Introduction to theoretical foundations

(from experimenter’s viewpoint)

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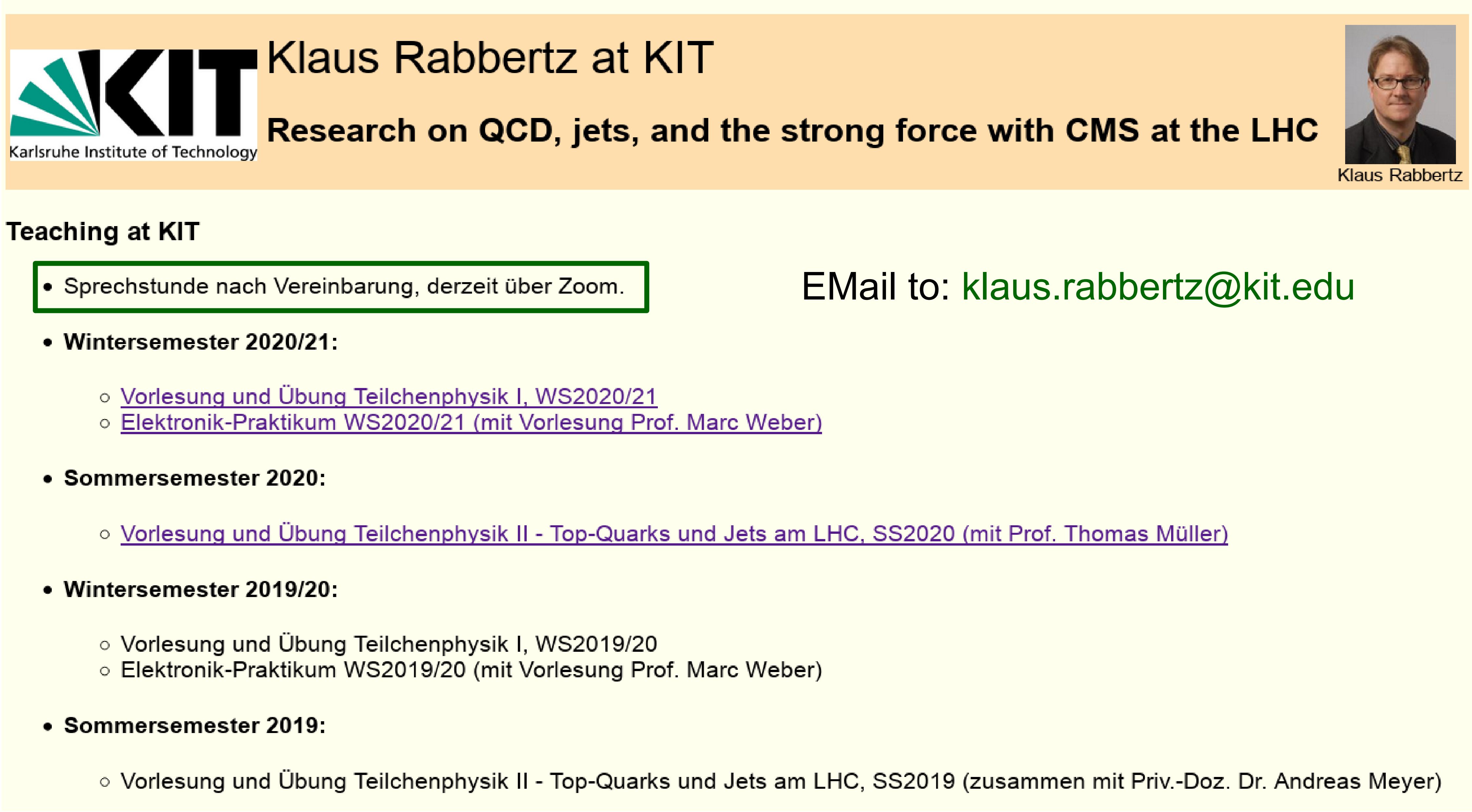

Klaus Rabbertz at KIT

Research on QCD, jets, and the strong force with CMS at the LHC

Teaching at KIT

- Sprechstunde nach Vereinbarung, derzeit über Zoom.

