Exercises for "Superconductivity, Josephson ..." WS 2023/2024

Prof. Dr. A. Shnirman
Dr. K. Piasotski

Exercise 5
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1. **Andreev bound states in an ideal long SNS contact. (40 Punkte)**

In the lecture we have considered a short SNS contact. The derivation for the long contact of length L is sketched in the script on page 93 (Equation 550). There the Andreev reflection amplitude is derived for the right NS interface. Provide an analogous derivation for the left interface. Obtain the quantisation condition and try to estimate the energies and the number of the Andreev bound states.

We consider

$$H = \tau_z \left(\frac{p^2}{2m} - \mu \right) + (\Delta(x)\tau_+ + \text{h.c.}), \tag{1}$$

$$\Delta(x) = \Delta[\Theta(-x) + \Theta(x-L)e^{i\varphi}] \tag{2}$$

Left and right of the interface we have the following fundamental solutions

$$\psi_L(x) = \psi_k e^{ikx}, \quad \psi_R(x) = U\psi_k e^{ikx}, \tag{3}$$

where

$$\psi_k = \begin{pmatrix} \Delta \\ \epsilon + \mu - \frac{k^2}{2m} \end{pmatrix}, \quad \left(\frac{k^2}{2m} - \mu \right)^2 = -\Delta^2 + \epsilon^2, \quad U = e^{i\varphi\tau_z/2}. \tag{4}$$

We search for the bound state solutions, such that

$$\lim_{x\to-\infty} \psi_L(x) = \lim_{x\to\infty} \psi_R(x) = 0. \tag{5}$$

This forces us to choose

$$\psi_L(x) = c_+^L \psi_{-k_+} e^{-ik_+x} + c_-^L \psi_{-k_-} e^{-ik_-x} = \begin{pmatrix} \psi_{-k_+} e^{-ik_+x} & \psi_{-k_-} e^{-ik_-x} \end{pmatrix} \begin{pmatrix} c_+^L \\ c_-^L \end{pmatrix}, \tag{6}$$

$$\psi_R(x) = c_+^R U \psi_{k_+} e^{ik_+(x-L)} + c_-^R U \psi_{k_-} e^{ik_-(x-L)} \tag{7}$$

$$= U \begin{pmatrix} \psi_{k_+} e^{ik_+(x-L)} & \psi_{k_-} e^{ik_-(x-L)} \end{pmatrix} \begin{pmatrix} c_+^R \\ c_-^R \end{pmatrix}, \tag{8}$$

$$k_\pm = \pm \sqrt{2m\mu \pm 2mi\sqrt{\Delta^2 - \epsilon^2}}. \tag{9}$$

In the central section, all four fundamental solutions are admissible, such that

$$\psi_C(x) = [c_{p+} e^{ik_N^p x} + c_{p-} e^{-ik_N^p x}] \begin{pmatrix} 1 \\ 0 \end{pmatrix} + [c_{h+} e^{ik_N^h x} + c_{h-} e^{-ik_N^h x}] \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{10}$$

$$k_N^p = \sqrt{2m(\mu + \epsilon)}, \quad k_N^h = \sqrt{2m(\mu - \epsilon)}. \tag{11}$$

Now we apply the boundary conditions on the lhs

$$\begin{pmatrix} 0 & \begin{pmatrix} \psi_{-k_+} & \psi_{-k_-} \end{pmatrix} \\ 0 & -i\begin{pmatrix} k_+\psi_{-k_+} & k_-\psi_{-k_-} \end{pmatrix} \end{pmatrix} \begin{pmatrix} c_+^R \\ c_-^R \\ c_+^L \\ c_-^L \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ ik_N^p & -ik_N^p & 0 & 0 \\ 0 & 0 & ik_N^h & -ik_N^h \end{pmatrix} \begin{pmatrix} c_{p+} \\ c_{p-} \\ c_{h+} \\ c_{h-} \end{pmatrix}. \tag{12}$$

Now we apply the boundary conditions on the rhs

$$\begin{pmatrix} U\begin{pmatrix} \psi_{k_+} & \psi_{k_-} \end{pmatrix} & 0 \\ iU\begin{pmatrix} k_+\psi_{k_+} & k_-\psi_{k_-} \end{pmatrix} & 0 \end{pmatrix} \begin{pmatrix} c_+^R \\ c_-^R \\ c_+^L \\ c_-^L \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} e^{ik_N^pL} & e^{-ik_N^pL} & 0 & 0 \\ 0 & 0 & e^{ik_N^hL} & e^{-ik_N^hL} \\ ik_N^p e^{ik_N^pL} & -ik_N^p e^{-ik_N^pL} & 0 & 0 \\ 0 & 0 & ik_N^h e^{ik_N^hL} & -ik_N^h e^{-ik_N^hL} \end{pmatrix} \begin{pmatrix} c_{p+} \\ c_{p-} \\ c_{h+} \\ c_{h-} \end{pmatrix}. \tag{14}$$

