Exercises for "Superconductivity, Josephson ..." WS 2023/2024

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Solutions of the Exercise 3
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1. Finite temperature Josephson current in a tunnel junction (30 Points)

In the lecture you have derived an expression for the Josephson's current in a tunnel junction at the absolute zero temperature $T = 0$. Repeat the derivation for the case of finite $T \neq 0$.
NB: You are not supposed to evaluate the resulting momentum integrals.

We proceed in the manner analogous to the lecture, albeit now we replace the zero-temperature averages with finite temperature ones. In particular, in the BCS theory we have

$$
\begin{aligned}
c_{k,\sigma}(t) =& u_k e^{-iE_k t}\alpha_{k,\sigma} + \sigma v_k e^{iE_k t}\alpha^\dagger_{-k,-\sigma}, &(1)\\
\langle \alpha^\dagger_{k,\sigma}\alpha_{k',\sigma'}\rangle =& \delta_{k,k'}\delta_{\sigma,\sigma'}n_F(E_k). &(2)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\langle c_{k,\sigma}(t)c_{k',\sigma'}(t')\rangle =& \sigma u_k v_k \delta_{k,-k'}\delta_{\sigma,-\sigma'}\left(e^{iE_k(t-t')}n_F(E_k) - e^{-iE_k(t-t')}(1-n_F(E_k))\right), &(3)\\
\langle c^\dagger_{k,\sigma}(t)c^\dagger_{k',\sigma'}(t')\rangle =& \sigma u_k v_k \delta_{k,-k'}\delta_{\sigma,-\sigma'}\left(e^{-iE_k(t-t')}(1-n_F(E_k)) - e^{iE_k(t-t')}n_F(E_k)\right). &(4)
\end{aligned}
$$

Such that

$$
\begin{aligned}
&\langle R^\dagger_{k_1,\sigma}(t)R^\dagger_{-k_1,-\sigma}(t')\rangle \langle L_{k_2,\sigma}(t)L_{-k_2,-\sigma}(t')\rangle - \langle R^\dagger_{k_1,\sigma}(t')R^\dagger_{-k_1,-\sigma}(t)\rangle \langle L_{k_2,\sigma}(t')L_{-k_2,-\sigma}(t)\rangle\\
&= u_{k_1}v_{k_1}u_{k_2}v_{k_2}\left(e^{-iE_{k_1}(t-t')}(1-n_F(E_{k_1})) - e^{iE_{k_1}(t-t')}n_F(E_{k_1})\right)\\
&\times\left(e^{iE_{k_2}(t-t')}n_F(E_{k_2}) - e^{-iE_{k_2}(t-t')}(1-n_F(E_{k_2}))\right)\\
&- u_{k_1}v_{k_1}u_{k_2}v_{k_2}\left(e^{iE_{k_1}(t-t')}(1-n_F(E_{k_1})) - e^{-iE_{k_1}(t-t')}n_F(E_{k_1})\right)\\
&\times\left(e^{-iE_{k_2}(t-t')}n_F(E_{k_2}) - e^{iE_{k_2}(t-t')}(1-n_F(E_{k_2}))\right)\\
&= -2iu_{k_1}v_{k_1}u_{k_2}v_{k_2}[(2n_F(E_{k_1})-1)\cos(E_{k_1}[t-t'])\sin(E_{k_2}[t-t'])\\
&+ (2n_F(E_{k_2})-1)\cos(E_{k_2}[t-t'])\sin(E_{k_1}[t-t'])]. \qquad (5)
\end{aligned}
$$

Likewise we obtain

$$
\begin{aligned}
&\langle L^\dagger_{k_2,\sigma}(t)L^\dagger_{-k_2,-\sigma}(t')\rangle \langle R_{k_1,\sigma}(t)R_{-k_1,-\sigma}(t')\rangle - \langle L^\dagger_{k_2,\sigma}(t')L^\dagger_{-k_2,-\sigma}(t)\rangle \langle R_{k_1,\sigma}(t')R_{-k_1,-\sigma}(t)\rangle\\
&= -2iu_{k_1}v_{k_1}u_{k_2}v_{k_2}[(2n_F(E_{k_1})-1)\cos(E_{k_1}[t-t'])\sin(E_{k_2}[t-t'])\\
&+ (2n_F(E_{k_2})-1)\cos(E_{k_2}[t-t'])\sin(E_{k_1}[t-t'])]. \qquad (6)
\end{aligned}
$$