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- **Wintersemester 2020/21:**
  - Vorlesung und Übung Teilchenphysik I, WS2020/21
  - Elektronik-Praktikum WS2020/21 (mit Vorlesung Prof. Marc Weber)
- **Sommersemester 2020:**
  - Vorlesung und Übung Teilchenphysik II - Top-Quarks und Jets am LHC, SS2020 (mit Prof. Thomas Müller)
- **Wintersemester 2019/20:**
  - Vorlesung und Übung Teilchenphysik I, WS2019/20
  - Elektronik-Praktikum WS2019/20 (mit Vorlesung Prof. Marc Weber)
- **Sommersemester 2019:**
  - Vorlesung und Übung Teilchenphysik II - Top-Quarks und Jets am LHC, SS2019 (zusammen mit Priv.-Doz. Dr. Andreas Meyer)

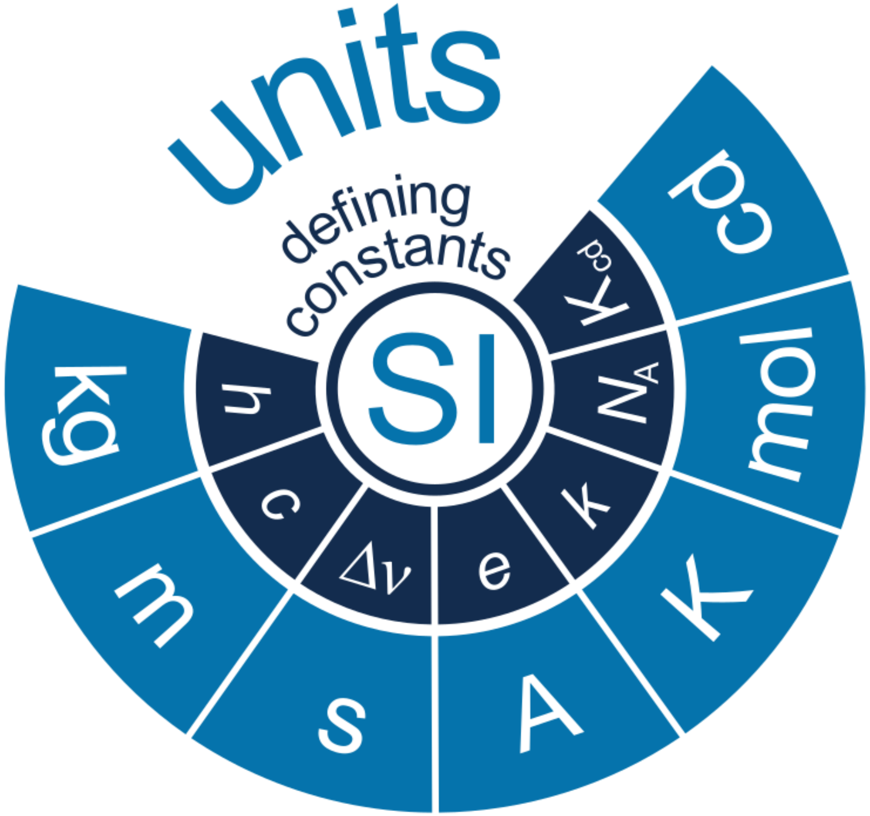
What: Literature

- See also slides of previous editions from U. Husemann, M. Schröder, R. Wolf from which parts have been used
- Theory: Oldies but Goldies
  - **J.D. Bjorken, S.D. Drell: Relativistic Quantum Fields, McGraw-Hill College (1965).**
  - **F. Halzen, A.D. Martin: Quarks & Leptons, Wiley (1984).**
  - **O. Nachtmann: Phänomene und Konzepte der Elementarteilchenphysik, Vieweg+Teubner (1992).**
  - **M.E. Peskin, D.V. Schröder, “An Introduction to Quantum Field Theory”, Westview Press, 1995.**
  - **V. D. Barger, R. J. N. Phillips: Collider Physics, Westview Press (1996).**
- Theory: Newer
  - **W.N. Cottingham, D.A. Greenwood: An Introducton to the Standard Model of Particle Physics, Cambridge UP (2007).**
  - **I.J.R. Aitchison, A.J.G. Hey: Gauge Theories in Particle Physics, 4th ed., CRC Press (2012).**

Updated definition of SI base units

Like for the “`second`” or “`meter`” all other base units are redefined by fixing the numerical values of seven universally valid constants of nature:

→ No dependence any more of
- local noble persons
- an artifact deposed in Paris
- measures of earth or our solar system



Has been decided in November 2018 in Versailles and is officially in force since World Metrology Day (20. May) 2019.

