Introduction to theoretical foundations II

(from experimenter's viewpoint)

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Summary

- **Relativistic quantum mechanics incorporates relativistic energy-momentum relation:**

$$E^2 = p^2 + m^2 \qquad \text{or} \qquad p^\mu p_\mu = E^2 - p^2 = m^2$$

- **Most important equations of motion:**
  - **Spin-0 particles (scalars): Klein-Gordon** $\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0$
  - **Spin-1/2 particles: Dirac** $\left(i\gamma^\mu \partial_\mu - m\right) \psi = 0$
  - **Spin-1 particles (vectors): Proca** $\left(\partial_\nu \partial^\nu + m^2\right) A^\mu = 0$

    for completeness

- **From canonical operator replacement:**

$$\hat{E} \rightarrow i\frac{\partial}{\partial t} \qquad \hat{\vec{p}} \rightarrow -i\vec{\nabla} \qquad \text{or} \qquad \hat{p^\mu} \rightarrow i\partial^\mu$$

Discrete symmetries

- Transformation of spinors under ***discrete symmetry operations C, P, T*** important in particle physics
  - ***Charge conjugation C***: particle → anti-particle

$$\psi(x) \rightarrow \psi'(x) = i\gamma^2\psi^*(x)$$

  - ***Parity P***: mirroring at origin $x = (t, \vec{x}) \rightarrow x' = (t, -\vec{x})$

$$\psi(x) \rightarrow \psi'(x') = i\gamma^0\psi(x)$$

  - ***Time reversal T***: $x = (t, \vec{x}) \rightarrow x' = (-t, \vec{x})$

$$\psi(x) \rightarrow \psi'(x') = i\gamma^1\gamma^3\psi(x)$$

- ***CPT theorem*** (Pauli, Lüders 1957):

Every locally Lorentz-invariant quantum-field theory is invariant under CPT symmetry

Observables from spinors

- Physical observables: ***bilinear forms of spinors***
- Classification by ***transformation behaviour*** under *C* and *P* transformations

Bilinear Form		***C***	***P***	***T***
$\overline{\psi}\psi$	scalar	+	+	+
$\overline{\psi}\gamma^5\psi$	pseudo-scalar	+	-	-
$\overline{\psi}\gamma^\mu\psi$	vector	-	γ^0: +, γ^i: -	γ^0: +, γ^i: -
$\overline{\psi}\gamma^\mu\gamma^5\psi$	axial-vector	+	γ^0: -, γ^i: +	γ^0: +, γ^i: -
$\overline{\psi}\Sigma^{\mu\nu}\psi$	tensor ($\Sigma^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu\gamma^\nu]$)	-	σ^{0j}: -, σ^{ij}: + σ^{0j}: -, σ^{ij}: +	σ^{0j}: +, σ^{ij}: + σ^{0j}: +, σ^{ij}: +

Helicity and chirality

- ***Helicity* $\boldsymbol{\lambda}$** ('direction of rotation')
  - Projection of spin onto unit vector in direction of momentum

$$\lambda = \vec{\Sigma} \cdot \frac{\vec{p}}{|\vec{p}|} \quad \text{with spin operator } \vec{\Sigma},\ \Sigma_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

  - Commutes with Hamilton operator: λ is ***'good' quantum number***
  - ***Not Lorentz-invariant***: for massive particles, there is always a reference frame in which momentum but not spin is in opposite direction
- ***Chirality*** ('handedness')
  - Important particle property in electroweak interaction
  - ***Eigenvalue of γ^5 operator*** (+: right handed, -: left handed)
  - u, v are chirality eigenstates: $\gamma^5 u = \pm u$, $\gamma^5 v = \pm v$
  - Any spinor can decomposed into left- and right-handed components

$$\psi = (\psi_R + \psi_L) \quad \text{with} \quad \psi_{R/L} = P_{R/L}\psi,\ P_{R/L} = \frac{1}{2}\left(1 \pm \gamma^5\right)$$

