

# Exercise sheet 2. Theoretical Nanooptics. WS 2022/2023

Karlsruhe Institute of Technology  
Ivan Fernandez-Corbaton and Maxim Vavilin

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## 1 True or false?

State whether the following statements are true or false and write a short reasoning for your answer.

- a) (1 Point) The distinction between  $\mathbf{E}$  and  $\mathbf{H}$  is relativistically invariant.
- b) (1 Point) The distinction between  $\mathbf{G}_+$  and  $\mathbf{G}_-$  is relativistically invariant.
- c) (1 Point) The largest group for which the distinction between  $\mathbf{G}_+$  and  $\mathbf{G}_-$  is invariant is the Poincaré group of relativistic transformations.
- d) (1 Point) The operators  $\frac{\nabla \times}{\omega}$  and  $i\hat{\mathbf{k}} \times$  have different physical meanings.
- e) (1 Point) The operators  $\frac{\nabla \times}{\omega}$  and  $i\hat{\mathbf{k}} \times$  apply to the same representation of fields.

## 2 Hilbert space of solutions of Maxwell's equations in vacuum and the photon wave function

In the lecture the Hilbert space of solutions of Maxwell's equations  $\mathbb{M}$  was introduced, where any vector  $|\phi\rangle \in \mathbb{M}$  can be decomposed into positive and negative helicity parts

$$|\phi\rangle = |G_+\rangle + |G_-\rangle.$$

The scalar product in  $\mathbb{M}$  can be defined as

$$\langle\phi'|\phi\rangle = \int \frac{d^3k}{|\mathbf{k}|} \begin{bmatrix} \mathbf{G}'_+(\mathbf{k}) \\ \mathbf{G}'_-(\mathbf{k}) \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{G}_+(\mathbf{k}) \\ \mathbf{G}_-(\mathbf{k}) \end{bmatrix}. \quad (1)$$

Using [1, Eq. (17)] we can introduce the basis of circularly polarized plane waves, such that any vector  $|\phi\rangle \in \mathbb{M}$  can be represented as

$$|\phi\rangle = \sum_{\lambda=\pm 1} \int \frac{d^3k}{|\mathbf{k}|} \phi_\lambda(\mathbf{k}) |\mathbf{k}, \lambda\rangle.$$

a) (1 Point) Write  $\phi_\lambda(\mathbf{k})$  as function of  $\mathbf{G}_\lambda(\mathbf{k})$ , where

$$\mathbf{G}_\lambda(\mathbf{r}, t) = \int d^3k \mathbf{G}_\lambda(\mathbf{k}) e^{-ikt + i\mathbf{k}\cdot\mathbf{r}}.$$

b) (1 Point) Prove that the scalar product (1) is equivalent to

$$\langle \phi' | \phi \rangle = \sum_{\lambda=\pm 1} \int \frac{d^3k}{|\mathbf{k}|} \phi'_\lambda{}^*(\mathbf{k}) \phi_\lambda(\mathbf{k}).$$

c) (3 Points) Prove that the scalar product in coordinate representation is

$$\langle \phi' | \phi \rangle = \frac{4\pi}{(2\pi)^6} \sum_{\lambda=\pm 1} \iint \frac{d^3\mathbf{r} d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \phi'_\lambda{}^*(\mathbf{r}') \phi_\lambda(\mathbf{r}), \quad (2)$$

where

$$\phi(\mathbf{r}) := \int d^3k \phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (3)$$

Hint: prove that

$$\int_{|\mathbf{k}| < R} \frac{d^3k}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{a}} = 4\pi \frac{1 - \cos(R|\mathbf{a}|)}{|\mathbf{a}|^2} \quad (4)$$

and apply the Riemann-Lebesgue lemma.

### 3 Electromagnetic angular momentum: Orbital and Spin ?

As of 16/12/20, you can read the following statement in the Wikipedia article devoted to the angular momentum of light [https://en.wikipedia.org/wiki/Angular\\_momentum\\_of\\_light](https://en.wikipedia.org/wiki/Angular_momentum_of_light):

*There are two distinct forms of rotation of a light beam, one involving its polarization and the other its wavefront shape. These two forms of rotation are therefore associated with two distinct forms of angular momentum, respectively named light spin angular momentum (SAM) and light orbital angular momentum (OAM).*

This exercise is a critical analysis of such statement.

We will work in  $\mathbb{M}$ , the Hilbert space of Maxwell fields in vacuum, and use the representation

$$|G\rangle = \begin{bmatrix} \mathbf{G}_+(\mathbf{r}, t) \\ \mathbf{G}_-(\mathbf{r}, t) \end{bmatrix} \quad (5)$$

for vectors  $|G\rangle \in \mathbb{M}$ .

In this representation, the expression of the operator for the  $z$ -component of angular momentum is

$$J_z \equiv \begin{bmatrix} J_z^+ & 0 \\ 0 & J_z^- \end{bmatrix}, \quad (6)$$

where

$$J_z^+ = J_z^- = \underbrace{\begin{pmatrix} -i\partial_\psi & 0 & 0 \\ 0 & -i\partial_\psi & 0 \\ 0 & 0 & -i\partial_\psi \end{pmatrix}}_{L_z} + \underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{S_z}, \quad (7)$$

where  $\psi = \arctan(y, x)$ . The matrix operators in Eq. (7) act on the 3-vector of cartesian components  $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$  of the  $\mathbf{G}_\pm(\mathbf{r}, t)$  fields. Formally, the first matrix is the  $z$ -component of the orbital angular momentum operator  $L_z$ , and the second one the  $z$ -component of the spin angular momentum operator  $S_z$ .

- a) (3 Points) Choose a plane-wave whose wavevector is **not**  $\mathbf{k} = [0, 0, \pm|\mathbf{k}|]$ . You can use [1, Eq. (17)]. Make it simple for yourself and choose a pure helicity plane wave, either  $\lambda = +1$  or  $\lambda = -1$ . Check that you have chosen the polarization vector  $\hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}})$  correctly by verifying the transversality of the field: That is  $\mathbf{k} \cdot \hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}}) = 0$ , or  $\nabla \cdot \mathbf{G}_\lambda(\mathbf{r}, t) = 0$ .
- b) (1 Point) Apply  $S_z$  to your plane-wave and
- c) (1 Points) check its transversality again. What has happened? Does the resulting object belong to  $\mathbb{M}$ ? Why?
- d) (1 Point) We know that applying  $J_z$  to the plane-wave would keep it in  $\mathbb{M}$ . What can you say about the result of applying  $L_z$  to the plane-wave?
- e) (3 Points) Repeat now the previous three calculations for  $\mathbf{k} = [0, 0, |\mathbf{k}|]$ . This time apply  $L_z$  explicitly as well. Discuss the results.
- f) (1 Point) What can you say about the statement in the Wikipedia article?
- g) (1 Point) What are the symmetries **separately** generated by  $L_z$  and  $S_z$ ?

## References

- [1] I. Fernandez-Corbaton. *An alternative starting point for electromagnetism*. Lecture Notes (2019).