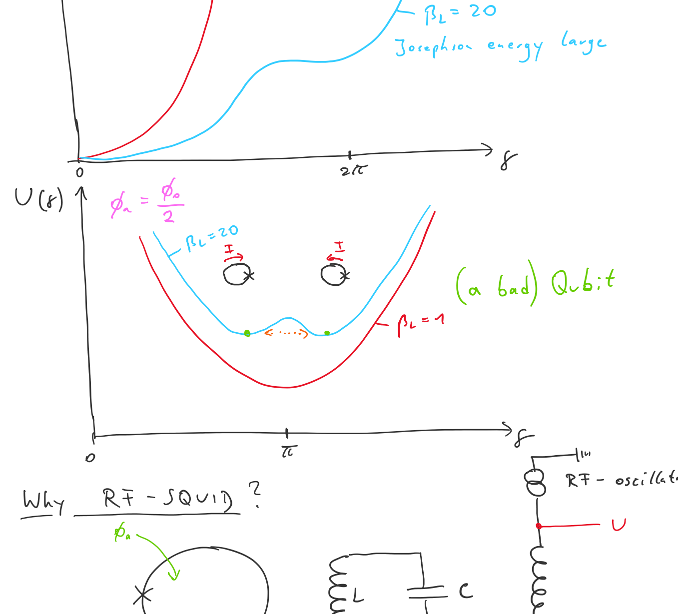


$$\phi_i = \phi_a + LI_s = \phi_a + L I_c \sin\left(2\pi\frac{\phi_i}{\phi_0}\right)$$

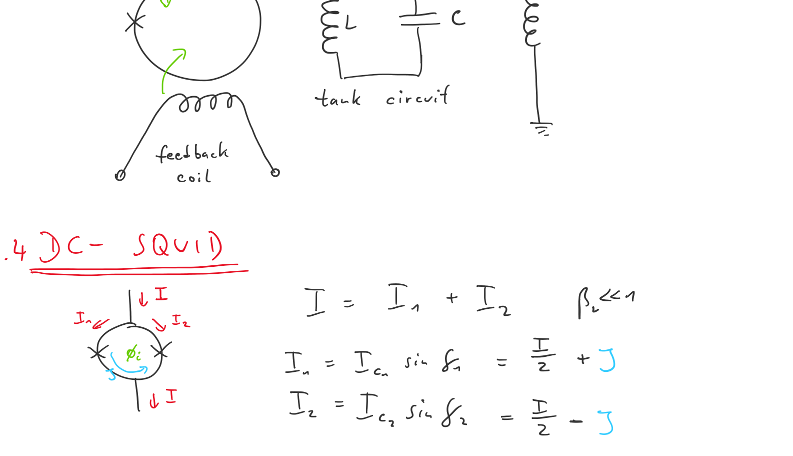
$$U(\gamma) = \underbrace{\frac{1}{2L}\left(\frac{\gamma}{2\pi}\phi_0 - \phi_a\right)^2}_{\text{inductive energy}} - \underbrace{E_J\cos\gamma}_{\text{Josephson energy}}$$

$$U(\gamma) = E_J\left[\frac{1}{2\beta_L}\left(\gamma - 2\pi\frac{\phi_a}{\phi_0}\right)^2 - \cos\gamma\right]$$

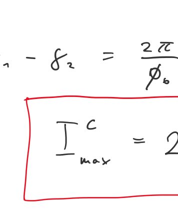
$$\beta_L = \frac{2\pi L I_c}{\phi_0}$$



Why RF-SQUID?



2.4 DC-SQUID



$$I = I_1 + I_2 \qquad \beta_L \ll 1$$

$$I_1 = I_{c_1}\sin\gamma_1 = \frac{I}{2} + J$$

$$I_2 = I_{c_2}\sin\gamma_2 = \frac{I}{2} - J$$

$$\gamma_1 - \gamma_2 = \frac{2\pi}{\phi_0}\phi_i = \frac{2\pi}{\phi}\left(\phi_a + LJ\right)$$

$$I^c_{max} = 2I_c\left|\cos\left(\frac{\pi\phi}{\phi_0}\right)\right|$$