  - For ***massless*** particles: chirality = helicity

Classical mechanics

- **All information of a physical system is contained in the action**

$$S = \int \mathrm{d}t \, L(\vec{q}, \dot{\vec{q}}, t)$$

$\vec{q}$ generalised coordinates

Lagrange function L: $L = T - U = (E_{\mathrm{kin}} - E_{\mathrm{pot}})$

- **Equations of motion from principle of stationary action, dS = 0:**

$$\frac{d}{\mathrm{d}t}\frac{\partial L}{\partial(\partial \dot{q}_i)} - \frac{\partial L}{\partial q_i} = 0$$

Euler-Lagrange equations

Lagrange formalism

Relativistic quantum mechanics

- **All information is contained in the action integral**

$$\mathrm{S} = \int \mathrm{d}t \int \mathrm{d}^3 x \mathcal{L}(\phi(x), \partial_\mu \phi(x))$$

Field $\phi(x)$ here scalar:
Separate coordinate at each x
(generalisation of canonical coords.)

Lagrange density
('Lagrangian') : $\mathcal{L}$ $\quad T - U||_{\delta x} \equiv (E_{\mathrm{kin}} - E_{\mathrm{pot}})|_{\delta x}$

- **Equations of motion from principle of stationary action, dS = 0:**

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} - \frac{\partial \mathcal{L}}{\partial \phi(x)} = 0$$

Euler-Lagrange equations

Lagrangian: Examples

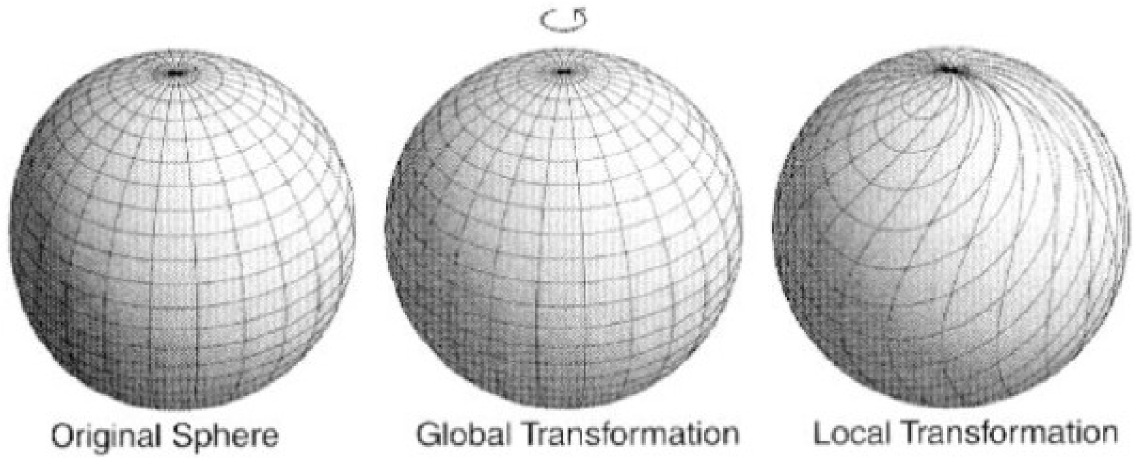
- **Lagrange densities for various fields of particles with mass m**

field	Lagrange density $\mathcal{L}$	equation of motion
scalar field $\phi(x)$ $(S = 0)$	$\frac{1}{2}\left[(\partial_\mu \phi)(\partial^\mu \phi)^* - m^2\phi^2\right]$	Klein-Gordon eq.
fermion field $\psi(x)$ $(S = \frac{1}{2})$	$\overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$	Dirac eq.
vector field $A_\mu(x)$ $(S = 1)$	$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$	Proca eq.