It follows that

$$\begin{aligned}\langle[I(t),H_T(t')]\rangle_0 =&\ldots-\frac{ie}{\hbar}T^2e^{-i\varphi}\sum_{k_1,k_2,\sigma}\bigg(\langle R^\dagger_{k_1,\sigma}(t)R^\dagger_{-k_1,-\sigma}(t')\rangle\,\langle L_{k_2,\sigma}(t)L_{-k_2,-\sigma}(t')\rangle\\
&-\langle R^\dagger_{k_1,\sigma}(t')R^\dagger_{-k_1,-\sigma}(t)\rangle\,\langle L_{k_2,\sigma}(t')L_{-k_2,-\sigma}(t)\rangle\bigg)\\
&+\frac{ie}{\hbar}T^2e^{i\varphi}\sum_{k_1,k_2,\sigma}\bigg(\langle L^\dagger_{k_2,\sigma}(t)L^\dagger_{-k_2,-\sigma}(t')\rangle\,\langle R_{k_1,\sigma}(t)R_{-k_1,-\sigma}(t')\rangle\\
&-\langle L^\dagger_{k_2,\sigma}(t')L^\dagger_{-k_2,-\sigma}(t)\rangle\,\langle R_{k_1,\sigma}(t')R_{-k_1,-\sigma}(t)\rangle\bigg)\\
=&-\frac{16ie}{\hbar}T^2\sin(\varphi)\sum_{k_1,k_2}u_{k_1}v_{k_1}u_{k_2}v_{k_2}\cos(E_{k_1}[t-t'])\sin(E_{k_2}[t-t'])\\
&\times\tanh\left(\frac{\beta E_{k_1}}{2}\right).\end{aligned}\tag{7}$$

Here the dots stand for the uninteresting φ-independent terms, and we used the identity

$$2n_F(\omega)-1=-\tanh\left(\frac{\beta\omega}{2}\right).\tag{8}$$

We hence find

$$\begin{aligned}I(t)=&-i\int_{-\infty}^{t}dt'\,\langle[I(t),H_T(t')]\rangle_0\underbrace{e^{\eta t'}}_{\text{adiabatic switching}}\\
=&-\frac{16e}{\hbar}T^2\sin(\varphi)\sum_{k_1,k_2}u_{k_1}v_{k_1}u_{k_2}v_{k_2}\int_{-\infty}^{t}dt'\cos(E_{k_1}[t-t'])\sin(E_{k_2}[t-t'])e^{\eta t'}\tanh\left(\frac{\beta E_{k_1}}{2}\right)\\
=&-\frac{16e}{\hbar}T^2\sin(\varphi)\sum_{k_1,k_2}u_{k_1}v_{k_1}u_{k_2}v_{k_2}\frac{E_{k_2}}{E_{k_2}^2-E_{k_1}^2}\tanh\left(\frac{\beta E_{k_1}}{2}\right)\\
=&-\frac{4e}{\hbar}T^2\sin(\varphi)\sum_{k_1,k_2}\frac{1}{E_{k_1}}\frac{\Delta^2}{E_{k_2}^2-E_{k_1}^2}\tanh\left(\frac{\beta E_{k_1}}{2}\right),\end{aligned}\tag{9}$$

where we used the fact that

$$\int_{-\infty}^{t}dt'\cos(\omega[t-t'])\sin(\omega'[t-t'])e^{\eta t'}=\frac{\omega'}{\omega'^2-\omega^2}\tag{10}$$

