Exercises for "Superconductivity, Josephson ..." SS 2023

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Exercise 1: Solution
Tutorial on 13.10.2023

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1. **Free energy of the Abrikosov vortex: (50 Punkte)**
In the lecture and the script we have discussed the lower critical field H_{c1} for a type II superconductor with large Ginzburg-Landau parameter, $\kappa \gg 1$. At this field vortices start to penetrate the bulk of the superconductor. In order to obtain H_{c1} one needs the free energy of the vortex line per unit length, $\mathcal{F}_{\text{vortex}} \sim \epsilon L$. We used the following result (in the reduced GL units)

$$\epsilon \sim \frac{2\pi}{\kappa^2} \ln \kappa \ . \tag{1}$$

Here we prove this result using the Ginzburg-Landau free energy functional and the solution of the Ginzburg-Landau equations for a single Abrikosov vortex. Consult if necessary the books by Abrikosov or Tinkham.

1) The free energy functional in the GL reduced units reads

$$\mathcal{F} = \int dV\, F_n + \int dV \left\{ -|\Psi|^2 + \frac{1}{2}|\Psi|^4 + \left| \left(-\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right) \Psi \right|^2 + \vec{B}^2 \right\} \ . \tag{2}$$

Integrate by parts and use the Ginsburg-Landau equation to obtain the contribution of the kinetic energy

$$\mathcal{F}_{\text{kin}} = \int dV \left| \left(-\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right) \Psi \right|^2 = \int dV \ \left\{ |\Psi|^2 - |\Psi|^4 \right\} \ . \tag{3}$$

The Ginsburg-Landau equation reads

$$\left(-i\kappa^{-1}\vec{\nabla} + \vec{A} \right)^2 \Psi - \Psi + |\Psi|^2\Psi = 0 \ , \tag{4}$$

This gives

$$\begin{aligned} \mathcal{F}_{\text{kin}} &= \int dV \left| \left(-\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right) \Psi \right|^2 = \int dV \left[\left(-\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right) \Psi \right] \left[\left(\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right) \Psi^* \right] \\ &= \int dV \left[\left(-\frac{i\vec{\nabla}}{\kappa} + \vec{A} \right)^2 \Psi \right] [\Psi^*] = \int dV \ \left\{ |\Psi|^2 - |\Psi|^4 \right\} \ . \end{aligned} \tag{5}$$

2) Combine $\mathcal{F}_{\text{kin}}$ with the condensation energy

$$\mathcal{F}_{\text{cond}} = \int dV \left\{ -|\Psi|^2 + \frac{1}{2}|\Psi|^4 \right\} \tag{6}$$

and the field energy

$$\mathcal{F}_{\text{field}} = \int dV\, \vec{B}^2 \ . \tag{7}$$

Take into account he condensation energy without the vortex and prove that

$$\mathcal{F}_{\text{vortex}} = \int dV \left\{ \frac{1}{2}(1 - |\Psi|^4) + \vec{B}^2 \right\} \ . \tag{8}$$

The condensation energy with no vortex reads

$$\mathcal{F}_{\text{cond},0} = \int dV \left\{ -\frac{1}{2} \right\} \ . \tag{9}$$

The vortex energy is the difference of $\mathcal{F}$ with a vortex and $\mathcal{F}$ without a vortex, i.e., with $\vec{B} = 0$ and $|\Psi| = 1$. Thus

$$\mathcal{F}_{\text{vortex}} = [\mathcal{F}_{\text{cond}} - \mathcal{F}_{\text{cond},0}] + \mathcal{F}_{\text{kin}} + \mathcal{F}_{\text{field}} \ . \tag{10}$$

Combining all together we obtain

$$\mathcal{F}_{\text{vortex}} = \int dV \left\{ \frac{1}{2}(1 - |\Psi|^4) + \vec{B}^2 \right\} \ . \tag{11}$$

3) Use the solutions obtained in the lecture (script) for Ψ and $\vec{B}$. Distinguish between the contribution of the vertex core $r < 1/\kappa$ and that from the interval $1 > r > 1/\kappa$. Estimate the vortex energy.