- $\mathcal{L}$ **has dimension GeV^{-1}**
- $\mathcal{L}$ **is a Lorentz scalar: Lorentz-invariant without 'free' indices μ**

Symmetries

- **Global and local symmetries**
  - **Global:** same everywhere
  - **Local:** varies with x



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- **Example: quantum-mechanical phase**
  - **Global:** physics unchanged for $\psi(x) \to U\,\psi(x) = \exp[i\alpha]\,\psi(x)$
  - **Local:** physics unchanged for $\psi(x) \to U(x)\,\psi(x) = \exp[i\alpha(x)]\,\psi(x)$
- **Connection to group theory**
  - Continuous transformations $\psi(x) \to U\psi(x)$ form Abelian group U(1) under multiplication: Group of unitary transformations
  - Abelian group: U commute, i.e. $[U_i, U_j] = 0$

Global phase transformations

- **Lagrangians of fermions and bosons are invariant**
  - **Global: The phase α = const is the same for any space-time point x**

- **Example: Lagrangian of free fermions**

$$\psi(x) \rightarrow \psi^{'}(x) = e^{i\alpha}\psi(x)$$

$$\overline{\psi}(x) \rightarrow \overline{\psi}^{'}(x) = \overline{\psi}(x)e^{-i\alpha}$$

- **Proof**

$$\mathcal{L}^{'} = \overline{\psi}^{'}(i\gamma^{\mu}\partial_{\mu} - m)\psi^{'}$$

$$= \overline{\psi}e^{-i\alpha}(i\gamma^{\mu}\partial_{\mu} - m)e^{i\alpha}\psi$$

$$= \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi = \mathcal{L}$$

Time out

Local phase transformations

- **Let's allow different phases at each point in space-time … α = α(x)?**

  → physics still should stay invariant!

taken from A. Quandt

Local phase transformations

- **But: Lagrangian is NOT invariant under local phase transformations α = α(x)!**

$$\psi(x) \to \psi^{'}(x) = e^{i\alpha(x)}\psi(x)$$

$$\overline{\psi}(x) \to \overline{\psi}^{'}(x) = \overline{\psi}(x)e^{-i\alpha(x)}$$

- **Proof:**

$$\mathcal{L}^{'} = \overline{\psi}^{'}(i\gamma^{\mu}\partial_{\mu} - m)\psi^{'}$$

$$= \overline{\psi}e^{-i\alpha(x)}(i\gamma^{\mu}\partial_{\mu} - m)e^{i\alpha(x)}\psi$$

$$= \overline{\psi}(i\gamma^{\mu}\left(\partial_{\mu} + i\partial_{\mu}\alpha(x)\right) - m)\psi \neq \mathcal{L}$$

Covariant derivative

- **Invariance can be restored at the cost of introducing artificially the covariant derivative $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ and an additional gauge field A_μ with transformation behaviour:**

$$\psi(x) \to \psi^{'}(x) = e^{i\alpha(x)}\psi(x)$$

$$\overline{\psi}(x) \to \overline{\psi}^{'}(x) = \overline{\psi}(x)e^{-i\alpha(x)}$$

