Exercise sheet 2
Theoretical Nanooptics WS 2023/2024

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Please create one PDF file with all of your solutions and submit it *only* by uploading it to the designated ILIAS page by 10 am on the day of the tutorial. Make sure that the contents of all the scanned/photographed pages are clearly visible.

1 True or false?

State whether the following statements are true or false and write a short reasoning for your answer.

a) (1 Point) The distinction between $\mathbf{E}$ and $\mathbf{H}$ is relativistically invariant.

b) (1 Point) The distinction between $\mathbf{G}_+$ and $\mathbf{G}_-$ is relativistically invariant.

c) (1 Point) The largest group for which the distinction between $\mathbf{G}_+$ and $\mathbf{G}_-$ is invariant is the Poincaré group of relativistic transformations.

d) (1 Point) The operators $\frac{\nabla\times}{\omega}$ and $\mathrm{i}\hat{\mathbf{k}}\times$ have different physical meanings.

e) (1 Point) The operators $\frac{\nabla\times}{\omega}$ and $\mathrm{i}\hat{\mathbf{k}}\times$ apply to the same representation of fields.

2 Hilbert space of solutions of Maxwell's equations in vacuum and the photon wave function

In the lecture the Hilbert space of solutions of Maxwell's equations $\mathbb{M}$ was introduced, where any vector $|\phi\rangle \in \mathbb{M}$ can be decomposed into positive and negative helicity parts

$$|\phi\rangle = |g_+\rangle + |g_-\rangle \,.$$

The scalar product in $\mathbb{M}$ can be defined as

$$\langle \phi' | \phi \rangle = \int \frac{d^3k}{|\mathbf{k}|} \begin{bmatrix} \mathbf{G}'_+(\mathbf{k}) \\ \mathbf{G}'_-(\mathbf{k}) \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{G}_+(\mathbf{k}) \\ \mathbf{G}_-(\mathbf{k}) \end{bmatrix} . \tag{1}$$

Using [1, Eq. (17)] we can introduce the basis of circularly polarized plane waves, such that any vector $|\phi\rangle \in \mathbb{M}$ can be represented as

$$|\phi\rangle = \sum_{\lambda=\pm 1} \int \frac{d^3k}{|\mathbf{k}|} \phi_\lambda(\mathbf{k}) \, |\mathbf{k}\ \lambda\rangle \, .$$

a) (1 Point) Write $\phi_\lambda(\mathbf{k})$ as function of $\mathbf{G}_\lambda(\mathbf{k})$, where

$$\mathbf{G}_\lambda(\mathbf{r}, t) = \int d^3k \, \mathbf{G}_\lambda(\mathbf{k}) e^{-ikt + i\mathbf{k}\cdot\mathbf{r}}.$$

b) (1 Point) Prove that the scalar product (1) is equivalent to

$$\langle \phi' | \phi \rangle = \sum_{\lambda=\pm 1} \int \frac{d^3k}{|\mathbf{k}|} \phi'^*_\lambda(\mathbf{k}) \phi_\lambda(\mathbf{k}).$$

c) (3 Points) Prove that the scalar product in coordinate representation is

$$\langle \phi' | \phi \rangle = \frac{1}{2\pi^2} \sum_{\lambda=\pm 1} \iint \frac{d^3\mathbf{r} d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \phi'^*_\lambda(\mathbf{r}') \phi_\lambda(\mathbf{r}), \tag{2}$$

where

$$\phi(\mathbf{r}) := \int \frac{d^3k}{\sqrt{(2\pi)^3}} \phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3}$$

Hints:

Option 1: Prove that

$$\int_{|\mathbf{k}|<R} \frac{d^3k}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{a}} = 4\pi \frac{1 - \cos(R|\mathbf{a}|)}{|\mathbf{a}|^2} \tag{4}$$

and apply the Riemann-Lebesgue lemma.

Option 2: Use Parseval's theorem to switch between **k**- and **r**-space representations. As a next step, use a certain property of the Fourier transform of two functions multiplied with each other. You might find the inverse Fourier transform of $\frac{1}{|\mathbf{k}|}$ useful, which is given by

$$\mathcal{FT}^{-1}\left(\frac{1}{|\mathbf{k}|}\right) = \frac{1}{|\mathbf{r}|^2} \, . \tag{5}$$

3 Electromagnetic angular momentum: Orbital and Spin ?

As of 16/12/20, you can read the following statement in the Wikipedia article devoted to the angular momentum of light https://en.wikipedia.org/wiki/Angular_momentum_of_light:

There are two distinct forms of rotation of a light beam, one involving its polarization and the other its wavefront shape. These two forms of rotation are therefore associated with two distinct forms of angular momentum, respectively named light spin angular momentum (SAM) and light orbital angular momentum (OAM).

This exercise is a critical analysis of such statement.