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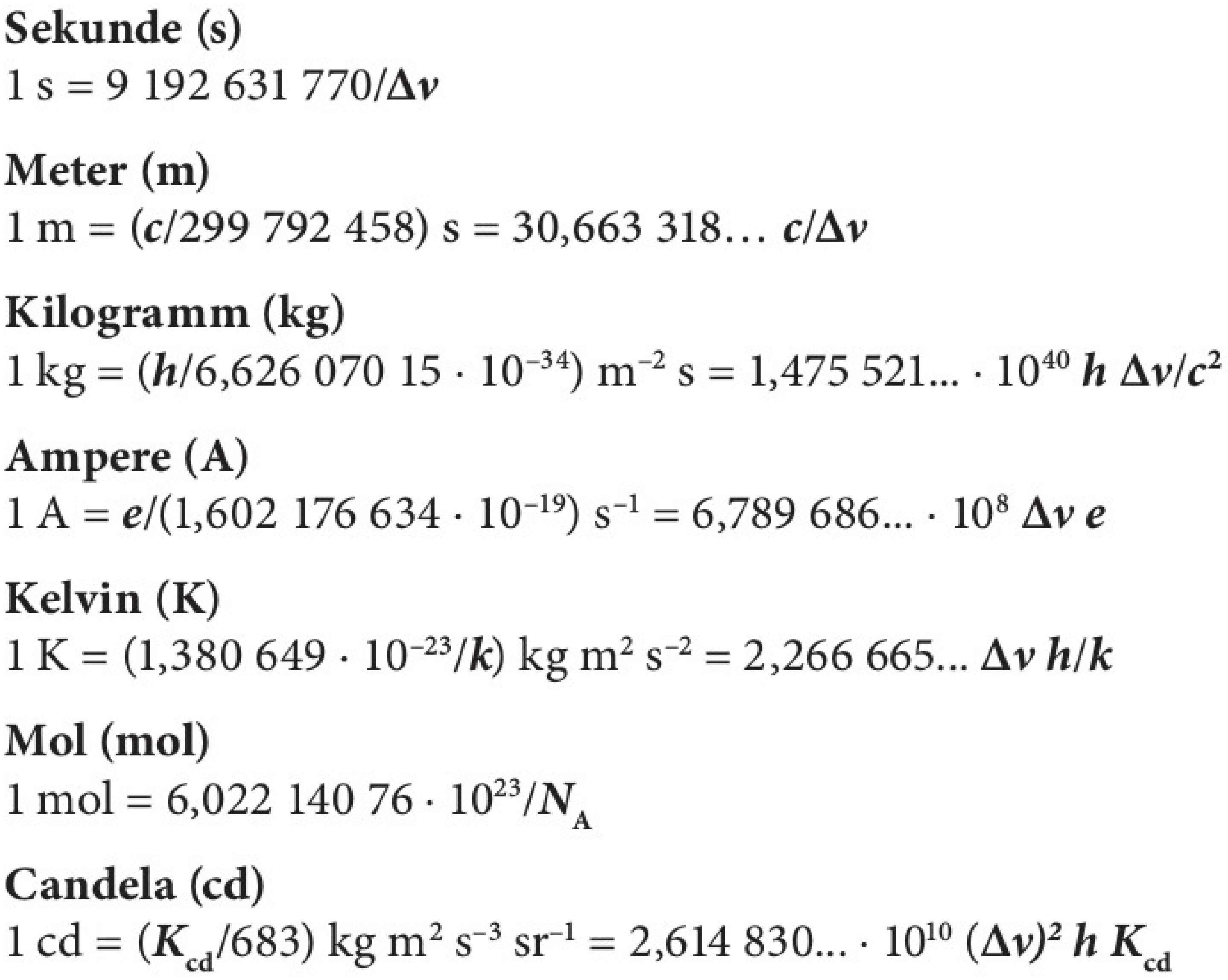
https://www.bipm.org/