$$A_\mu(x) \to A^{'}_\mu(x) = A_\mu(x) - \frac{1}{q}\partial_\mu\alpha(x)$$

- **Proof:**

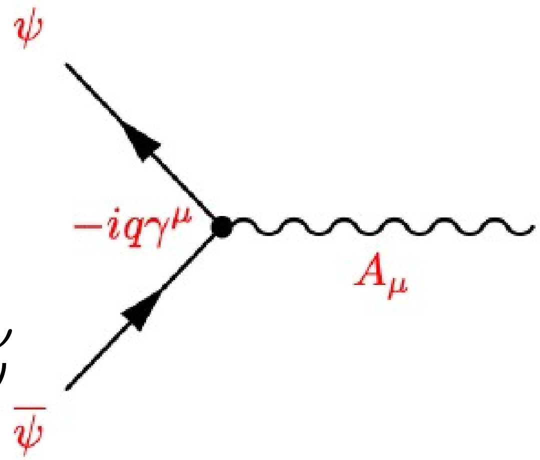
$$\begin{aligned}
\mathcal{L}^{'} &= \overline{\psi}^{'}(i\gamma^\mu D^{'}_\mu - m)\psi^{'} \\
&= \overline{\psi}^{'}(i\gamma^\mu \left(\partial_\mu + iqA^{'}_\mu\right) - m)\psi^{'} \\
&= \overline{\psi}e^{-i\alpha(x)}(i\gamma^\mu (\partial_\mu + iqA_\mu - i\partial_\mu\alpha(x)) - m)e^{i\alpha(x)}\psi \\
&= \overline{\psi}(i\gamma^\mu (\partial_\mu + i\partial_\mu\alpha(x) + iqA_\mu - i\partial_\mu\alpha(x)) - m)\psi \\
&= \overline{\psi}(i\gamma^\mu D_\mu - m)\psi = \mathcal{L}
\end{aligned}$$

The "artificial" gauge field

- Covariant derivative introduces gauge field A_μ
- Allows arbitrary phase α(x) of ψ(x)
  - Gauge field transports phase information from x to x' (no instantaneous information exchange)
- A_μ couples to property q of spinor field ψ(x)
  - q can be identified with electric charge

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$$\rightarrow \underbrace{\overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi}_{\text{free fermion}} - \underbrace{q(\overline{\psi}\gamma^\mu\psi)A_\mu}_{\text{interaction}} - \underbrace{\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{photon}}$$



- A_μ can be identified with photon field
- Dynamics of A_μ given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{q}[D_\mu, D_\nu]$

$\mathrm{L}_{\mathrm{kin}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ → Proca equation for massless vector boson

Lagrange density of QED

- **Postulation of local U(1) gauge symmetry → Lagrangian of QED**

$$\mathcal{L}_{\mathrm{QED}} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$= \underbrace{\overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi}_{\text{free fermion}} - \underbrace{q(\overline{\psi}\gamma^\mu\psi)A_\mu}_{\text{interaction}} - \underbrace{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{photon gauge field}}$$

- **Euler-Lagrange eq. for $\overline{\psi}$:** $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0$

  - **Dirac equation for interacting fermion**

$$\rightarrow (i\gamma^\mu\partial_\mu - m)\psi = q\gamma^\mu A_\mu \psi$$

Lagrange density of QED

- **Postulation of local U(1) gauge symmetry → Lagrangian of QED**

$$\mathcal{L}_{\mathrm{QED}} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

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- **Euler-Lagrange eq. for A_ν** $\quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial_\mu F^{\mu\nu} = 0$$

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \left(\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu\right) = 0$$

$$\partial_\mu A^\mu = 0$$

Lorenz gauge

  - **Proca equation for massless vector boson**

$$\rightarrow \left(\partial_\mu \partial^\mu - 0\right) A^\nu = 0$$

Lagrange density of QED

- **Postulation of local U(1) gauge symmetry → Lagrangian of QED**

$$\mathcal{L}_{\mathrm{QED}} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$= \underbrace{\overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi}_{\text{free fermion}} - \underbrace{q(\overline{\psi}\gamma^\mu\psi)A_\mu}_{\text{interaction}} - \underbrace{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{photon gauge field}}$$

- **Electromagnetic interaction derived as a consequence of local gauge invariance**
- **Lagrangian must not have boson mass term** $m^2 A_\mu A^\mu$
  - **massive gauge bosons break local gauge invariance**

Lagrange density → Observable?

