Exercises for "Superconductivity, Josephson ..." WS 2023/2024

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Exercise 2

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1. **Expectation value of operators in the BCS ground state. (30 Points)**

Following the dream team of Bardeen, Cooper, and Schrieffer, in the class we considered the following family of states as our variational ansatz for the ground state of the BCS Hamiltonian:

$$|\mathrm{BCS}\rangle = \prod_k \left(u_k + v_k c^\dagger_{k,\,\uparrow} c^\dagger_{-k,\,\downarrow}\right)|0\rangle\,. \tag{1}$$

Here $|0\rangle$ is the Fermionic vacuum state $c_{k,\sigma}|0\rangle = 0$, $c^\dagger_{k,\sigma}$ and $c_{k,\sigma}$ are the Fermionic creation and annihilation operators obeying the standard canonical commutation relations

$$\{c_{k,\sigma},\ c_{k',\sigma'}\} = \{c^\dagger_{k,\sigma},\ c^\dagger_{k',\sigma'}\} = 0, \quad \{c_{k,\sigma},\ c^\dagger_{k',\sigma'}\} = \delta_{k,k'}\delta_{\sigma,\sigma'}. \tag{2}$$

Prove the following result

$$\left\langle \mathrm{BCS}\,\middle|\, c^\dagger_{k',\uparrow}c^\dagger_{-k',\downarrow}c_{-k,\downarrow}c_{k,\uparrow}\,\middle|\,\mathrm{BCS}\right\rangle = u_k v_k u_{k'} v_{k'}. \tag{3}$$

2. **Generalized Cooper problem. (35 Points)**

In the lecture we have considered the Cooper problem, in which two electrons are created on top of the full Fermi sea. Consider now a generalized Cooper problem. In this case the excited state is assumed to be a superposition of either of two electrons slightly above the Fermi level or of two holes slightly below the Fermi level (the total number of particles is, thus, not sharply defined). That is consider an ansatz of the form

$$|\Psi\rangle = \sum_{-\hbar\omega_D<\xi_k<0} \alpha(k)\chi(\sigma_1,\sigma_2)c_{-k,\sigma_2}c_{k,\sigma_1}|\Psi_0\rangle + \sum_{0<\xi_k<\hbar\omega_D} \alpha(k)\chi(\sigma_1,\sigma_2)c^\dagger_{k,\sigma_1}c^\dagger_{-k,\sigma_2}|\Psi_0\rangle \tag{4}$$

where $|\Psi_0\rangle = \prod_{k\le k_F} c^\dagger_{k,\sigma}|0\rangle$ stands for the fully occupied Fermi sea, and $\xi_k = \epsilon_k - \mu$ (note that in the grand-canonical description the energies of both electrons and holes are positive). Find the bound state wave function and the binding energy per particle Δ. Compare with Δ of the Cooper problem and with Δ of the BCS theory.

3. **Better intuition behind the Bogulubov buisness. (35 Points)**

Part1: Constructive transformation of the field operators

Consider the following toy Hamiltonian

$$H = \epsilon(c^\dagger_\uparrow c_\uparrow + c^\dagger_\downarrow c_\downarrow) + V(c^\dagger_\uparrow c^\dagger_\downarrow + c_\downarrow c_\uparrow), \tag{5}$$

with the single mode Fermionic field operators

$$\{c_\sigma,\ c_{\sigma'}\} = \{c^\dagger_\sigma,\ c^\dagger_{\sigma'}\} = 0, \quad \{c_\sigma,\ c^\dagger_{\sigma'}\} = \delta_{\sigma,\sigma'}. \tag{6}$$

Defining the following objects

$$J_+ = c_\uparrow^\dagger c_\downarrow^\dagger, \quad J_- = c_\downarrow c_\uparrow, \quad J_z = \frac{1}{2}\left(N - 1\right), \quad N = n_\uparrow + n_\downarrow, \quad n_\sigma = c_\sigma^\dagger c_\sigma, \tag{7}$$

demonstrate the following relations hold:

$$[J_+,\ J_-] = 2J_z, \quad [J_\pm,\ J_z] = \mp J_\pm. \tag{8}$$

In other words, we may assert that the operators $J_x = \frac{1}{2}(J_+ + J_-)$, $J_y = \frac{1}{2i}(J_+ - J_-)$, and J_z form a representation of the $\mathfrak{su}(2)$ Lie algebra (note that the representation is reducible since the parity operator $e^{i\pi N}$ is conserved).

