Exercises for "Superconductivity, Josephson ..." SS 2023

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Exercise 1
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1. Free energy of the Abrikosov vortex: (50 Punkte)

In the lecture and the script we have discussed the lower critical field H_{c1} for a type II superconductor with large Ginzburg-Landau parameter, $\kappa \gg 1$. At this field vortices start to penetrate the bulk of the superconductor. In order to obtain H_{c1} one needs the free energy of the vortex line per unit length, $\mathcal{F}_{\text{vortex}} \sim \epsilon L$. We used the following result (in the reduced GL units)

$$\epsilon \sim \frac{2\pi}{\kappa^2} \ln \kappa \ . \tag{1}$$

Here we prove this result using the Ginzburg-Landau free energy functional and the solution of the Ginzburg-Landau equations for a single Abrikosov vortex. Consult if necessary the books by Abrikosov or Tinkham.

1) The free energy functional in the GL reduced units reads

$$\mathcal{F} = \int dV\, F_n + \int dV \left\{ -|\Psi|^2 + \frac{1}{2}|\Psi|^4 + \left| \left(-\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right) \Psi \right|^2 + \vec{B}^2 \right\} \ . \tag{2}$$

Integrate by parts and use the Ginsburg-Landau equation to obtain the contribution of the kinetic energy

$$\mathcal{F}_{\text{kin}} = \int dV \left| \left(-\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right) \Psi \right|^2 = \int dV \ \left\{ |\Psi|^2 - |\Psi|^4 \right\} \ . \tag{3}$$

2) Combine $\mathcal{F}_{\text{kin}}$ with the condensation energy

$$\mathcal{F}_{\text{cond}} = \int dV \left\{ -|\Psi|^2 + \frac{1}{2}|\Psi|^4 \right\} \tag{4}$$

and the field energy

$$\mathcal{F}_{\text{field}} = \int dV\, \vec{B}^2 \ . \tag{5}$$

Take into account he condensation energy without the vortex and prove that

$$\mathcal{F}_{\text{vortex}} = \int dV \left\{ \frac{1}{2}(1 - |\Psi|^4) + \vec{B}^2 \right\} \ . \tag{6}$$

3) Use the solutions obtained in the lecture (script) for Ψ and $\vec{B}$. Distinguish between the contribution of the vertex core $r < 1/\kappa$ and that from the interval $1 > r > 1/\kappa$. Estimate the vortex energy.

2. Pearl vortex **(50 Punkte)**

In the lecture and the script we have considered a single vortex in a thin superconducting film (Pearl vortex).

1) We have introduced the field $\vec{\Phi}(x,y) \equiv \frac{\hbar c}{2e}\vec{\nabla}\phi$, where ϕ is the phase of the superconducting order parameter. For a single vortex we have: $\phi(x,y) = -\varphi = \cos^{-1}(x/\sqrt{x^2+y^2})$. Here φ is the angle in the polar coordinates. Prove the following relations

$$\vec{\Phi} = -\frac{\Phi_0}{2\pi r}\vec{e}_\varphi \ . \tag{7}$$

$$\vec{\nabla} \times \vec{\Phi} = -\Phi_0 \delta(x)\delta(y)\vec{e}_z \ . \tag{8}$$

Show that in the Fourier representation this relation reads

$$\vec{\Phi}(\vec{q}) = -i\Phi_0[\vec{q} \times \vec{e}_z]/q^2 \ . \tag{9}$$

2) In the lecture (script) we obtained for the current density in the Fourier representation

$$\vec{J}(\vec{q}) = -\frac{c}{4\pi\Lambda}\vec{\Phi}(\vec{q})\,\frac{2\Lambda q}{1+2\Lambda q} \ , \tag{10}$$

Analyse the current density in the coordinate representation $\vec{J}(x,y)$ (or in polar coordinates $\vec{J}(r,\varphi)$ in two limits: a) $r \ll \Lambda$; b) $r \gg \Lambda$.