- ***Process rate*** given by (cf. Fermi's gold rule)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{|\text{matrix element}|^2}{\text{flux of incoming particles}} \cdot \text{phase space}$$

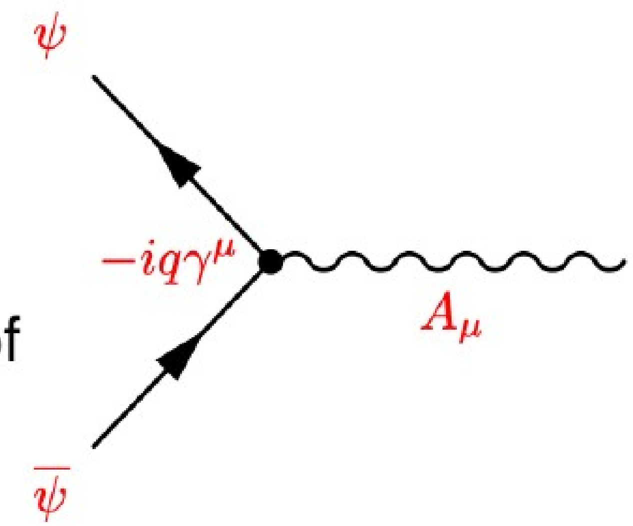
- ***Dynamics*** of process encoded in ***matrix element***
  - Element of scattering matrix that transforms initial state into outgoing state ('scattered wave')
- From Lagrange density: rules how to compute matrix element in perturbation theory → ***Feynman rules***
  - Graphical representation: ***Feynman graphs***

Perturbative series

- Process dynamics given by ***matrix element***

$$\mathcal{M}_{fi} = \psi_f^\dagger \psi_{\text{scat}} = \psi_f^\dagger \mathcal{S} \psi_i$$

  - ψ_f: final state after scattering of initial state ψ_i



- E.g. fermion scattering: ψ_{scat} given by solution of inhomogeneous Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi_{\text{scat}} = -e\gamma^\mu A_\mu \psi_{\text{scat}}$$

Cannot be solved analytically!

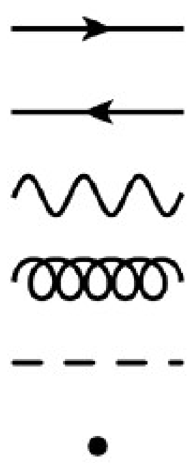
- Possible to ***expand solution*** in orders of coupling constant $\alpha \equiv e^2 \ll 1$

$$\psi_{\text{scat}} = \mathcal{S}\psi_i = \left[\sum_{n=0}^{\infty} \alpha^n S_n\right] \psi_i$$

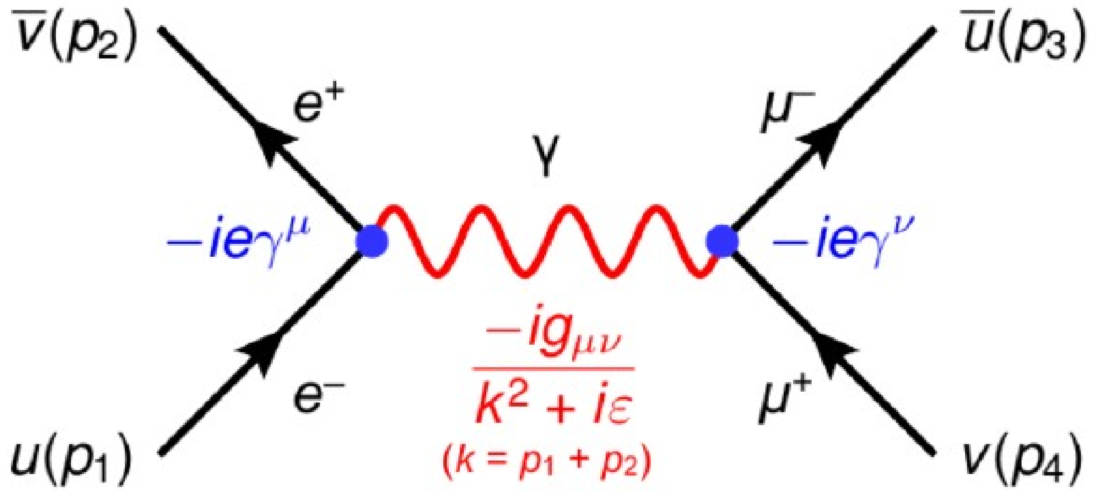
  - Each term in ***perturbation series*** associated with distinct process
  - S_n can be computed with ***Feynman rules***