Using our definitions (7) we write the Hamiltonian in the language of Js:

$$H = 2(\epsilon J_z + V J_x) + \epsilon = 2\sqrt{\epsilon^2 + V^2}(\cos\vartheta J_z + \sin\vartheta J_x) + \epsilon, \tag{9}$$

$$\cos\vartheta = \frac{\epsilon}{\sqrt{\epsilon^2 + V^2}}, \quad \sin\vartheta = \frac{V}{\sqrt{\epsilon^2 + V^2}}. \tag{10}$$

Show that the Hamiltonian (9) is diagonalised by the following unitary transformation $U = e^{i\vartheta J_y}$, that is

$$H_d = UHU^\dagger = 2\sqrt{\epsilon^2 + V^2} J_z + \epsilon. \tag{11}$$

Next we define the Bogolubov operators

$$\gamma_\sigma = U^\dagger c_\sigma U, \quad \gamma_\sigma^\dagger = U^\dagger c_\sigma^\dagger U \equiv (\gamma_\sigma)^\dagger. \tag{12}$$

These guys are helpful since by virtue of Eq. (11) we have

$$H = U^\dagger H_d U = 2\sqrt{\epsilon^2 + V^2} U^\dagger J_z U + \epsilon = \sqrt{\epsilon^2 + V^2}(\gamma_\uparrow^\dagger \gamma_\uparrow + \gamma_\downarrow^\dagger \gamma_\downarrow) + \epsilon - \sqrt{\epsilon^2 + V^2}. \tag{13}$$

Using BCH formula, or otherwise, show that

$$\gamma_\uparrow = \cos\frac{\vartheta}{2} c_\uparrow + \sin\frac{\vartheta}{2} c_\downarrow^\dagger, \quad \gamma_\downarrow = \cos\frac{\vartheta}{2} c_\downarrow - \sin\frac{\vartheta}{2} c_\uparrow^\dagger. \tag{14}$$

Part2: BCS ground state

The ground state is defined via the following condition

$$\gamma_\sigma \left|0\right\rangle_\gamma = 0. \tag{15}$$

Show that this implies the following relation

$$U \left|0\right\rangle_\gamma = \left|0\right\rangle, \tag{16}$$

where

$$c_\sigma \left|0\right\rangle = 0, \tag{17}$$

is the vacuum state of original Fermions.

Consider the following representation

$$U^{\dagger} = e^{-i\vartheta J_y} = e^{a_+ J_+} e^{a_z J_z} e^{a_- J_-}, \tag{18}$$

where $a_{\pm},\ a_z$ are functions of the angle ϑ. By differentiating the Eq. (18) with respect to ϑ, establish the following system of non-linear differential equations

$$\frac{d}{d\vartheta} a_+ = -\frac{1}{2} e^{a_z}, \quad \frac{d}{d\vartheta} a_z = a_-, \quad \frac{d}{d\vartheta} a_- = \frac{1}{2}(1 + a_-^2), \tag{19}$$

$$a_+(0) = a_-(0) = a_z(0) = 0. \tag{20}$$

Show that these are solved by

$$a_{\pm}(\vartheta) = \mp \tan\frac{\vartheta}{2}, \quad a_z(\vartheta) = -2 \log \cos\frac{\vartheta}{2}. \tag{21}$$

Using the equation (16) along with the decomposition (18) and the result (21), prove that

$$|0\rangle_{\gamma} = (1 + a_+ J_+) e^{-\frac{1}{2} a_z} |0\rangle \equiv \left(\cos\frac{\vartheta}{2} - \sin\frac{\vartheta}{2} c_{\uparrow}^{\dagger} c_{\downarrow}^{\dagger} \right) |0\rangle \,. \tag{22}$$