New definition of SI base units

The new definitions based on constants of nature



Sekunde (s)

$1\ \mathrm{s} = 9\,192\,631\,770/\boldsymbol{\Delta\nu}$

Meter (m)

$1\ \mathrm{m} = (\boldsymbol{c}/299\,792\,458)\ \mathrm{s} = 30{,}663\,318\ldots\ \boldsymbol{c}/\boldsymbol{\Delta\nu}$

Kilogramm (kg)

$1\ \mathrm{kg} = (\boldsymbol{h}/6{,}626\,070\,15 \cdot 10^{-34})\ \mathrm{m}^{-2}\ \mathrm{s} = 1{,}475\,521\ldots \cdot 10^{40}\ \boldsymbol{h}\ \boldsymbol{\Delta\nu}/\boldsymbol{c}^2$

Ampere (A)

$1\ \mathrm{A} = \boldsymbol{e}/(1{,}602\,176\,634 \cdot 10^{-19})\ \mathrm{s}^{-1} = 6{,}789\,686\ldots \cdot 10^{8}\ \boldsymbol{\Delta\nu}\ \boldsymbol{e}$

Kelvin (K)

$1\ \mathrm{K} = (1{,}380\,649 \cdot 10^{-23}/\boldsymbol{k})\ \mathrm{kg\ m^2\ s^{-2}} = 2{,}266\,665\ldots\ \boldsymbol{\Delta\nu}\ \boldsymbol{h}/\boldsymbol{k}$

Mol (mol)

$1\ \mathrm{mol} = 6{,}022\,140\,76 \cdot 10^{23}/\boldsymbol{N}_{\mathrm{A}}$

Candela (cd)

$1\ \mathrm{cd} = (\boldsymbol{K}_{\mathrm{cd}}/683)\ \mathrm{kg\ m^2\ s^{-3}\ sr^{-1}} = 2{,}614\,830\ldots \cdot 10^{10}\ (\boldsymbol{\Delta\nu})^2\ \boldsymbol{h}\ \boldsymbol{K}_{\mathrm{cd}}$

“Natural” units: $\hbar = c = 1$

Energy, momentum, mass have the same unit, e.g. elektronvolt eV:

$$[E] = [p] = [m] = \mathrm{eV}$$

$$1\mathrm{eV} = e \cdot 1\mathrm{Volt} = 1.6 \cdot 10^{-19}\mathrm{J}$$

The rel. energy-momentum relation simply gets:

$$\boxed{E^2 = p^2 + m^2}$$

Length and time can be expressed in units of 1/eV ausdrücken:

$$E = \hbar\omega \rightarrow \omega \qquad \vec{p} = \hbar\vec{k} \rightarrow \vec{k}$$

$$\Rightarrow \Delta E \cdot \Delta t = 1 \Rightarrow [t] = \mathrm{eV}^{-1}$$

$$\Rightarrow \Delta p \cdot \Delta x = 1 \Rightarrow [x] = \mathrm{eV}^{-1}$$

$$\hbar\, c = 1 \approx 200\,\mathrm{MeV} \cdot \mathrm{fm}$$

Areas correspond to units in 1/eV²:

$$(\hbar\, c)^2 = 1 \approx 40000\,\mathrm{MeV}^2 \cdot \mathrm{fm}^2$$

$$1\mathrm{b} = 10^{-28} m^2$$

Barn = Scheune

Proton radius squared

$$1\mathrm{fm}^2 \approx \frac{1}{40000\,\mathrm{MeV}^2} = \frac{25}{\mathrm{GeV}^2} = 0.01\mathrm{barn}$$

$$\frac{1}{\mathrm{GeV}^2} \approx 0.4\mathrm{mb}$$

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Institut für Experimentelle Teilchenphysik