Feynman rules

- Elements of Feynman rules
  - External lines: incoming/outgoing particles
  - Vertices: coupling between particles
  - Propagators (=internal lines): exchange of virtual particles during scattering process (Green's function of free field equation in momentum space)



- For example, $e^+e^- \to \mu^+\mu^-$ scattering



$$\mathcal{M} = \overline{v}(p_2)(-ie\gamma^\mu)u(p_1)\frac{-ig_{\mu\nu}}{(p_1+p_2)^2 + i\varepsilon}\overline{u}(p_3)(-ie\gamma^\nu)v(p_4)$$

Summary

- Symmetries are a basic principle of physics
- Principle of local gauge invariance
  - Postulate of local gauge invariance of the Lagrange density
    → leads to interaction terms with gauge bosons as mediators
- QED: Symmetry under U(1) gauge transformation → photon exchange
- Cross section from Lagrange density
  - Fermi's golden rule:
    matrix element squared X phase space → cross section
  - Feynman rules:
    - set of rules how to calculate matrix elements
    - can be read off Lagrange density (at leading order …)
    - represented by Feynman graphs

Time out

The standard model so far

- Postulation of local U(1) gauge symmetry → Lagrangian of QED

$$\mathcal{L}_{\rm QED} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$= \underbrace{\overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi}_{\text{free fermion}} - \underbrace{q(\overline{\psi}\gamma^\mu\psi)A_\mu}_{\text{interaction}} - \underbrace{\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{photon gauge field}}$$

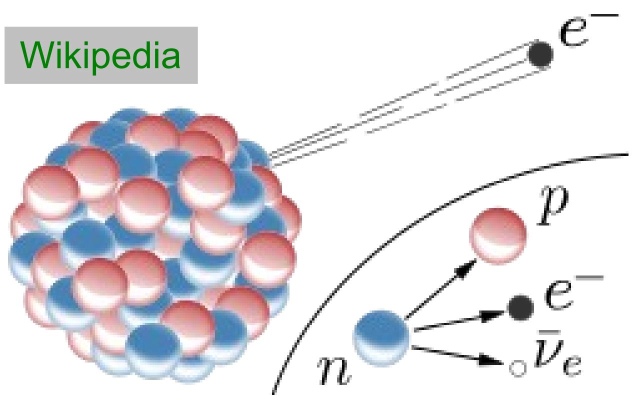
- We have fermions and (massless) vector bosons
- We have a conserved quantum number q identifiable with the electric charge
- We have electromagnetic interactions

→ Quantum version of Maxwell's unified theory of electricity and magnetism

Fermi's four-fermion coupling

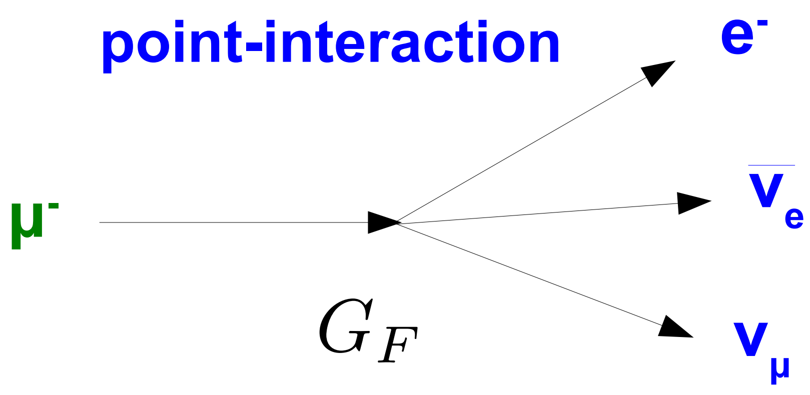
β-decay

Wikipedia



μ-decay

point-interaction



Versuch einer Theorie der β-Strahlen. I[1]).