We will work in $\mathbb{M}$, the Hilbert space of Maxwell fields in vacuum, and use the representation

$$|G\rangle = \begin{bmatrix} \mathbf{G}_+(\mathbf{r}, t) \\ \mathbf{G}_-(\mathbf{r}, t) \end{bmatrix} \tag{6}$$

for vectors $|G\rangle \in \mathbb{M}$.

In this representation, the expression of the operator for the z-component of angular momentum is

$$J_z \equiv \begin{bmatrix} J_z^+ & 0 \\ 0 & J_z^- \end{bmatrix}, \tag{7}$$

where

$$J_z^+ = J_z^- = \underbrace{\begin{pmatrix} -i\partial_\psi & 0 & 0 \\ 0 & -i\partial_\psi & 0 \\ 0 & 0 & -i\partial_\psi \end{pmatrix}}_{L_z} + \underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{S_z}, \tag{8}$$

where $\psi = \arctan(y, x)$. The matrix operators in Eq. (8) act on the 3-vector of Cartesian components $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ of the $\mathbf{G}_\pm(\mathbf{r}, t)$ fields. Formally, the first matrix is the z-component of the orbital angular momentum operator L_z, and the second one the z-component of the spin angular momentum operator S_z.

a) (3 Points) Choose a plane-wave whose wave vector **is not** $\mathbf{k} = [0, 0, \pm|\mathbf{k}|]$. You can use [1, Eq. (17)]. Make it simple for yourself and choose a pure helicity plane wave, either $\lambda = +1$ or $\lambda = -1$. Check that you have chosen the polarization vector $\hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}})$ correctly by verifying the transversality of the field: That is $\mathbf{k} \cdot \hat{\mathbf{e}}_\lambda(\hat{\mathbf{k}}) = 0$, or $\nabla \cdot \mathbf{G}_\lambda(\mathbf{r}, t) = 0$.

b) (1 Point) Apply S_z to your plane-wave and...

c) (1 Points) ... check its transversality again. What has happened? Does the resulting object belong to $\mathbb{M}$? Why?

d) (1 Point) We know that applying J_z to the plane-wave would keep it in $\mathbb{M}$. What can you say about the result of applying L_z to the plane-wave?

e) (3 Points) Repeat now the previous three calculations for $\mathbf{k} = [0, 0, |\mathbf{k}|]$. This time apply L_z explicitly as well. Discuss the results.

f) (1 Point) What can you say about the statement in the Wikipedia article?

g) (1 Point) What are the symmetries **separately** generated by L_z and S_z?

4 True or false?

State whether the following statements are true or false and write a short reasoning for your answer.

a) Consider $X(\theta) = \exp(-i\theta\Gamma)$, where Γ is an operator and θ a real number:

   i) (1 Point) There are eigenstates of Γ that are not eigenstates of $X(\theta)$.

   ii) (1 Point) $X(\theta)$ is always unitary.

   iii) (1 Point) When Γ is self-adjoint ($\Gamma^H = \Gamma$, also often written $\Gamma^\dagger = \Gamma$), it is possible to find a vector $|G\rangle$ such that $\langle G|G\rangle \neq \langle G|X(\theta)^H X(\theta)|G\rangle$.

b) Consider the transition of a two level system. In units of $\hbar = 1$, the difference between the energies of the two levels is ω_0, and the difference between their angular momentum is 1:

   i) (1 Point) Any light beam with average energy equal to ω_0 can induce the transition

   ii) (1 Point) Any light beam with average angular momentum equal to 1 can induce the transition

   iii) (1 Point) Any light beam with average energy equal to ω_0 and average angular momentum equal to 1 can induce the transition

c) (1 Point) The algebraic tools that we develop in class (inner products, average of operators, etc ...) can be used to treat the physics of single photons

d) (1 Point) In classical electromagnetism, we cannot understand the outcomes of measurements using inner products

e) (1 Point) $\mathbb{M}$, the Hilbert space of Maxwell solutions in vacuum, is a useless construct for the study of light-matter interactions

5 Symmetry and commutators

Consider S and $X(\theta)$, two linear operators which map elements of $\mathbb{M}$ onto elements in $\mathbb{M}$. $X(\theta)$ is a unitary transformation corresponding to a continuous symmetry, which is generated by a Hermitian operator Γ: $X(\theta) = \exp(-\mathrm{i}\theta\Gamma)$ for real θ. Consider the two following equations

$$\begin{aligned} &\text{A: } X(\theta)SX(\theta)^{-1} = S \ \forall\ \theta \\ &\text{B: } [S, \Gamma] = 0 \end{aligned} \tag{9}$$

a) (1 Point) Show that B $\Longrightarrow$ A

b) (2 Points) Show that A $\Longrightarrow$ B

References

[1] I. Fernandez-Corbaton. *An alternative starting point for electromagnetism.* Lecture Notes (2019).