Exercise sheet 3. Theoretical Nanooptics. WS 2020/2021

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1 True or false?

State whether the following statements are true or false and write a short reasoning for your answer.

a) Consider $X(\theta) = \exp(-i\theta\Gamma)$, where Γ is an operator and θ a real number:

   i) (1 Point) There are eigenstates of Γ that are not eigenstates of $X(\theta)$.

   ii) (1 Point) $X(\theta)$ is always unitary.

   iii) (1 Point) When Γ is self-adjoint ($\Gamma^H = \Gamma$, also often written $\Gamma^\dagger = \Gamma$), it is possible to find a vector $|G\rangle$ such that $\langle G|G\rangle \neq \langle G|X(\theta)^H X(\theta)|G\rangle$.

b) Consider the transition of a two level system. In units of $\hbar = 1$, the difference between the energies of the two levels is ω_0, and the difference between their angular momentum is 1:

   i) (1 Point) Any light beam with average energy equal to ω_0 can induce the transition

   ii) (1 Point) Any light beam with average angular momentum equal to 1 can induce the transition

   iii) (1 Point) Any light beam with average energy equal to ω_0 and average angular momentum equal to 1 can induce the transition

c) (1 Point) The algebraic tools that we develop in class (inner products, average of operators, etc ...) can be used to treat the physics of single photons

d) (1 Point) In classical electromagnetism, we cannot understand the outcomes of measurements using inner products

e) (1 Point) $\mathbb{M}$, the Hilbert space of Maxwell solutions in vacuum, is a useless construct for the study of light-matter interactions

2 Symmetry and commutators

Consider S and $X(\theta)$, two linear operators which map elements of $\mathbb{M}$ onto elements in $\mathbb{M}$. $X(\theta)$ is a unitary transformation corresponding to a continuous symmetry, which is generated by a Hermitian operator Γ: $X(\theta) = \exp(-\mathrm{i}\theta\Gamma)$ for real θ. Consider the two following equations

$$\begin{aligned} &\text{A: } X(\theta)SX(\theta)^{-1} = S \; \forall \; \theta \\ &\text{B: } [S, \Gamma] = 0 \end{aligned} \tag{1}$$

a) (1 Point) Show that B $\Longrightarrow$ A

b) (2 Points) Show that A $\Longrightarrow$ B

3 Discrete Translational Invariance

In this exercise, you will investigate some of the consequences of discrete translational invariance. Such kind of symmetry appears, for example, in natural and artificial crystals. Before we start, here are some preliminary short questions:

a) Consider a general plane-wave with momentum $\mathbf{k} = [k_x, k_y, k_z]$:

   i) (1 Point) Is the plane-wave an eigenvector of the linear momentum operator along the y-axis, P_y?

   ii) (1 Point) If the answer is yes, what is the eigenvalue?

   iii) (1 Point) What is the result of $P_y|\mathbf{k}\rangle = ?$

   iv) (1 Point) What is the result of a translation by Δ along the y-axis? $T_y(\Delta)|\mathbf{k}\rangle = ?$

b) (1 Point) What is the angular frequency ω of the plane-wave? *(Recall that we set the speed of light to c_0=1)*

c) The frequency ω is the eigenvalue of a generator:

   i) (1 Point) Write down two names for such generator.

   ii) (1 Point) What kind of transformations does it generate?

   iii) (1 Point) What does a time-invariant system not mix?

d) With our conventions, it is always true that, for a plane-wave $k_x^2 + k_y^2 + k_z^2 = \omega^2$, and that ω is a positive real number.

   i) (1 Point) Assume that $k_x = 0$ and that k_y is real. What needs to happen with k_z if $|k_y| > \omega$?

ii) (1 Point) This kind of plane-waves are called "evanescent". How does such an evanescent plane-wave depend on z?