From Eq.(12) we find

$$\begin{pmatrix} c_{p+} \\ c_{p-} \\ c_{h+} \\ c_{h-} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ ik_N^p & -ik_N^p & 0 & 0 \\ 0 & 0 & ik_N^h & -ik_N^h \end{pmatrix}^{-1} \begin{pmatrix} 0 & \begin{pmatrix} \psi_{-k_+} & \psi_{-k_-} \end{pmatrix} \\ 0 & -i\begin{pmatrix} k_+\psi_{-k_+} & k_-\psi_{-k_-} \end{pmatrix} \end{pmatrix} \begin{pmatrix} c_+^R \\ c_-^R \\ c_+^L \\ c_-^L \end{pmatrix}, \tag{15}$$

such that Eq. (14) gives

$$\begin{pmatrix} U\begin{pmatrix} \psi_{k_+} & \psi_{k_-} \end{pmatrix} & 0 \\ iU\begin{pmatrix} k_+\psi_{k_+} & k_-\psi_{k_-} \end{pmatrix} & 0 \end{pmatrix} \begin{pmatrix} c_+^R \\ c_-^R \\ c_+^L \\ c_-^L \end{pmatrix} - M \begin{pmatrix} 0 & \begin{pmatrix} \psi_{-k_+} & \psi_{-k_-} \end{pmatrix} \\ 0 & -i\begin{pmatrix} k_+\psi_{-k_+} & k_-\psi_{-k_-} \end{pmatrix} \end{pmatrix} \begin{pmatrix} c_+^R \\ c_-^R \\ c_+^L \\ c_-^L \end{pmatrix} = 0 \tag{16}$$

where

$$M = \begin{pmatrix} \cos(k_N^pL) & 0 & \sin(k_N^pL)/k_N^p & 0 \\ 0 & \cos(k_N^hL) & 0 & \sin(k_N^hL)/k_N^h \\ -k_N^p\sin(k_N^pL) & 0 & \cos(k_N^pL) & 0 \\ 0 & -k_N^h\sin(k_N^hL) & 0 & \cos(^h_NL) \end{pmatrix}. \tag{17}$$

It follows that we have the condition

$$\det\begin{pmatrix} UJ & -CJ + iSK^{-1}JK_S \\ iUJK_S & SKJ + iCJK_S \end{pmatrix} = 0, \tag{18}$$

where

$$C = \begin{pmatrix} \cos(k_N^pL) & 0 \\ 0 & \cos(k_N^hL) \end{pmatrix}, \quad S = \begin{pmatrix} \sin(k_N^pL) & 0 \\ 0 & \sin(k_N^hL) \end{pmatrix}, \tag{19}$$

$$K = \begin{pmatrix} k_N^p & 0 \\ 0 & k_N^h \end{pmatrix}, \quad K_S = \begin{pmatrix} k_+ & 0 \\ 0 & k_- \end{pmatrix}, \tag{20}$$

$$J = \begin{pmatrix} \psi_{k_+} & \psi_{k_-} \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta}{\epsilon - i\sqrt{\Delta^2-\epsilon^2}} & \dfrac{\Delta}{\epsilon + i\sqrt{\Delta^2-\epsilon^2}} \end{pmatrix}. \tag{21}$$

To obtain the approximate result from the lecture we approximate $k_N^p \approx k_N^h \approx k_+ \approx -k_- \approx k_F$ everywhere apart from the phases. In this connection, the secular equation (18) simplifies to

$$\det\begin{pmatrix} UJ & -CJ + iSJ\tau_z \\ ik_F UJ\tau_z & k_F[SJ + iCJ\tau_z] \end{pmatrix} = \det[UJk_F] \tag{22}$$

$$\times \det[SJ + iCJ\tau_z + iUJ\tau_z(J)^{-1}U^\dagger(CJ - iSJ\tau_z)] = 0. \tag{23}$$

This yields the condition

$$(\Delta^2 - 2\epsilon^2)\cos([k_N^p - k_N^h]L) + \Delta^2\cos\varphi - 2\epsilon\sqrt{\Delta^2 - \epsilon^2}\sin([k_N^p - k_N^h]L) = 0. \tag{24}$$

