Exercise sheet 1
Theoretical Nanooptics WS 2023/2024

Karlsruhe Institute of Technology
Ivan Fernandez-Corbaton
Ivanina Ilieva

November 17th 2023

In order for the exercise sheet to count towards obtaining the course credits, you must upload your solutions to the designated ILIAS page by 10 am on the day of the tutorial. Make sure that the contents of all the scanned/photographed pages are clearly visible. Please create one PDF file with all of your solutions.

1 The $\sqrt{2}\mathbf{G}_\pm = \mathbf{E} \pm i\mathbf{H}$ fields

1.1 True or false?

State whether the following statements are true or false and write a short reasoning for your answer.

a) There is no difference in the physical content between Maxwell equations written with $\mathbf{E}$ and $\mathbf{H}$, and Maxwell equations written with $\mathbf{G}_\pm$.

b) There is no obvious advantage of writing Maxwell equations using $\mathbf{G}_\pm$.

c) In the definition $\mathbf{G}_\pm(\mathbf{r},t) = \frac{\mathbf{E}(\mathbf{r},t) \pm i\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}$, it does not matter whether $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ are real or complex fields. In both cases the $\mathbf{G}_\pm(\mathbf{r},t)$ splits the two handedness of the electromagnetic field.

2 One-sided vs. two-sided Fourier transforms

Consider a **real** field $\boldsymbol{\mathcal{X}}(\mathbf{r},t) \in \mathbb{R}^3\ \forall(\mathbf{r},t)$:

a) Write down the expression for its Fourier transform $\mathbf{X}(\mathbf{r},\omega)$ using a range of $\omega \in [-\infty,\infty]$.

b) Does $\mathbf{X}(\mathbf{r},\omega)$ contain all the information from $\boldsymbol{\mathcal{X}}(\mathbf{r},t)$? Why?

c) Show that, with such definition of the Fourier transform, the following identity is met $\mathbf{X}(\mathbf{r},\omega) = [\mathbf{X}(\mathbf{r},-\omega)]^*$, where * means complex conjugation. Discuss what the identity means regarding the information contained in $\mathbf{X}(\mathbf{r},\omega)$ for $\omega \in [-\infty,\infty]$.

d) Write down an inverse Fourier transform that uses $\mathbf{X}(\mathbf{r},\omega) \in (0,\infty]$ to produce a $(\mathbf{r},t)$-dependent field $\mathbf{X}(\mathbf{r},t)$.

e) Can $\mathbf{X}(\mathbf{r},t)$ be real? Why?

f) Discuss the possible differences between the information contained in $\boldsymbol{\mathcal{X}}(\mathbf{r},t)$ and $\mathbf{X}(\mathbf{r},t)$

g) Give at least two reasons why discarding the $\omega = 0$ point is appropriate

h) In the lectures, we use the one-sided Fourier transform with $\omega \in (0,\infty]$. Can such Fourier transform be used to represent a general $(\mathbf{r},t)$-dependent complex field? Why?

3 The helicity of a plane-wave

Consider the expression for a plane-wave of well-defined helicity $\lambda = \pm 1$ and wavevector $\mathbf{k}$:

$$|\mathbf{k}\ \lambda\rangle \equiv k\hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t) = \frac{k}{\sqrt{2}}\begin{bmatrix}\mathrm{i}\sin\phi - \lambda\cos\phi\cos\theta \\ -\mathrm{i}\cos\phi - \lambda\sin\phi\cos\theta \\ \lambda\sin\theta\end{bmatrix}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}kt), \quad (1)$$

where $k = \omega = +\sqrt{\mathbf{k}\cdot\mathbf{k}}$, $\phi = \arctan(k_y, k_x)$, $\theta = \arccos(k_z/k)$, $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$, and the $\hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}})$ are polarization vectors corresponding to the two handedness.

a) Write the xyz coordinates of the wavevector as a function of ω, θ, and ϕ.

b) Show that the plane-wave is transverse, that is, that its polarization is perpendicular to its wavevector.

c) Show that the plane-wave is an eigenstate of the $\frac{\nabla\times}{\omega}$ operator with eigenvalue λ.