Exercise sheet 2. Theoretical Nanooptics. WS 2021/2022

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1 True or false?

State whether the following statements are true or false and write a short reasoning for your answer.

a) (1 Point) The distinction between $\mathbf{E}$ and $\mathbf{H}$ is relativistically invariant.

b) (1 Point) The distinction between $\mathbf{G}_+$ and $\mathbf{G}_-$ is relativistically invariant.

c) (1 Point) The largest group for which the distinction between $\mathbf{G}_+$ and $\mathbf{G}_-$ is invariant is the Poincaré group of relativistic transformations.

d) (1 Point) The operators $\frac{\nabla\times}{\omega}$ and $\mathrm{i}\hat{\mathbf{k}}\times$ have different physical meanings.

e) (1 Point) The operators $\frac{\nabla\times}{\omega}$ and $\mathrm{i}\hat{\mathbf{k}}\times$ apply to the same representation of fields.

2 Hilbert space of solutions of Maxwell's equations in vacuum and the photon wave function

In the lecture the Hilbert space of solutions of Maxwell's equations $\mathbb{M}$ was introduced, where any vector $|\phi\rangle \in \mathbb{M}$ can be decomposed into positive and negative helicity parts

$$|\phi\rangle = |G_+\rangle + |G_-\rangle \,.$$

The scalar product in $\mathbb{M}$ can be defined as

$$\langle\phi'|\phi\rangle = \int \frac{d^3k}{|\boldsymbol{k}|} \begin{bmatrix} \boldsymbol{G}'_+(\boldsymbol{k}) \\ \boldsymbol{G}'_-(\boldsymbol{k}) \end{bmatrix}^\dagger \begin{bmatrix} \boldsymbol{G}_+(\boldsymbol{k}) \\ \boldsymbol{G}_-(\boldsymbol{k}) \end{bmatrix} . \tag{1}$$

Using [1, Eq. (17)] we can introduce the basis of circularly polarized plane waves, such that any vector $|\phi\rangle \in \mathbb{M}$ can be represented as

$$\begin{aligned} |\phi\rangle &= \int \frac{d^3k}{|\boldsymbol{k}|}\, \phi_+(\boldsymbol{k})\, |\boldsymbol{k}\, +\rangle + \int \frac{d^3k}{|\boldsymbol{k}|}\, \phi_-(\boldsymbol{k})\, |\boldsymbol{k}\, -\rangle \\ &= \sum_{\lambda=\pm1} \int \frac{d^3k}{|\boldsymbol{k}|}\, \phi_\lambda(\boldsymbol{k})\, |\boldsymbol{k}\, \lambda\rangle\,. \end{aligned}$$

a) (1 Point) Write $\phi_\lambda(\boldsymbol{k})$ as function of $\boldsymbol{G}_\lambda(\boldsymbol{k})$.

b) (1 Point) Prove that the scalar product (1) is equivalent to

$$\langle\phi'|\phi\rangle = \sum_{\lambda=\pm1} \int \frac{d^3k}{|\boldsymbol{k}|}\, \phi'^*_\lambda(\boldsymbol{k})\phi_\lambda(\boldsymbol{k}).$$

c) (3 Points) Prove that the scalar product in coordinate representation is

$$\langle\phi'|\phi\rangle = \frac{1}{2\pi^2} \sum_{\lambda=\pm1} \iint \frac{d^3\boldsymbol{r}d^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\, \phi'^*_\lambda(\boldsymbol{r}')\phi_\lambda(\boldsymbol{r}), \tag{2}$$

where

$$\phi(\boldsymbol{r}) := \int \frac{d^3k}{\sqrt{(2\pi)^3}} \phi(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{3}$$

Hint: prove that

$$\int_{|\boldsymbol{k}|<R} \frac{d^3k}{|\boldsymbol{k}|}\, e^{i\boldsymbol{k}\cdot\boldsymbol{a}} = 4\pi \frac{1-\cos(R|\boldsymbol{a}|)}{|\boldsymbol{a}|^2} \tag{4}$$

and apply the Riemann-Lebesgue lemma.