Solving the condition for ϵ we find

$$\frac{\epsilon^2}{\Delta^2} = \cos^2\left(\frac{[k_N^p - k_N^h]L \pm \varphi}{2}\right) \simeq \cos^2\left(\frac{L}{\xi}\frac{\epsilon}{\Delta} \pm \frac{\varphi}{2}\right) \tag{25}$$

2. A non-ideal NS contact. (60 Punkte)

In the lecture (see script) we have considered an ideal NS (normal metal - superconductor) contact and concentrated on one transverse channel (effectively one-dimensional wire). In this contact the superconducting order parameter vanishes for $x < 0$ and is equal to $|\Delta|e^{i\phi}$ for $x > 0$. Here we consider a non-ideal contact, which is modeled by adding a potential barrier at $x = 0$, namely $V(x) = H\delta(x)$. Write down the Bogoliubov-de Gennes equations for this case and find the scattering amplitudes in all possible scattering channels for an incoming (from the normal side) electron with energy E. Consider both cases $E < |\Delta|$ and $E > |\Delta|$. Consult if necessary the paper: G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B 25, 4515 (1982).

The BdG equations read

$$\begin{pmatrix} -\frac{\hbar^2\nabla_x^2}{2m} + V(x) - \tilde{\mu} & \Delta(x) \\ \Delta^*(x) & \frac{\hbar^2\nabla_x^2}{2m} - V(x) + \tilde{\mu} \end{pmatrix}\begin{pmatrix} u \\ v^* \end{pmatrix} = E\begin{pmatrix} u \\ v^* \end{pmatrix} . \tag{26}$$

This gives

$$\begin{pmatrix} -\frac{\hbar^2\nabla_x^2}{2m} + H\delta(x) - \tilde{\mu} & \Delta(x) \\ \Delta^*(x) & \frac{\hbar^2\nabla_x^2}{2m} - H\delta(x) + \tilde{\mu} \end{pmatrix}\begin{pmatrix} u \\ v^* \end{pmatrix} = E\begin{pmatrix} u \\ v^* \end{pmatrix} . \tag{27}$$

The delta-function can only be compensated by a jump in the first derivative. Thus, instead of continuity of the first derivative we obtain a new boundary condition

$$\left.\frac{du}{dx}\right|_{x=+0} - \left.\frac{du}{dx}\right|_{x=-0} = \frac{2mH}{\hbar^2}u(x=0) , \tag{28}$$

$$\left.\frac{dv^*}{dx}\right|_{x=+0} - \left.\frac{dv^*}{dx}\right|_{x=-0} = \frac{2mH}{\hbar^2}v^*(x=0) . \tag{29}$$

These are in addition to the continuity of the wave functions themselves:

$$u(x=+0) = u(x=-0) \quad , \quad v^*(x=+0) = v^*(x=-0) . \tag{30}$$

There are two scattering channels from left to right

$$\psi_+^L(x) = \begin{pmatrix}1\\0\end{pmatrix} e^{ik_e x} + r_{++}\begin{pmatrix}1\\0\end{pmatrix} e^{-ik_e x} + r_{-+}\begin{pmatrix}0\\1\end{pmatrix} e^{-ik_h x}, \tag{31}$$

$$\psi_-^L(x) = \begin{pmatrix}0\\1\end{pmatrix} e^{ik_h x} + r_{+-}\begin{pmatrix}1\\0\end{pmatrix} e^{-ik_e x} + r_{--}\begin{pmatrix}0\\1\end{pmatrix} e^{-ik_h x}, \tag{32}$$

where

$$k_e^2 = 2m\mu_N + 2m\epsilon, \quad k_h^2 = 2m\mu_N - 2m\epsilon. \tag{33}$$

Note that in the matrix form we have

$$\Psi^L(x) = \left(\psi_+^L(x)|\psi_-^L(x)\right) = \begin{pmatrix}e^{ik_e x} & 0\\ 0 & e^{ik_h x}\end{pmatrix} + \begin{pmatrix}e^{-ik_e x} & 0\\ 0 & e^{-ik_h x}\end{pmatrix}\begin{pmatrix}r_{++} & r_{+-}\\ r_{-+} & r_{--}\end{pmatrix}. \tag{34}$$