Further calculation (see e.g. [Phys. Rev. Lett. **11**, 104 (1963)]) shows that

$$I(t)=-\frac{2\Delta e\pi^2T^2\rho_F}{\hbar}\tanh\left(\frac{\beta\Delta}{2}\right)\sin(\varphi),\tag{11}$$

where ρ_F is the normal state density of states at the Fermi level.
We note that as $\beta\to\infty$, we obtain

$$I(t)=-\frac{4e}{\hbar}T^2\sin(\varphi)\sum_{k_1,k_2}\frac{1}{E_{k_1}}\frac{\Delta^2}{E_{k_2}^2-E_{k_1}^2}=-\frac{2e}{\hbar}T^2\sin(\varphi)\sum_{k_1,k_2}\frac{1}{E_{k_1}E_{k_2}}\frac{\Delta^2}{E_{k_2}+E_{k_1}},\tag{12}$$

that is, the result presented in the lecture.

2. SN junction in the tunnelling regime **(70 Points)**

As opposed to the lecture notes, suppose now that you have a junction between a superconductor and a normal metal ($\Delta = 0$).

Calculate different contributions to the total current in the tunnelling approximation (including the normal contribution omitted in the Josephson's problem studied during the lecture).

Convince yourself that the resulting current is independent of the phase of the order parameter in the superconducting region of the setup.

Using the current-voltage relation $I(V)$ derived above, define the differential conductance of the setup as

$$G(V) = \frac{dI(V)}{dV}. \tag{13}$$

Compare your results with that of the celebrated paper by Blonder, Tinkham, and Klapwijk [Phys. Rev. B **25***, 4515 (1982)], namely Fig. 7. Interpret your findings.*

Now we consider the metal-insulator interface. Without loss of generality, we assume that the right system is metallic. The anomalous correlators $\langle RR \rangle$, $\langle R^\dagger R^\dagger \rangle$ are zero in this case, such that

$$\begin{aligned}\langle [I(t), H_T(t')] \rangle =& \frac{ie}{\hbar} T^2 \sum_{k_1,k_2,q_1,q_2,\sigma,\gamma} \Bigg(\langle [R^\dagger_{k_1,\sigma}(t) L_{k_2,\sigma}(t), L^\dagger_{q_2,\gamma}(t') R_{q_1,\gamma}(t')] \rangle \\ & - \langle [L^\dagger_{k_2,\sigma}(t) R_{k_1,\sigma}(t), R^\dagger_{q_1,\gamma}(t') L_{q_2,\gamma}(t')] \rangle \Bigg). \end{aligned} \tag{14}$$

Applying the Wick's theorem we find

$$\begin{aligned}\langle [R^\dagger_{k_1,\sigma}(t) L_{k_2,\sigma}(t), L^\dagger_{q_2,\gamma}(t') R_{q_1,\gamma}(t')] \rangle =& \langle L_{k_2,\sigma}(t) L^\dagger_{q_2,\gamma}(t') \rangle \langle R^\dagger_{k_1,\sigma}(t) R_{q_1,\gamma}(t') \rangle \\ & - \langle L^\dagger_{q_2,\gamma}(t') L_{k_2,\sigma}(t) \rangle \langle R_{q_1,\gamma}(t') R^\dagger_{k_1,\sigma}(t) \rangle, \end{aligned} \tag{15}$$

$$\begin{aligned}\langle [L^\dagger_{k_2,\sigma}(t) R_{k_1,\sigma}(t), R^\dagger_{q_1,\gamma}(t') L_{q_2,\gamma}(t')] \rangle =& \langle R_{k_1,\sigma}(t) R^\dagger_{q_1,\gamma}(t') \rangle \langle L^\dagger_{k_2,\sigma}(t) L_{q_2,\gamma}(t') \rangle \\ & - \langle R^\dagger_{q_1,\gamma}(t') R_{k_1,\sigma}(t) \rangle \langle L_{q_2,\gamma}(t') L^\dagger_{k_2,\sigma}(t) \rangle. \end{aligned} \tag{16}$$

