Exercise sheet 3
Theoretical Nanooptics WS 2023/2024

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1 Discrete Translational Invariance

In this exercise, you will investigate some of the consequences of discrete translational invariance. Such kind of symmetry appears, for example, in natural and artificial crystals. Before we start, here are some preliminary short questions:

a) Consider a general plane-wave with momentum $\mathbf{k} = [k_x, k_y, k_z]$:

   i) (1 Point) Is the plane-wave an eigenvector of the linear momentum operator along the y-axis, P_y?

   ii) (1 Point) If the answer is yes, what is the eigenvalue?

   iii) (1 Point) What is the result of $P_y|\mathbf{k}\rangle = ?$

   iv) (1 Point) What is the result of a translation by Δ along the y-axis? $T_y(\Delta)|\mathbf{k}\rangle =?$

b) (1 Point) What is the angular frequency ω of the plane-wave? *(Recall that we set the speed of light to c_0=1)*

c) The frequency ω is the eigenvalue of a generator:

   i) (1 Point) Write down two names for such generator.

   ii) (1 Point) What kind of transformations does it generate?

   iii) (1 Point) What does a time-invariant system not mix?

d) With our conventions, it is always true that, for a plane-wave $k_x^2 + k_y^2 + k_z^2 = \omega^2$, and that ω is a positive real number.

   i) (1 Point) Assume that $k_x = 0$ and that k_y is real. What needs to happen with k_z if $|k_y| > \omega$?

ii) (1 Point) This kind of plane-waves are called “evanescent”. How does such an evanescent plane-wave depend on z?

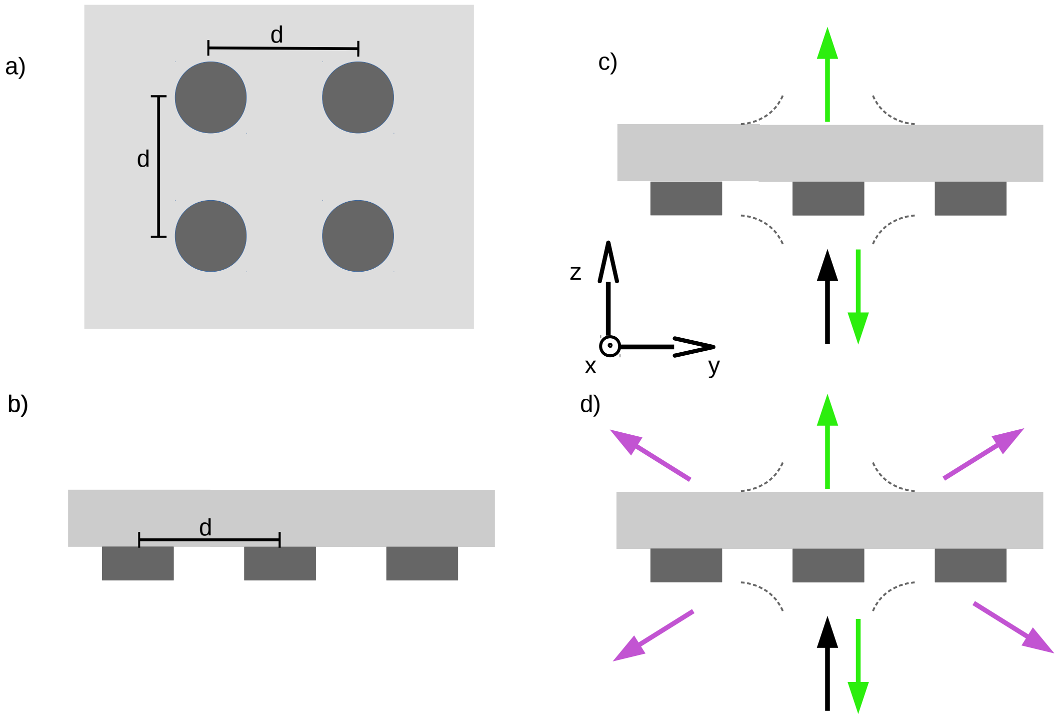

Figure 1: Array of silicon disks.

Look now at Fig. 1. Consider first the a) and b) panels. They show two different views of an array made of silicon disks on top of a glass substrate. The disks are arranged in a “square”-symmetric way (see Fig. 1a)). The coordinate axes at the center of the figure apply to the array views in panels b), c), and d). The shortest distance between the centers of two disks is d. Both the substrate and the disk arrangement extend to infinity in the x and y directions. The system is invariant under the following translations: $T_x(nd)$ and $T_y(md)$, where n and m are any integer number. In panels c) and d), the black arrow represents the plane-wave whose momentum lies along the z-axis: $\mathbf{k} = [0, 0, 2\pi]$. This plane-wave illuminates the system, and we label it $|0\ 0\ 2\pi\rangle$. We ignore its polarization for now. The green and magenta arrows in panels c) and d) represent propagating (non-evanescent) plane-waves that arise upon light-matter interaction. The dotted curves in panels c) and d) represent evanescent plane-waves that arise upon light-matter interaction.

e) (1 Point) The response of the array does not change with time: What will be the angular frequency $\bar{\omega}$ of any outgoing field?

f) (1 Point) Write down the formula that the discrete translational symmetry $T_y(md)$ implies for the scattering operator S of the system. *Hint: Use the formula that constitutes the basic definition of symmetry from the lectures.*

g) (2 Points) Consider the $m = 1$ case in $T_y(md)$. **Derive** the conditions imposed on $\bar{k}_y$ by this symmetry so that the scattering amplitude $\langle \bar{k}_x\ \bar{k}_y\ \bar{k}_z|S|0\ 0\ 2\pi\rangle$ is not equal to zero.