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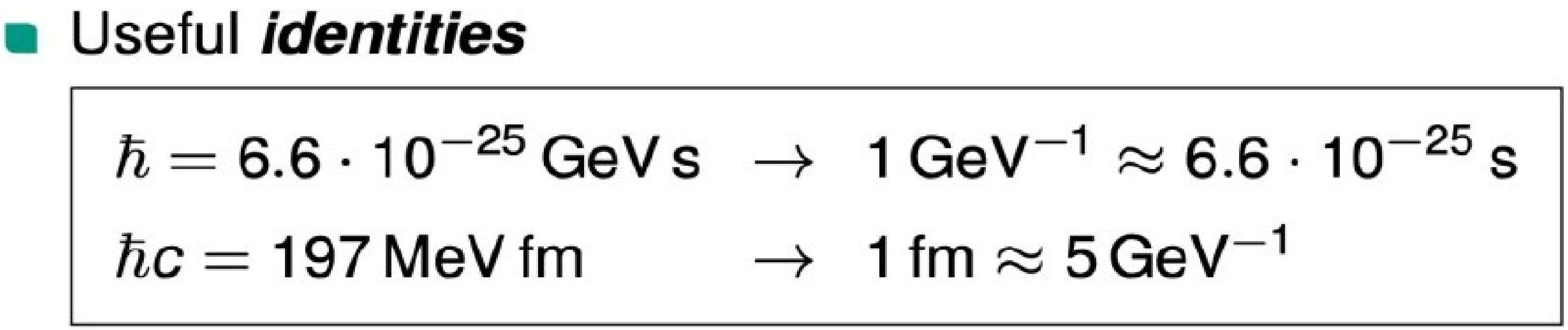
Secret name for nuclear unit in 2nd World War! → Los Alamos
Also proverb: “... couldn't hit the broad side of a barn ...”
→ “... könnte nicht mal'n Scheunentor treffen ...”

Natural units

Previously:



- Useful ***identities***

$$\hbar = 6.6 \cdot 10^{-25}\,\mathrm{GeV\,s} \quad \rightarrow \quad 1\,\mathrm{GeV}^{-1} \approx 6.6 \cdot 10^{-25}\,\mathrm{s}$$

$$\hbar c = 197\,\mathrm{MeV\,fm} \quad \rightarrow \quad 1\,\mathrm{fm} \approx 5\,\mathrm{GeV}^{-1}$$

Also: Rydberg energy (ionisation energy of hydrogen):

$$E = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2} = -\frac{1}{2} m_e \alpha^2 = -\frac{1}{2} 511\mathrm{keV}/137^2 = -13.6\,\mathrm{eV}$$

Here, instead of elementary charge e resp. e^2 the dimensionless fine structure constant α appears that later becomes the (running) coupling constant of the elektromagnetic interaction:

$$\alpha = e^2/4\pi\epsilon_0\hbar c$$

Similarly, we'll encounter the strong coupling constant! α_S

Notations

- **Vectors** **3-vector:** $x^i = \vec{x}$ $(i = 1, 2, 3)$

  4-vector: $x^\mu = (t, \vec{x})$ $(\mu = 0, 1, 2, 3)$

- **Contravariant x^μ and covariant x_μ representation connected via metric tensor:**

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad x_\mu = g_{\mu\nu} x^\nu \equiv \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu$$

- **Einstein convention: Summation over identical indices is implied!**
  - **Lowercase Roman indices:** **a, b, c, … = 1, 2, 3**
  - **Lowercase Greek indices:** **μ, v, ρ, … = 0, 1, 2, 3**
  - **Uppercase Roman indices:** **A, B, C, … = 1, 2, … , 8**

Notations

- **4-vectors:**
  - **Time-space:** $x^\mu = (t, \vec{x})$
  - **4-momentum:** $p^\mu = (E, \vec{p})$
  - **4-current:** $j^\mu = (\rho, \vec{j})$
  - **electromagnetic 4-potential:** $A^\mu = (\phi, \vec{A})$
  - **4-gradient: contravariant** $\partial^\mu = \frac{\partial}{\partial_\mu} = \left(\frac{\partial}{\partial_t}, -\vec{\nabla}\right)$

    covariant $\partial_\mu = \frac{\partial}{\partial^\mu} = \left(\frac{\partial}{\partial^t}, \vec{\nabla}\right)$
- **Laplace & d'Alembert operators:** $\Delta = \nabla^2 \qquad \Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial_t^2} - \nabla^2$
- **Field-strength tensor:** $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Notations