At zero temperature we find

$$\langle R^\dagger_{k,\sigma}(t) R_{k',\sigma'}(t') \rangle = \delta_{k,k'} \delta_{\sigma,\sigma'} e^{i(\epsilon_k - eV)(t-t')} \Theta(-\epsilon_k), \tag{17}$$

$$\langle R_{k,\sigma}(t) R^\dagger_{k',\sigma'}(t') \rangle = \delta_{k,k'} \delta_{\sigma,\sigma'} e^{-i(\epsilon_k - eV)(t-t')} \Theta(\epsilon_k), \tag{18}$$

$$\langle L^\dagger_{k',\sigma'}(t') L_{k,\sigma}(t) \rangle = v_k^2 e^{iE_k(t-t')} \delta_{k,k'} \delta_{\sigma,\sigma'}, \tag{19}$$

$$\langle L_{k,\sigma}(t) L^\dagger_{k',\sigma'}(t') \rangle = u_k^2 e^{-iE_k(t-t')} \delta_{k,k'} \delta_{\sigma,\sigma'}, \tag{20}$$

where ϵ_k is the excitation energy of particles in the normal metal, and Heaviside theta functions stem from the zero-temperature limit of the Fermi functions $\lim_{T\to 0} n_F(\omega) = \Theta(-\omega)$.

Gathering all of the results we find

$$\begin{aligned}\langle [R^\dagger_{k_1,\sigma}(t) L_{k_2,\sigma}(t), L^\dagger_{q_2,\gamma}(t') R_{q_1,\gamma}(t')] \rangle =& \delta_{k_1,q_1} \delta_{\sigma,\gamma} \delta_{k_2,q_2} e^{i(\epsilon_{k_1} - eV)(t-t')} [\Theta(-\epsilon_{k_1}) u_{k_2}^2 e^{-iE_{k_2}(t-t')} \\ & - \Theta(\epsilon_{k_1}) v_{k_2}^2 e^{iE_{k_2}(t-t')}], \end{aligned} \tag{21}$$

$$\begin{aligned}\langle [L^\dagger_{k_2,\sigma}(t) R_{k_1,\sigma}(t), R^\dagger_{q_1,\gamma}(t') L_{q_2,\gamma}(t')] \rangle =& \delta_{k_1,q_1} \delta_{k_2,q_2} \delta_{\sigma,\gamma} e^{-i(\epsilon_{k_1} - eV)(t-t')} [\Theta(\epsilon_{k_1}) v_{k_2}^2 e^{-iE_{k_2}(t-t')} \\ & - \Theta(-\epsilon_{k_1}) u_{k_2}^2 e^{iE_{k_2}(t-t')}]. \end{aligned} \tag{22}$$

It follows that

$$\langle [I(t), H_T(t')] \rangle = \frac{4ie}{\hbar} T^2 \sum_{k_1,k_2} \Bigg(u_{k_2}^2 \cos([\epsilon_{k_1} - E_{k_2} - eV](t-t'))\Theta(-\epsilon_{k_1}) \\ - v_{k_2}^2 \cos([\epsilon_{k_1} + E_{k_2} - eV](t-t'))\Theta(\epsilon_{k_1}) \Bigg), \tag{23}$$

and hence

$$\begin{aligned} I(t) =& - i \int_{-\infty}^{t} dt' \, \langle [I(t), H_T(t')] \rangle \, e^{\eta t'} \\ =& \frac{4e}{\hbar} T^2 \int_{-\infty}^{t} dt' \sum_{k_1,k_2} \Bigg(u_{k_2}^2 \cos([\epsilon_{k_1} - E_{k_2} - eV](t-t'))\Theta(-\epsilon_{k_1}) \\ & - v_{k_2}^2 \cos([\epsilon_{k_1} + E_{k_2} - eV](t-t'))\Theta(\epsilon_{k_1}) \Bigg) e^{-\eta t'} \\ =& \frac{4\pi e}{\hbar} T^2 \sum_{k,q} \Bigg(u_q^2 \delta(\epsilon_k - E_q - eV)\Theta(-\epsilon_k) - v_q^2 \delta(\epsilon_k + E_q - eV)\Theta(\epsilon_k) \Bigg) \end{aligned} \tag{24}$$

where we used

$$\int_{-\infty}^{t} dt' \cos(\omega(t-t')) e^{\eta t'} = \pi \delta(\omega). \tag{25}$$

For simplicity, assume that

$$E_k = \sqrt{\Delta^2 + \epsilon_k^2}, \tag{26}$$

that is that there is no so-called normal state conductance mismatch at the interface (the dispersion of the electrons in the normal state is the same).