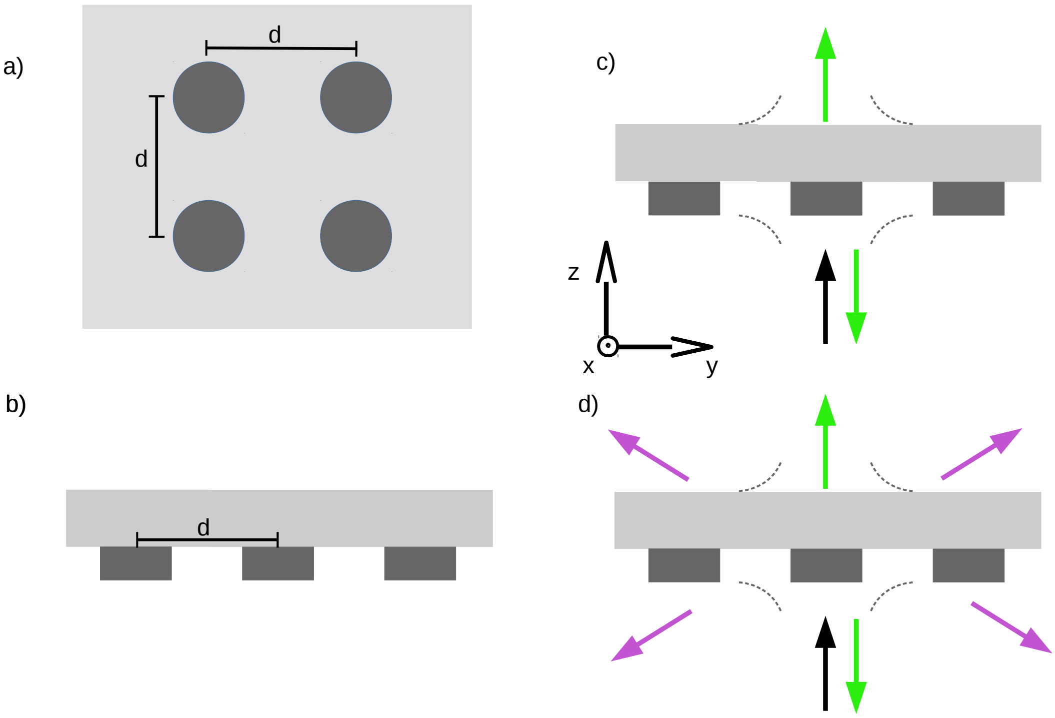

Figure 1: Array of silicon disks.

Look now at Fig. 1. Consider first the a) and b) panels. They show two different views of an array made of silicon disks on top of a glass substrate. The disks are arranged in a "square"-symmetric way (see Fig. 1a)). The coordinate axes at the center of the figure apply to the array views in panels b), c), and d). The shortest distance between the centers of two disks is d. Both the substrate and the disk arrangement extend to infinity in the x and y directions. The system is invariant under the following translations: $T_x(nd)$ and $T_y(md)$, where n and m are any integer number. In panels c) and d), the black arrow represents the plane-wave whose momentum lies along the z-axis: $\mathbf{k} = [0, 0, 2\pi]$. This plane-wave illuminates the system, and we label it $|0\ 0\ 2\pi\rangle$. We ignore its polarization for now. The green and magenta arrows in panels c) and d) represent propagating (non-evanescent) plane-waves that arise upon light-matter interaction. The dotted curves in panels c) and d) represent evanescent plane-waves that arise upon light-matter interaction.

e) (1 Point) The response of the array does not change with time: What will be the angular frequency $\bar{\omega}$ of any outgoing field?

f) (1 Point) Write down the formula that the discrete translational symmetry $T_y(md)$ implies for the scattering operator S of the system. *Hint: Use the formula that constitutes the basic definition of symmetry from the lectures.*

g) (2 Points) Consider the $m = 1$ case in $T_y(md)$. **Derive** the conditions imposed on $\bar{k}_y$ by this symmetry so that the scattering amplitude $\langle \bar{k}_x \; \bar{k}_y \; \bar{k}_z | S | 0 \; 0 \; 2\pi \rangle$ is not equal to zero.

The answer to the previous question is $\bar{k}_y = \frac{2\pi}{d} s_y$ for integer s_y. Similarly, one can derive from the $T_x(d)$ symmetry that $\bar{k}_x = \frac{2\pi}{d} s_x$ for integer s_x.

h) (1 Point) Write down a formula for the value of $\bar{k}_z^2$ as a function of d, s_x and s_y

i) The case $(s_x, s_y) = (0, 0)$ is often called "zeroth diffraction order".

  i) (1 Point) What color corresponds to the "zeroth diffraction order" in panels c) and d)?

  ii) (1 Point) How many plane-waves in each panel correspond to the "zeroth diffraction order"?

Consider the case $(s_x = 0, s_y = 1)$, which is a "first diffraction order". The first order(s) can be evanescent like in panel c) or propagating like in panel d)

j) (1 Point) What does the periodicity d have to meet so that the scattering behavior looks like panel c) and **not** panel d)?

Let us change gears a bit to finish up.

k) (2 Points) What other geometrical symmetries does the array have? Consider discrete rotations and mirror symmetries.

l) (1 Point) Assume that the illuminating plane-wave is linearly polarized along the x-axis: What is the polarization of the green plane-waves in panel c)? Why?

4 Continuous Translational Invariance

Consider a thin dielectric slab, described by S, which is infinite in the x- and y-directions and is surrounded by free space. The fields in the free space outside of the slab can be described by eigenvectors of the translation operator $|\mathbf{k}\rangle$ (arbitrary polarization is implied). For a given incoming state $|\mathbf{k}\rangle$ we wish to find properties of the outgoing states $|\mathbf{k}'\rangle$ using the symmetry of the problem. Fields inside the slab are not of interest. Apply the S-operator formalism and the symmetry arguments to complete the following tasks:

a) (2 Points) The quantity $\langle \mathbf{k}' | S | \mathbf{k} \rangle$ is the probability amplitude of the given incoming state $|\mathbf{k}\rangle$ to be scattered into a state $|\mathbf{k}'\rangle$. Use the translational symmetry of the scatterer to derive the corresponding conservation law.

b) (1 Point) Is the state $|\mathbf{k}\rangle$ an eigenvector of the Hamiltonian operator H? If yes, give its eigenvalue.

c) (1 Point) Assume that the scatterer conserves energy and derive the corresponding conservation quantity.

d) (2 Points) Given an incoming state $|\mathbf{k}\rangle$, what are the scattered states $|\mathbf{k}'\rangle$ that are allowed by the conservation constraints? Give their physical interpretation.

Now consider two semi-infinite spaces, joined at a flat interface, that are characterized by refractive indices n and n'. A plane wave with momentum $\mathbf{k}$ is refracted into a plane wave with momentum $\mathbf{k}'$. *Without* using the S-operator formalism but using symmetry considerations derive the Snell's law:

e) (1 Point) How does the frequency of the plane wave change under refraction?

f) (2 Points) How does the momentum change under refraction?

g) (2 Points) Derive the Snell's law.