For $1 > r > 1/\kappa$ we obtained in the lecture $|\Psi|^2 = f^2 \approx 1 - (\kappa r)^{-2}$. Then $|\Psi|^4 \approx 1 - 2(\kappa r)^{-2} + (\kappa r)^{-4}$ and $(1/2)(1 - |\Psi|^4) \approx (\kappa r)^{-2} - (1/2)(\kappa r)^{-4}$. The integration gives (with logarithmic accuracy, i.e., 1 is neglected in comparison with $\ln \kappa$)

$$\int\limits_{1/\kappa}^{1} 2\pi r dr\, \frac{1}{2}(1 - |\Psi|^4) \approx \frac{2\pi}{\kappa^2}\, \ln \kappa \ . \tag{12}$$

For the magnetic field at $1 > r > 1/\kappa$ we use (see script)

$$B(r) \approx \frac{1}{\kappa}\, \ln(1/r) \ , \tag{13}$$

It is easy to see that

$$\int\limits_{1/\kappa}^{\infty} 2\pi r dr\, B^2(r) \tag{14}$$

is not logarithmically diverging at $r \to 0$. Thus the field energy contribution is subleading and can be neglected with logarithmic accuracy.

Let us now estimate the core energy. For the estimate let's take $|\Psi| \approx \kappa r$ for $0 < r < 1/\kappa$. Then

$$\int\limits_{0}^{1/\kappa} 2\pi r dr\, \frac{1}{2}(1 - |\Psi|^4) \approx \frac{\pi}{3\kappa^2} \ . \tag{15}$$

There is no $\ln\kappa$, thus, this can be neglected. For the field part we can take for the estimate $B(r < 1/\kappa) \approx B(1/\kappa) \approx (1/\kappa)\ln\kappa$. This gives

$$\int_0^{1/\kappa} 2\pi r dr \, \ln^2\kappa/\kappa^2 \sim \kappa^{-4}\ln^2\kappa \; . \tag{16}$$

This can definitely be neglected. Thus, the core energy in negligible with logarithmic accuracy.

2. **Pearl vortex** **(50 Punkte)**

   In the lecture and the script we have considered a single vortex in a thin superconducting film (Pearl vortex).

   1) We have introduced the field $\vec{\Phi}(x,y) \equiv \frac{\hbar c}{2e}\vec{\nabla}\phi$, where ϕ is the phase of the superconducting order parameter. For a single vortex we have: $\phi(x,y) = -\varphi = \cos^{-1}(x/\sqrt{x^2+y^2})$. Here φ is the angle in the polar coordinates. Prove the following relations

$$\vec{\Phi} = -\frac{\Phi_0}{2\pi r}\vec{e}_\varphi \; . \tag{17}$$

$$\vec{\nabla} \times \vec{\Phi} = -\Phi_0 \delta(x)\delta(y)\vec{e}_z \; . \tag{18}$$

   Show that in the Fourier representation this relation reads

$$\vec{\Phi}(\vec{q}) = -i\Phi_0[\vec{q} \times \vec{e}_z]/q^2 \; . \tag{19}$$

First of all we remind ourselves that

$$\vec{\nabla} f(r,\varphi) = \frac{\partial}{\partial r} f(r,\varphi)\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\varphi} f(r,\varphi)\vec{e}_\varphi, \tag{20}$$

for some function $f(r,\varphi)$. It follows that

$$\vec{\Phi}(x,y) \equiv \frac{\hbar c}{2e}\vec{\nabla}\phi = -\underbrace{\frac{hc}{2e}}_{=\Phi_0}\frac{1}{2\pi r}\vec{e}_\varphi \tag{21}$$