- **4-vectors:**
  - **Time-space:** $x^\mu = (t, \vec{x})$ $\quad [x^\mu] = \mathrm{GeV}^{-1}$
  - **4-momentum:** $p^\mu = (E, \vec{p})$ $\quad [p^\mu] = \mathrm{GeV}$
  - **4-current:** $j^\mu = (\rho, \vec{j})$
  - **electromagnetic 4-potential:** $A^\mu = (\phi, \vec{A})$

Remark: Units

  - **4-gradient: contravariant** $\partial^\mu = \frac{\partial}{\partial_\mu} = \left(\frac{\partial}{\partial_t}, -\vec{\nabla}\right)$ $\quad [\partial^\mu] = \mathrm{GeV}$

    covariant $\partial_\mu = \frac{\partial}{\partial^\mu} = \left(\frac{\partial}{\partial^t}, \vec{\nabla}\right)$

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Time out

Quantum mechanics

Classical: Energy-momentum of free particle: $E = \frac{p^2}{2m}$

Canonical operator replacement: $\boxed{\hat{E} \rightarrow i\frac{\partial}{\partial t} \qquad \hat{\vec{p}} \rightarrow -i\vec{\nabla}}$

→ Schrödinger equation (free particle):

$$\hat{H}\phi = \frac{\hat{\vec{p}}^2}{2m}\phi = -\frac{\Delta}{2m}\phi = \hat{E}\phi = i\frac{\partial}{\partial t}\phi$$

2nd order derivative in space

1st order derivative in time

Observation: Cannot be Lorentz invariant → no solution for relativistic QM

Relativistic extension

Recall 3-dim. scalar product: **invariant under rotations in 3-dim. space**

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x^i y_i = x^i y_i = |\vec{x}| \cdot |\vec{y}| \cdot \cos(\angle[\vec{x}, \vec{y}])$$

“4-dim.” scalar product: **invariant under Lorentz transformations**

$$p^\mu p_\mu = E^2 - p^2 = m^2$$

→ Relativistic: Energy-momentum of free particle: $E^2 = p^2 + m^2$

Operator replacement → $-\partial^2 t\phi = (-\Delta + m^2)\phi$

$$\Longleftrightarrow \ (\partial^2 t - \Delta + m^2)\phi = (\Box + m^2)\phi = 0$$

Klein-Gordon Equation

Klein-Gordon Equation

Spin-0 particle wave eq.: $(\Box + m^2)\phi = (\partial_\mu \partial^\mu + m^2)\phi = 0$

Relativistic, i.e. Lorentz-invariant? **OK**

Free wave solutions: $\phi(\vec{x}, t) = N e^{\pm i(p^\mu x_\mu)}$

But:

Possibility of negative energy $E = \pm\sqrt{(p^2 + m^2)}$

Probability current $\partial^\mu j_\mu = 0$

has non-positive definite probability density ρ = j⁰,
e.g. for plane wave solution with E < 0: $\rho = 2\,|N|^2\,E < 0$

Meaning for single-particle states … ???

Dirac's Approach 1927

Find representation of energy-momentum conservation that is linear in both, space and time derivatives!

Ansatz: $\hat{H} = \vec{\alpha}\hat{\vec{p}} + \beta m$ **with** $\vec{\alpha} = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$

$$\rightarrow i\partial t\,\psi = \hat{H}\psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)\psi$$

What are these α and β? They cannot be simple numbers … !

Re-apply both operators and require Klein-Gordon eq. to be fulfilled:

$$(i\partial t)^2\,\psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)^2\psi$$

$$= \left(-\sum_{i,j=1}^{3}\underbrace{\left(\frac{\alpha_i\alpha_j+\alpha_j\alpha_i}{2}\right)}\partial_i\partial_j - im\sum_{i=1}^{3}\underbrace{\left(\alpha_i\beta+\beta\alpha_i\right)}\partial_i + \underbrace{(\beta m)}^2\right)\psi$$

$$\equiv \left(-\vec{\nabla}^2 + m^2\right)\psi$$

Anti-commutator relations $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ $\{\alpha_i, \beta\} = 0$ $\beta^2 = 1$