3 Electromagnetic angular momentum: Orbital and Spin ?

As of 16/12/20, you can read the following statement in the Wikipedia article devoted to the angular momentum of light https://en.wikipedia.org/wiki/Angular_momentum_of_light:

There are two distinct forms of rotation of a light beam, one involving its polarization and the other its wavefront shape. These two forms of rotation are therefore associated with two distinct forms of angular momentum, respectively named light spin angular momentum (SAM) and light orbital angular momentum (OAM).

This exercise is a critical analysis of such statement.

We will work in $\mathbb{M}$, the Hilbert space of Maxwell fields in vacuum, and use the representation

$$|G\rangle = \begin{bmatrix} \mathbf{G}_+(\mathbf{r},t) \\ \mathbf{G}_-(\mathbf{r},t) \end{bmatrix} \tag{5}$$

for vectors $|G\rangle \in \mathbb{M}$.

In this representation, the expression of the operator for the z-component of angular momentum is

$$J_z \equiv \begin{bmatrix} J_z^+ & 0 \\ 0 & J_z^- \end{bmatrix}, \tag{6}$$

where

$$J_z^+ = J_z^- = \underbrace{\begin{pmatrix} -i\partial_\psi & 0 & 0 \\ 0 & -i\partial_\psi & 0 \\ 0 & 0 & -i\partial_\psi \end{pmatrix}}_{L_z} + \underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{S_z}, \tag{7}$$

where $\psi = \arctan(y, x)$. The matrix operators in Eq. (7) act on the 3-vector of cartesian components $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ of the $\mathbf{G}_\pm(\mathbf{r},t)$ fields. Formally, the first matrix is the z-component of the orbital angular momentum operator L_z, and the second one the z-component of the spin angular momentum operator S_z.

a) (3 Points) Choose a plane-wave whose wavevector **is not** $\mathbf{k} = [0, 0, \pm|\mathbf{k}|]$. You can use [1, Eq. (17)]. Make it simple for yourself and choose a pure helicity plane wave, either $\lambda = +1$ or $\lambda = -1$. Check that you have chosen the polarization vector $\hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}})$ correctly by verifying the transversality of the field: That is $\mathbf{k} \cdot \hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}}) = 0$, or $\nabla \cdot \mathbf{G}_\lambda(\mathbf{r},t) = 0$.

b) (1 Point) Apply S_z to your plane-wave and

c) (1 Points) check its transversality again. What has happened? Does the resulting object belong to $\mathbb{M}$? Why?

d) (1 Point) We know that applying J_z to the plane-wave would keep it in $\mathbb{M}$. What can you say about the result of applying L_z to the plane-wave?

e) (3 Points) Repeat now the previous three calculations for $\mathbf{k} = [0, 0, |\mathbf{k}|]$. This time apply L_z explicitly as well. Discuss the results.

f) (1 Point) What can you say about the statement in the Wikipedia article?

g) (1 Point) What are the symmetries **separately** generated by L_z and S_z?

4 Generator of time translations

In this exercise, you will identify the expression of the generator of translations for functions of time $f : \mathbb{R} \to \mathbb{R}$. First, consider what the action of a time translation is, namely $f(t) \to f(t - \tau)$. You will achieve the goal of the exercise by writing such transformation as $\exp(-i\tau\Gamma)f(t)$, that is, the operator $\exp(-i\tau\Gamma)$ acting on $f(t)$. The goal of the exercise is to obtain an explicit expression for Γ. *Hint: There is a very famous expansion of functions which you can use, and then manipulate the result to get it looking like* $\exp(-i\tau\Gamma)f(t)$. (3 Points)

References

[1] I. Fernandez-Corbaton. *An alternative starting point for electromagnetism.* Lecture Notes (2019).