On the right hand side we have

$$\psi_\pm^R(x) = A_\pm \begin{pmatrix} U_g \\ e^{-i\phi}U_g^* \end{pmatrix} e^{ik_1 x} + B_\pm \begin{pmatrix} U_g^* \\ e^{-i\phi}U_g \end{pmatrix} e^{ik_2 x} , \tag{35}$$

where

$$U_g \equiv \sqrt{\frac{E + i\sqrt{|\Delta|^2 - E^2}}{2E}} . \tag{36}$$

Here

$$k_1 = k_F\sqrt{1 + i\frac{\sqrt{|\Delta|^2 - E^2}}{\tilde{\mu}}} . \tag{37}$$

and

$$k_2 = -k_F\sqrt{1 - i\frac{\sqrt{|\Delta|^2 - E^2}}{\tilde{\mu}}} . \tag{38}$$

In the matrix form we have

$$\Psi^R(x) = \left(\psi_+^R(x)|\psi_-^R(x)\right) = \begin{pmatrix}U_g e^{ik_1 x} & U_g^* e^{ik_2 x}\\ U_g^* e^{ik_1 x} & U_g e^{ik_2 x}\end{pmatrix}\begin{pmatrix}A_+ & A_-\\ B_+ & B_-\end{pmatrix} \tag{39}$$

The interface matching conditions give

$$\underbrace{\begin{pmatrix}U_g & U_g^*\\ U_g^* & U_g\end{pmatrix}}_{\hat{U}}\underbrace{\begin{pmatrix}A_+ & A_-\\ B_+ & B_-\end{pmatrix}}_{\hat{C}} = 1 + \underbrace{\begin{pmatrix}r_{++} & r_{+-}\\ r_{-+} & r_{--}\end{pmatrix}}_{\hat{R}}, \tag{40}$$

$$i\hat{U}\hat{K}_S\hat{C} - i\hat{K}_N(1 - \hat{R}) = \frac{2mH}{\hbar^2}(1 + \hat{R}), \tag{41}$$

$$\hat{K}_N = \begin{pmatrix}k_e & 0\\ 0 & k_h\end{pmatrix}, \quad \hat{K}_S = \begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}. \tag{42}$$

We find

$$\hat{R} = 2\hat{U}\left(\hat{U}\hat{K}_S + \hat{K}_N\hat{U} + i\frac{2mH}{\hbar^2}\hat{U}\right)^{-1}\hat{K}_N - 1, \tag{43}$$

$$\hat{C} = 2\left(\hat{U}\hat{K}_S + \hat{K}_N\hat{U} + i\frac{2mH}{\hbar^2}\hat{U}\right)^{-1}\hat{K}_N \tag{44}$$

In particular, in Andreev approximation, we have $\hat{K}_S \approx \hat{K}_N = k_F\tau_z$, such that

$$\hat{U}\hat{K}_S + \hat{K}_N\hat{U} + i\frac{2mH}{\hbar^2}\hat{U} = i\frac{2mH}{\hbar^2}U_g + 2k_F U_g\tau_z + i\frac{2mH}{\hbar^2}U_g^*\tau_x, \tag{45}$$

$$\left(\hat{U}\hat{K}_S + \hat{K}_N\hat{U} + i\frac{2mH}{\hbar^2}\hat{U}\right)^{-1} = \frac{iZU_g - U_g\tau_z - iZU_g^*\tau_x}{2k_F(Z^2U_g^{*2} - U_g^2(Z^2+1))}, \tag{46}$$

$$\frac{mH}{\hbar^2 k_F} = Z. \tag{47}$$

This gives us the reflection matrix

$$\hat{R} = 2\hat{U}\left(\hat{U}\hat{K}_S + \hat{K}_N\hat{U} + i\frac{2mH}{\hbar^2}\hat{U}\right)^{-1}\hat{K}_N - 1 = \frac{iZ\tau_z - 1 - Ze^{-i\gamma}\tau_y}{(Z^2e^{-2i\gamma} - (Z^2+1))} \tag{48}$$

$$+ \frac{-Ze^{i\gamma}\tau_y + e^{i\gamma}\tau_x + iZ\tau_z}{(e^{2i\gamma}(Z^2+1) - Z^2)} - 1. \tag{49}$$

We separate

$$\hat{R} = \hat{R}_N + \hat{R}_A, \tag{50}$$

$$\hat{R}_N = \frac{iZ\tau_z - 1}{(Z^2e^{-2i\gamma} - (Z^2+1))} + \frac{iZ\tau_z}{(e^{2i\gamma}(Z^2+1) - Z^2)} - 1 \approx 2Z\tau_z e^{-i\gamma}\sin(\gamma), \quad Z \ll 1, \tag{51}$$

$$\hat{R}_A = \frac{-Ze^{-i\gamma}\tau_y}{(Z^2e^{-2i\gamma} - (Z^2+1))} + \frac{-Ze^{i\gamma}\tau_y + e^{i\gamma}\tau_x}{(e^{2i\gamma}(Z^2+1) - Z^2)} \approx e^{-i\gamma}\tau_x, \quad Z \ll 1. \tag{52}$$