It follows that the first delta function is non-zero for

$$eV = - \left(\sqrt{\Delta^2 + \epsilon_q^2} + |\epsilon_k| \right) \leq -\Delta \tag{27}$$

while the other is non-zero for

$$eV = \left(\sqrt{\Delta^2 + \epsilon_q^2} + |\epsilon_k| \right) \geq \Delta, \tag{28}$$

this implies that the current $I(V)$ vanishes inside of the superconducting energy gap, and so does the conductance $G(V) = \frac{dI(V)}{dV}$.

Let us now consider the problem in greater detail. We rewrite

$$\begin{aligned} I(t) =& \frac{4\pi e}{\hbar} T^2 \int d\omega \int d\omega' \rho_\epsilon(\omega) \rho_\xi(\omega') u^2(\omega') \delta(\omega - \sqrt{\Delta^2 + \omega'^2} - eV) \Theta(-\omega) \\ & - \frac{4\pi e}{\hbar} T^2 \int d\omega \int d\omega' \rho_\epsilon(\omega) \rho_\xi(\omega') v^2(\omega') \delta(\omega + \sqrt{\Delta^2 + \omega'^2} - eV) \Theta(\omega) \end{aligned} \tag{29}$$

where we introduced

$$\rho_X(\omega) = \sum_k \delta(\omega - X_k) \approx \rho_X = \text{const}, \tag{30}$$

the associated normal state densities of states.

Employing the standard δ-function identities we discover

$$\begin{aligned}
\delta(\sqrt{\Delta^2+\omega'^2} - [eV-\omega]) =& \Theta(eV-\omega)\Theta(|eV-\omega|-\Delta)\frac{|\omega-eV|}{\lambda(\omega-eV)} \\
&\times [\delta(\omega' - \lambda(\omega-eV)) + \delta(\omega' + \lambda(\omega-eV))], \qquad (31)\\
\delta(\sqrt{\Delta^2+\omega'^2} - [\omega-eV]) =& \Theta(\omega-eV)\Theta(|eV-\omega|-\Delta)\frac{|\omega-eV|}{\lambda(\omega-eV)} \\
&\times [\delta(\omega' - \lambda(\omega-eV)) + \delta(\omega' + \lambda(\omega-eV))], \qquad (32)\\
\lambda(\omega) =& \sqrt{\omega^2-\Delta^2}. \qquad (33)
\end{aligned}$$

It follows that

$$\begin{aligned}
I(V) =& -\frac{eg_T}{h}\left(\Theta(eV-\Delta) - \Theta(-\Delta-eV)\right)\int_\Delta^{e|V|} d\omega \frac{\omega}{\sqrt{\omega^2-\Delta^2}} \qquad (34)\\
=& -\frac{eg_T}{h}\left(\Theta(eV-\Delta) - \Theta(-\Delta-eV)\right)\sqrt{e^2V^2-\Delta^2} \qquad (35)\\
\implies G(V) =& -\frac{dI(V)}{dV} = G_T\frac{e|V|}{\sqrt{e^2V^2-\Delta^2}}, \qquad (36)
\end{aligned}$$

where we introduced

$$g_T = 8\pi^2 T^2 \rho_\epsilon \rho_\xi, \quad , \quad G_T = \frac{1}{R_T} = \frac{g_T e^2}{h}. \tag{37}$$

Note that there is a minus sign in the definition of conductance, due to our definition of the voltage drop across the contact. We also see from Eq. (36) that the differential conductance provides a way to measure the density of states of the system.

Note that this result is in agreement with lower-right corner panel of Fig. 7 of Ref. [Phys. Rev. B **25**, 4515 (1982)], where the differential conductance is strongly suppressed inside of the superconducting energy gap in the tunnel junction limit $Z \sim T^{-2} \gg 1$.