Von **E. Fermi** in Rom.

Mit 3 Abbildungen. (Eingegangen am 16. Januar 1934.)

Eine quantitative Theorie des β-Zerfalls wird vorgeschlagen, in welcher man die Existenz des Neutrinos annimmt, und die Emission der Elektronen und Neutrinos aus einem Kern beim β-Zerfall mit einer ähnlichen Methode behandelt, wie die Emission eines Lichtquants aus einem angeregten Atom in der Strahlungstheorie. Formeln für die Lebensdauer und für die Form des emittierten kontinuierlichen β-Strahlenspektrums werden abgeleitet und mit der Erfahrung verglichen.

Fermi, Z. Phys., 1934, 88, 16; Nuovo Cim., 1934, 11, 1

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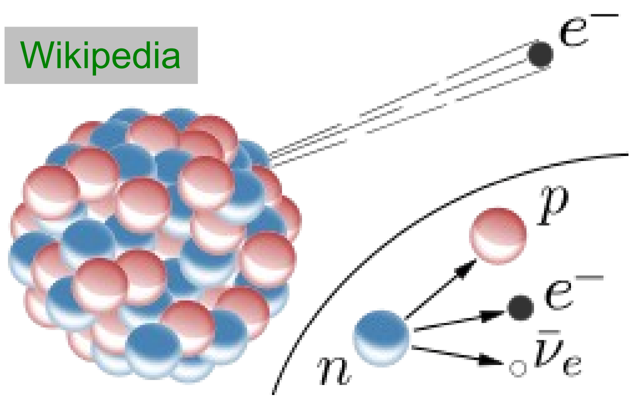

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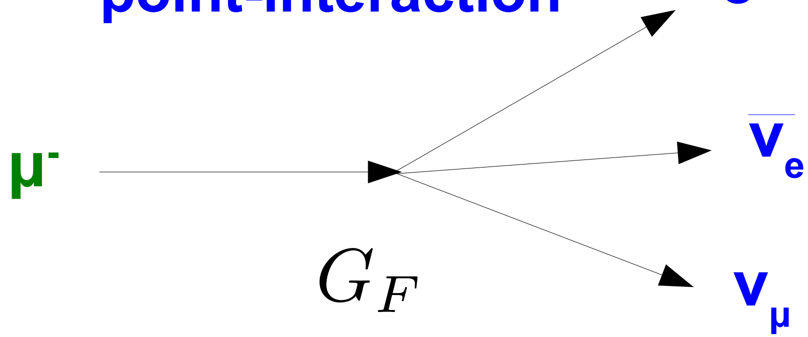
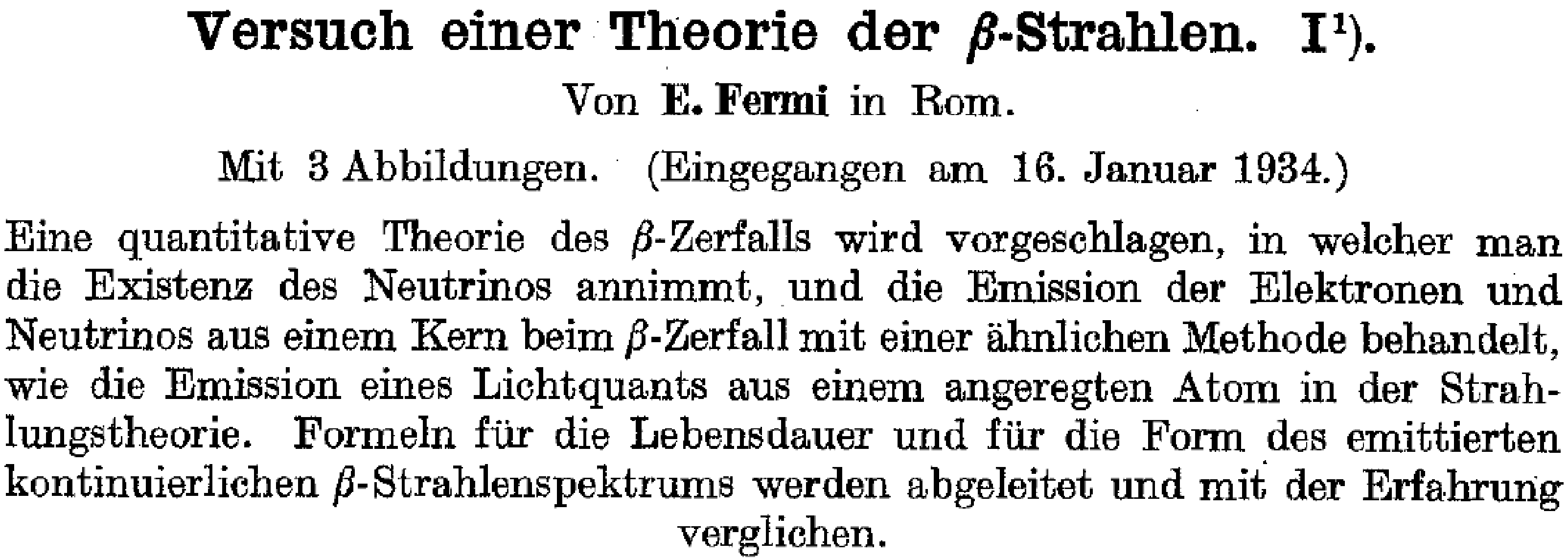