The answer to the previous question is $\bar{k}_y = \frac{2\pi}{d} s_y$ for integer s_y. Similarly, one can derive from the $T_x(d)$ symmetry that $\bar{k}_x = \frac{2\pi}{d} s_x$ for integer s_x.

h) (1 Point) Write down a formula for the value of $\bar{k}_z^2$ as a function of d, s_x and s_y

i) The case $(s_x, s_y) = (0, 0)$ is often called "zeroth diffraction order".

   i) (1 Point) What color corresponds to the "zeroth diffraction order" in panels c) and d)?

   ii) (1 Point) How many plane-waves in each panel correspond to the "zeroth diffraction order"?

Consider the case $(s_x = 0, s_y = 1)$, which is a "first diffraction order". The first order(s) can be evanescent like in panel c) or propagating like in panel d)

j) (1 Point) What does the periodicity d have to meet so that the scattering behavior looks like panel c) and **not** panel d)?

Let us change gears a bit to finish up.

k) (2 Points) What other geometrical symmetries does the array have? Consider discrete rotations and mirror symmetries.

l) (1 Point) Assume that the illuminating plane-wave is linearly polarized along the x-axis: What is the polarization of the green plane-waves in panel c)? Why?

2 Continuous Translational Invariance

Consider a thin dielectric slab, described by S, which is infinite in the x- and y-directions and is surrounded by free space. The fields in the free space outside of the slab can be described by eigenvectors of the translation operator $|\mathbf{k}\rangle$ (arbitrary polarization is implied). For a given incoming state $|\mathbf{k}\rangle$ we wish to find properties of the outgoing states $|\mathbf{k}'\rangle$ using the symmetry of the problem. Fields inside the slab are not of interest. Apply the S-operator formalism and the symmetry arguments to complete the following tasks:

a) (2 Points) The quantity $\langle \mathbf{k}'|S|\mathbf{k}\rangle$ is the probability amplitude of the given incoming state $|\mathbf{k}\rangle$ to be scattered into a state $|\mathbf{k}'\rangle$. Use the translational symmetry of the scatterer to derive the corresponding conservation law.

b) (1 Point) Is the state $|\mathbf{k}\rangle$ an eigenvector of the Hamiltonian operator H? If yes, give its eigenvalue.

c) (1 Point) Assume that the scatterer conserves energy and derive the corresponding conservation quantity.

d) (2 Points) Given an incoming state $|\mathbf{k}\rangle$, what are the scattered states $|\mathbf{k}'\rangle$ that are allowed by the conservation constraints? Give their physical interpretation.

Now consider two semi-infinite half-spaces (with a planar interface) that are characterized by refractive indices n and n'. A plane wave with momentum $\mathbf{k}$ is refracted into a plane wave with momentum $\mathbf{k}'$. *Without* using the S-operator formalism but using symmetry considerations derive the Snell's law:

e) (1 Point) How does the frequency of the plane wave change under refraction?

f) (2 Points) How does the momentum change under refraction? Explain your answer.

g) (2 Points) Derive the Snell's law.

3 Duality Transformation and Constitutive Relations

Consider Maxwell's equations in space-time representation for complex-valued fields

$$\partial_t \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{1}$$

together with constitutive relations

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \underline{\underline{\epsilon}} & 0 \\ 0 & \underline{\underline{\mu}} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{2}$$

a) (1 Point) What symmetry transformation does the helicity operator Λ generate? Write its action on a general plane wave with definite helicity $|\mathbf{k}\lambda\rangle$.

b) (2 Points) Write Maxwell's equations Eq.(1) in the $\mathbf{G}_\pm$ representation using the constitutive relations.

c) (2 Points) How do Maxwell's equation change under the action of the duality transformation $\mathbf{G}_\lambda \to \tilde{\mathbf{G}}_\lambda = \mathbf{D}(\theta)\tilde{\mathbf{G}}_\lambda$?

d) (1 Points) Consider a scatterer with $\underline{\underline{\epsilon}} = \underline{\underline{\mu}}$. What symmetry property does this condition imply for the scatterer? Explain your answer.

e) (1 Point) Consider transformation of a scatterer $\underline{\underline{\epsilon}} \to \underline{\underline{\epsilon}}' = \underline{\underline{\mu}}$, $\underline{\underline{\mu}} \to \underline{\underline{\mu}}' = \underline{\underline{\epsilon}}$. Which duality transformation is it equivalent to? Explain your answer.

4 Generator of Time Translations (3 Points)

In this exercise, you will identify the expression of the generator of time translations for functions $f(t) \in \mathbb{R}$. First, consider what the action of a time translation is, namely $f(t) \to f(t-\tau)$. You will achieve the goal of the exercise by writing such transformation as $\exp(-i\tau\Gamma)f(t)$, that is, the operator $\exp(-i\tau\Gamma)$ acting on $f(t)$. The goal of the exercise is to obtain an explicit expression for Γ. *Hint: There is a very famous expansion of real-valued one-parameter functions which you can use, and then manipulate the result to get it looking like* $\exp(-i\tau\Gamma)f(t)$.