Now we remind ourselves that

$$\vec{\nabla} \times \vec{A}(r) = -\vec{e}_\varphi \frac{\partial A_z(r)}{\partial r} + \vec{e}_z\left(\frac{\partial A_\varphi(r)}{\partial r} + \frac{A_\varphi(r)}{r}\right), \tag{22}$$

such that for $r \neq 0$ we find

$$\vec{\nabla} \times \vec{\Phi}(x,y) = -\frac{\Phi_0}{2\pi}\vec{e}_z\left(-\frac{1}{r^2} + \frac{1}{r^2}\right) = 0, \tag{23}$$

as one would expect for $\vec{\Phi}(x,y) \propto \nabla\phi$.

For $r \to 0$ our vector field features a singularity, with its rotor displaying the δ-function behaviour. To see that, we consider

$$\int_{|\vec{r}|\le\epsilon} d^{(2)}\vec{r}\left(\vec{\nabla} \times \vec{\Phi}(\vec{r})\right)\cdot\vec{e}_z = \oint_{|\vec{r}|=\epsilon} d\vec{r}\cdot\vec{\Phi}(x,y) = -\int_0^{2\pi} d\varphi\frac{\Phi_0}{2\pi} = -\Phi_0, \quad \forall\epsilon \in \mathbb{R}^+. \tag{24}$$

We see that the function $\vec{\nabla} \times \vec{\Phi}(x, y)$ possesses the properties:

$$\vec{\nabla} \times \vec{\Phi}(\vec{r}) = 0\vec{e}_z, \quad \vec{r} \neq \vec{0}, \tag{25}$$

$$\int_{\mathbb{R}^2} d^{(2)}\vec{r} \left(\vec{\nabla} \times \vec{\Phi}(\vec{r})\right) \cdot \vec{e}_z = -\Phi_0, \tag{26}$$

forcing us to conclude that

$$\vec{\nabla} \times \vec{\Phi}(x, y) = -\Phi_0 \delta^{(2)}(\vec{r})\vec{e}_z = -\Phi_0 \delta(x)\delta(y)\vec{e}_z. \tag{27}$$

In Fourier representation this equation reads

$$i\vec{q} \times \vec{\Phi}(\vec{q}) = -\Phi_0 \vec{e}_z, \tag{28}$$

whose solution is given by

$$\vec{\Phi}(\vec{q}) = -i\Phi_0 \frac{\vec{q} \times \vec{e}_z}{q^2}, \tag{29}$$

as is seen by the application of the vector identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}). \tag{30}$$

2) In the lecture (script) we obtained for the current density in the Fourier representation

$$\vec{J}(\vec{q}) = -\frac{c}{4\pi\Lambda} \vec{\Phi}(\vec{q}) \frac{2\Lambda q}{1 + 2\Lambda q} \;, \tag{31}$$

Analyse the current density in the coordinate representation $\vec{J}(x, y)$ (or in polar coordinates $\vec{J}(r, \varphi)$ in two limits: a) $r \ll \Lambda$; b) $r \gg \Lambda$.

We now calculate the current distribution. In the lecture (script) we got

$$\vec{J}(\vec{q}) = -\frac{c}{4\pi\Lambda} \vec{\Phi}(\vec{q}) \frac{2\Lambda q}{1 + 2\Lambda q} \;, \tag{32}$$

where

$$\vec{\Phi}(\vec{q}) = -i\Phi_0 [\vec{q} \times \vec{e}_z]/q^2 \;. \tag{33}$$

This gives

$$\vec{J}(\vec{r}) = -\frac{c}{4\pi\Lambda} \int \frac{dk_x dk_y}{4\pi^2} \, e^{i\vec{q}\vec{r}} \, \vec{\Phi}(\vec{q}) \, \frac{2\Lambda q}{1 + 2\Lambda q} \;. \tag{34}$$