Dirac: Properties of α_i and β

- Operators α_i and β can be expressed by **matrices**:
  - must be **hermitian** since $\hat{H}$ should have real eigenvalues
  - must be **traceless**

$$Tr(\alpha_i) = Tr(\alpha_i \underset{\mathbb{I}}{\beta\beta}) = Tr(\beta\alpha_i\beta) = -Tr(\beta\beta\alpha_i) = -Tr(\alpha_i) = 0$$

unity matrix　　　cyclic permutation　　　anti-commutation

  - must have **at least four dimensions**
    - $\alpha_i^2 = \mathbb{I}$, $\beta^2 = \mathbb{I}$ $\rightarrow$ has only eigenvalues ±1
    - Dimension must be even to obtain zero trace
    - In two dimensions only 3 independent traceless matrices: Pauli matrices σ_i, fourth matrix $\mathbb{I}$ is not traceless
    - four dimensions required

Representations of α_i and β

- **α_i and β matrices:**

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

($\sigma_i (i = 1,2,3)$ are the Pauli Matrices)

- **γ^μ matrices:** $\gamma^0 \equiv \beta \quad \gamma^i \equiv \beta\alpha_i$ ⇒ $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

compact notation of algebra

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$\mathbb{I}_4$	1 matrix
γ^μ	4 matrices
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	6 matrices
$\gamma^\mu \gamma^5$	4 matrices
$\gamma^5 \equiv \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$	1 matrix

- Basis of 4x4 matrices
- Orthonormal
- Traceless (apart from $\mathbb{I}_4$)

γ matrices

- Important tool for relativistic formulation of Dirac equation
  (NB: γ^μ is ***not a 4-vector*** but the same in each coordinate system)

$$\boxed{\begin{aligned} &\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) \\ &\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \\ &(\gamma^\mu)^\dagger \equiv (\gamma^\mu)^{T*} = \begin{cases} \gamma^0 \text{ for } \mu = 0 \\ -\gamma^\mu \text{ for } \mu = 1,2,3 \end{cases} \end{aligned}}$$

- γ matrices in common representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}$$

with Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad [\sigma_a, \sigma_b] = 2i\epsilon_{abc}$$

- Special combination: $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ with $\{\gamma^5, \gamma^0\} = 0$, $(\gamma^5)^2 = 1$
- Feynman dagger (‘d slash’): $\boxed{\partial\!\!\!/ = \gamma^\mu \partial_\mu}$

Dirac equation: Solution

- **Final formulation:** $$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0$$

- **Solutions:**

$$\psi_{+}(\vec{x}) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

$$\mathrm{E}(\vec{p}) = \sqrt{m^2+\vec{p}^2}$$ (free wave)

$$\psi_{-}(\vec{x}) = v(\vec{p})e^{-i(\vec{p}\vec{x}-Et)}$$

(at rest)

These are Dirac spinors

$$e^{\mp imt}\cdot\boxed{\begin{array}{ll} u_{\uparrow}(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & u_{\downarrow}(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ v_{\downarrow}(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} & v_{\uparrow}(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{array}}$$

+m solution

-m solution

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$$\mathrm{E}(\vec{p}) = \sqrt{m^2+\vec{p}^2}$$ (free wave)

(at rest)

Lorentz transformation $\Lambda : (m,0,0,0) \rightarrow (E, p_x, p_y, p_z)$

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$$\boxed{\begin{array}{ll} u_\uparrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}E+m\\0\\p_z\\p_x+ip_y\end{pmatrix} & u_\downarrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}0\\E+m\\p_x-ip_y\\-p_z\end{pmatrix} \\ v_\downarrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}p_z\\p_x+ip_y\\E+m\\0\end{pmatrix} & v_\uparrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}p_x-ip_y\\-p_z\\0\\E+m\end{pmatrix} \end{array}}$$

Time out

Transformation behaviour

- **Classification of physical objects according to transformation behaviour**
- **With Lorentz-Transformation Λ:**

(Lorentz-)Scalar:	$\Lambda: \quad m \rightarrow m' = m$
(Lorentz-)Vector:	$\Lambda: \quad x^\mu \rightarrow x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$
(Lorentz-)Tensor (2. order):	$\Lambda: \quad F^{\mu\nu} \rightarrow F^{\mu\nu\prime} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$
(Lorentz-)Spinor:	$\Lambda: \quad \psi^\alpha(x^\mu) \rightarrow \psi^{\alpha\prime}(x^{\mu\prime}) = S^\alpha_\beta \psi^\beta(\Lambda^\mu_\nu x^\nu)$