First publikation declined by "Nature" as too speculative

→ Appeared first in German and Italian!



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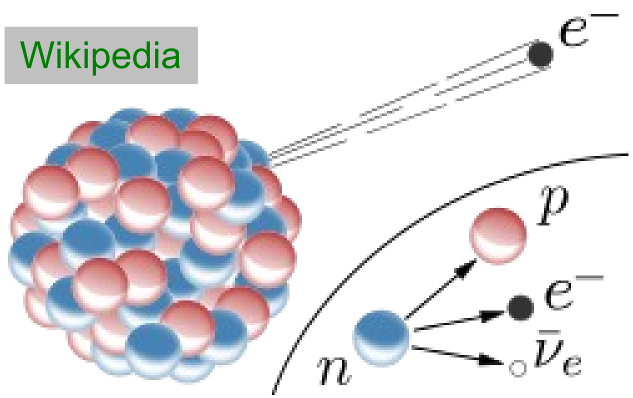
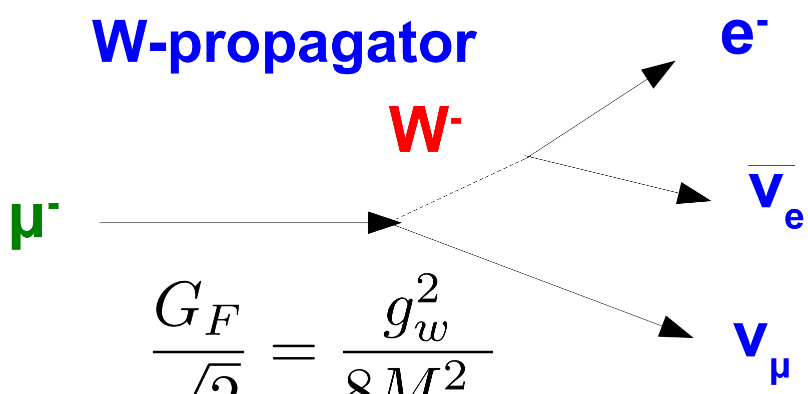


μ-decay

W-propagator



$$\frac{G_F}{\sqrt{2}} = \frac{g_w^2}{8M_W^2}$$

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