Assume in polar coordinates $\vec{r} = r(\cos\varphi, \sin\varphi)$ and $\vec{q} = q(\cos\theta, \sin\theta)$. Then $\vec{q}\vec{r} = qr\cos(\varphi - \theta)$ and

$$\vec{\Phi}(\vec{q}) = -i\Phi_0 \, \frac{1}{q} (\sin\theta, -\cos\theta) \;. \tag{35}$$

This gives

$$\vec{J}(r, \varphi) = \frac{i\Phi_0 c}{4\pi(4\pi^2)\Lambda} \int_0^\infty dq \int_0^{2\pi} d\vartheta e^{iqr\cos(\vartheta - \varphi)} \frac{2\Lambda q}{1 + 2\Lambda q} (\sin\vartheta, \; -\cos\vartheta) \tag{36}$$

$$= -\frac{\Phi_0 c}{4(4\pi^2)\Lambda^2} (\sin\varphi, \; -\cos\varphi) \int_0^\infty d\bar{q} \frac{\bar{q}}{1 + \bar{q}} \int_0^{2\pi} \frac{d\vartheta}{2\pi i} \cos\vartheta e^{i\bar{q}\frac{r}{2\Lambda}\cos\vartheta}. \tag{37}$$

Further we recognise the polar integral as the Bessel function of the first kind

$$J_1\left(\frac{\bar{q}r}{2\Lambda}\right) = \int_0^{2\pi} \frac{d\vartheta}{2\pi i} \cos\vartheta e^{i\bar{q}\frac{r}{2\Lambda}\cos\vartheta}, \tag{38}$$

such that

$$\vec{J}(r,\varphi) = -\frac{\Phi_0 c}{4(4\pi^2)\Lambda^2}(\sin\varphi,\ -\cos\varphi)\int_0^\infty d\bar{q}\frac{\bar{q}}{1+\bar{q}}J_1\left(\frac{\bar{q}r}{2\Lambda}\right) \tag{39}$$

$$= -\frac{\Phi_0 c}{4(4\pi^2)\Lambda^2}(\sin\varphi,\ -\cos\varphi)\frac{2\Lambda}{r}\int_0^\infty d\zeta\frac{\frac{2\Lambda}{r}\zeta}{1+\frac{2\Lambda}{r}\zeta}J_1(\zeta). \tag{40}$$

For $\frac{2\Lambda}{r} \gg 1$ we can write

$$\vec{J}(r,\varphi) \sim -(\sin\varphi,\ -\cos\varphi)\frac{\Phi_0 c}{2(2\pi)^2\Lambda}\frac{1}{r}\underbrace{\int_0^\infty d\zeta J_1(\zeta)}_{=1}. \tag{41}$$

Exploiting the fact that

$$\frac{\bar{q}}{2\Lambda}J_1\left(\frac{\bar{q}r}{2\Lambda}\right) = -\frac{\partial}{\partial r}J_0\left(\frac{\bar{q}r}{2\Lambda}\right), \tag{42}$$

we find

$$\vec{J}(r,\varphi) = \frac{\Phi_0 c}{2(4\pi^2)\Lambda}(\sin\varphi,\ -\cos\varphi)\frac{\partial}{\partial r}\int_0^\infty d\bar{q}\frac{J_0\left(\frac{\bar{q}r}{2\Lambda}\right)}{1+\bar{q}} \tag{43}$$

$$= \frac{\Phi_0 c}{2(4\pi^2)\Lambda}(\sin\varphi,\ -\cos\varphi)\frac{\partial}{\partial r}\frac{2\Lambda}{r}\int_0^\infty d\zeta\frac{J_0(\zeta)}{1+\frac{2\Lambda}{r}\zeta}. \tag{44}$$

It figures that

$$\vec{J}(r,\varphi) \sim -(\sin\varphi,\ -\cos\varphi)\frac{\Phi_0 c}{(2\pi)^2 r^2}\underbrace{\int_0^\infty d\zeta J_0(\zeta)}_{=1},\qquad \frac{2\Lambda}{r} \to 0. \tag{45}$$