Dirac equation: Solution

- Dirac equation in covariant notation

$$\boxed{(i\partial\!\!\!/ - m) = 0}$$

  - Hermitian conjugate spinor: $\psi^\dagger = \psi^{T*} \rightarrow$ adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$
  - Adjoint Dirac equation: $i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = 0$ (apply $|\{\}^\dagger\gamma^0$ to Dirac eq.)
- ***Solution*** of Dirac equation for ***free particles***
  - Ansatz: ***plane wave*** $\psi(x) = \chi(p)\exp[\mp ipx]$ (later: interaction as small perturbation) $\longrightarrow$ Dirac equation in momentum space

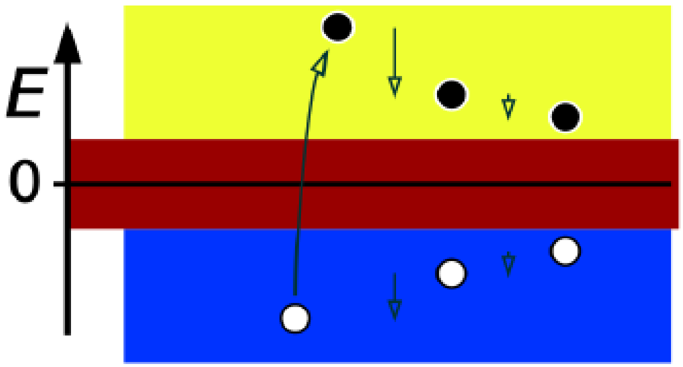
$$\begin{array}{ll} \text{positive energy} & (p\!\!\!/ - m)u(p) = 0 \\ \text{negative energy} & (p\!\!\!/ + m)v(p) = 0 \end{array} \quad \text{with} \quad p\!\!\!/ = \gamma^\mu p_\mu$$

  - Solutions with ***positive energy***: $\psi_{1,2} = u_{1,2}(p)\exp[-ipx]$
    - Particle with charge q and momentum $\vec{p}$
    - Spin $\vec{s}$ parallel ('up', u_1) and anti-parallel ('down', u_2)
  - Solutions with ***negative energy***: $\psi_{3,4} = v_{2,1}(p)\exp[ipx]$
    - Particle with charge q and momentum $-\vec{p}$
    - Spin $-\vec{s}$ parallel ('up', v_2) and anti-parallel ('down', v_1)

Dirac equation: Interpretation

- Interpretation as ***single-particle wave function***
  - *Problem*: negative-energy states → electrons can emit infinitely much energy via photons
- Solution: ***Dirac sea*** model
  - Ground state (‘vacuum’): all negative-energy states filled with electrons, following Pauli principle
    - No transitions from positive-energy state to negative-energy state
    - But the other way round: electron can be elevated from negative to positive-energy state

Dirac Sea



wikimedia.org

  - Hole in Dirac sea: ***anti particle*** with $-q$ but momentum $+\vec{p}$ and spin $+\vec{s}$
  - *Problems*: infinitely much charge, not applicable to bosons
- Solution (Feynman, Stückelberg): ***multi-particle system***
  - Requires ***quantised fields*** (‘2. quantisation’)
  - Particle with ***negative energy backward in time*** = anti-particle with ***positive energy forward in time***

Summary

- **Relativistic quantum mechanics incorporates relativistic energy-momentum relation:**

$$E^2 = p^2 + m^2$$

or

$$p^\mu p_\mu = E^2 - p^2 = m^2$$

- **Most important equations of motion:**
  - **Spin-0 particles (scalars): Klein-Gordon** $\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0$
  - **Spin-1/2 particles: Dirac** $\left(i\gamma^\mu \partial_\mu - m\right) \psi = 0$
  - **Spin-1 particles (vectors): Proca** $\left(\partial_\nu \partial^\nu + m^2\right) A^\mu = 0$

    for completeness

- **From canonical operator replacement:**

$$\boxed{\hat{E} \rightarrow i\frac{\partial}{\partial t} \qquad \hat{\vec{p}} \rightarrow -i\vec{\nabla}}$$

or

$$\boxed{\hat{p^\mu} \rightarrow i\partial^\mu}$$