# Exercise sheet 4. Theoretical Nanooptics. WS 2022/2023

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## 1 True or false?

State whether the following statements are true or false and write a short reasoning for your answer.

a) (1 Point) Duality transformation commutes with rotations around any axis.

b) Consider the state $|\Phi\rangle = |\mathbf{k}\ +\rangle + |\mathbf{k}\ -\rangle$:

   i) (1/2 Point) $|\Phi\rangle$ is an eigenstate of the helicity operator.

   ii) (1/2 Point) $|\Phi\rangle$ is an eigenstate of the parity operator.

   iii) (1 Point) $|\Phi\rangle$ can be an eigenstate of infinitely many mirror reflection operators.

c) (1/2 Point) A rotation commutes with all mirror reflections.

d) (1/2 Point) A rotation never commutes with a mirror reflection.

e) (1/2 Point) Parity followed by a mirror reflection anti-commutes with the helicity operator.

f) Consider a dual-symmetric object with a discrete rotational symmetry $R_z\left(\frac{2\pi}{4}\right)$

   (a) (1/2 Point) The illumination $|k\hat{\mathbf{z}}\ +\rangle + |2k\hat{\mathbf{z}}\ -\rangle$ produces some backscattering.

   (b) (1 Point) Such an object makes a good light sail, where a laser pushes the object away by means of the optical force.

## 2 Generator of time translations (3 Points)

In this exercise, you will identify the expression of the generator of time translations for functions $f(t) \in \mathbb{R}$. First, consider what the action of a time translation is, namely $f(t) \to f(t-\tau)$. You will achieve the goal of the exercise by writing such transformation as $\exp(-i\tau\Gamma)f(t)$, that is, the operator $\exp(-i\tau\Gamma)$ acting on $f(t)$. The goal of the exercise is to obtain an explicit expression for $\Gamma$. *Hint: There is a very famous expansion of real-valued one-parameter functions which you can use, and then manipulate the result to get it looking like* $\exp(-i\tau\Gamma)f(t)$.

## 3 Duality transformation and constitutive relations

Consider Maxwell's equations in space-time representation for complex-valued fields

$$\partial_t \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\nabla}\times \\ -\boldsymbol{\nabla}\times & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \tag{1}$$

together with constitutive relations

$$\begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \underline{\underline{\epsilon}} & 0 \\ 0 & \underline{\underline{\mu}} \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}. \tag{2}$$

a) (1 Point) What symmetry transformation does the helicity operator $\Lambda$ generate? Write its action on a general plane wave with definite helicity $|\boldsymbol{k}\lambda\rangle$.

b) (2 Points) Write Maxwell's equations Eq.(1) in the $\boldsymbol{G}_\pm$ representation using the constitutive relations.

c) (2 Points) How do Maxwell's equation change under the action of the duality transformation $\boldsymbol{G}_\lambda \to \tilde{\boldsymbol{G}}_\lambda = D(\theta)\tilde{\boldsymbol{G}}_\lambda$?

d) (1 Points) Consider a scatterer with $\underline{\underline{\epsilon}} = \underline{\underline{\mu}}$. What symmetry property does this condition imply for the scatterer? Explain your answer.

e) (1 Point) Consider transformation of a scatterer $\underline{\underline{\epsilon}} \to \underline{\underline{\epsilon}}' = \underline{\underline{\mu}}$, $\underline{\underline{\mu}} \to \underline{\underline{\mu}}' = \underline{\underline{\epsilon}}$. Which duality tranformation is it equivalent to? Explain your answer.

## 4 Zero back scattering: duality and rotation symmetries

Consider scattering by a 3D object, where we are interested in incident and scattered states along a fixed axis $\hat{\boldsymbol{z}}$. In other words, we will be only considering states $|\pm\hat{\boldsymbol{z}}, \lambda\rangle$. The incident field shall be directed in the positive $z$-direction.

a) (3 Points) Assume the scatterer is symmetric under rotations $R_z(\alpha)$ for some *fixed* $\alpha$ (not for all $\alpha$'s). Conduct analysis of what this condition implies for the scattering states.

b) (3 Points) Assume the scatterer is symmetric under duality transformation $D(\theta)$ for some *fixed* $\theta$ (not for all $\theta$'s). Conduct analysis of what this condition implies for the scattering states.

Now consider scattering by a two-dimensional square lattice. The lattice consists of two types of spheres that are placed in a checkered order. One type of spheres is described by $\epsilon = a$, $\mu = 1$ and the other is described by $\epsilon = 1$, $\mu = a$. Figure 1 is a sketch of the unit cell of the lattice.

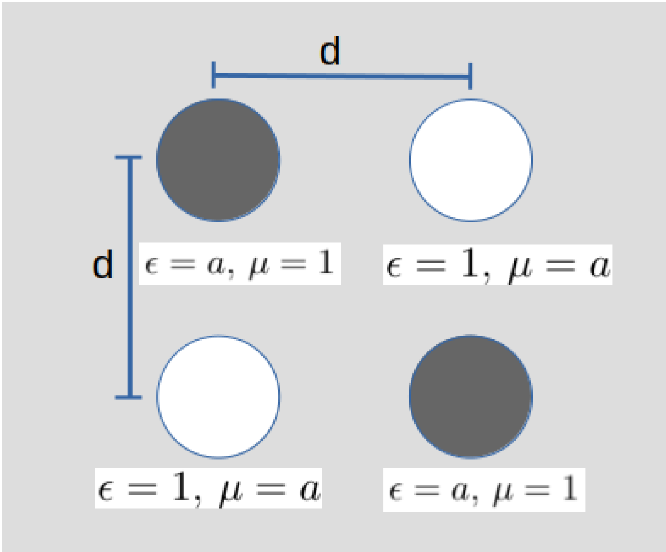


Figure 1: Unit cell of a two-dimensional square lattice.

c) (2 Points) Prove that the scatterer is symmetric under the $R(\frac{\pi}{2})D(\frac{\pi}{2})$ transformation.

d) (2 Points) What consequence does this symmetry have for the scattering states?