An alternative starting point for electromagnetism

Ivan Fernandez-Corbaton

Institute of Nanotechnology, Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany
ivan.fernandez-corbaton@kit.edu

Abstract. The aim of these lectures is to provide a simple introduction to an unconventional approach to electromagnetism. From the start, the prominent role of the electric and magnetic fields is taken over by two other fields. These fields represent the two handedness that Maxwell solutions can have. The use of this alternative set of fields has notable advantages over the electric and magnetic fields: Decoupled evolution equations, relativistic invariance, and the remarkable ability to split the two possible handedness in electromagnetism.

Keywords: Maxwell equations, Riemann-Silberstein vectors, Helicity

1 About these lectures

On the one hand, many of the ideas and concepts contained in these lectures are not new. The two handedness fields are Rieman-Silberstein-like combinations, and I would like to highlight the seminal works of I. Bialynicki-Birula about the Riemann-Silberstein vectors [1–3]. On the other hand, the versions of the Riemann-Silberstein vectors used in these lectures are not the ones that are typically found in the literature, and I am not aware of the existence of a gentle introduction to the formulism and its consequences, which is the main objective of these lectures.

My aim is that the lectures can be understood from a basic knowledge of electromagnetism and a few mathematical tools, namely Fourier transforms and some aspects of vector calculus. I explicitly write down even simple derivations, and, when faced with what I perceived as a choice between mathematical rigor and pedagogical value I have always opted for the latter, bluntly so in a few instances. I have intentionally not started by discussing the helicity operator and showing that it determines the structure of Maxwell equations. I have rather let the structure emerge naturally from Maxwell equations whey they are looked at in the right way.

2 An alternative form of Maxwell equations

We start with Maxwell equations in vacuum assuming $\epsilon_0 = \mu_0 = 1$, or, alternatively, $c_0 = 1/\sqrt{\epsilon_0\mu_0} = 1$ and $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 1$

$$\begin{aligned} &\text{ⓐ}: \nabla \cdot \mathbf{E}(\mathbf{r},t) = 0,\ \partial_t \mathbf{E}(\mathbf{r},t) = \nabla \times \mathbf{H}(\mathbf{r},t), \\ &\text{ⓑ}: \nabla \cdot \mathbf{H}(\mathbf{r},t) = 0,\ \partial_t \mathbf{H}(\mathbf{r},t) = -\nabla \times \mathbf{E}(\mathbf{r},t), \end{aligned} \tag{1}$$

and manipulate them as follows

$$\begin{aligned} &\frac{\text{ⓐ} + \mathrm{i}\text{ⓑ}}{\sqrt{2}}: \nabla \cdot \left[\frac{\mathbf{E}(\mathbf{r},t) + \mathrm{i}\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}\right] = 0, \\ &\qquad \partial_t \left[\frac{\mathbf{E}(\mathbf{r},t) + \mathrm{i}\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}\right] = \nabla \times \left[\frac{\mathbf{H}(\mathbf{r},t) - \mathrm{i}\mathbf{E}(\mathbf{r},t)}{\sqrt{2}}\right] = -\mathrm{i}\nabla \times \left[\frac{\mathbf{E}(\mathbf{r},t) + \mathrm{i}\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}\right], \\ &\frac{\text{ⓐ} - \mathrm{i}\text{ⓑ}}{\sqrt{2}}: \nabla \cdot \left[\frac{\mathbf{E}(\mathbf{r},t) - \mathrm{i}\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}\right] = 0, \\ &\qquad \partial_t \left[\frac{\mathbf{E}(\mathbf{r},t) - \mathrm{i}\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}\right] = \nabla \times \left[\frac{\mathbf{H}(\mathbf{r},t) + \mathrm{i}\mathbf{E}(\mathbf{r},t)}{\sqrt{2}}\right] = \mathrm{i}\nabla \times \left[\frac{\mathbf{E}(\mathbf{r},t) - \mathrm{i}\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}\right]. \end{aligned} \tag{2}$$

We now define a new set of fields

$$\boxed{\mathbf{G}_\pm(\mathbf{r},t) = \frac{\mathbf{E}(\mathbf{r},t) \pm \mathrm{i}\mathbf{H}(\mathbf{r},t)}{\sqrt{2}}}, \tag{3}$$

and use them to re-write the equations in (2)

$$\begin{aligned} \nabla \cdot \mathbf{G}_+(\mathbf{r},t) = 0,\ \partial_t \mathbf{G}_+(\mathbf{r},t) = -\mathrm{i}\nabla \times \mathbf{G}_+(\mathbf{r},t), \\ \nabla \cdot \mathbf{G}_-(\mathbf{r},t) = 0,\ \partial_t \mathbf{G}_-(\mathbf{r},t) = \mathrm{i}\nabla \times \mathbf{G}_-(\mathbf{r},t). \end{aligned} \tag{4}$$

The four equations in (4) are equivalent to the initial Maxwell equations in (1). The equivalence is clear since the manipulation $\frac{\text{ⓐ}\pm\mathrm{i}\text{ⓑ}}{\sqrt{2}}$ together with the definition in Eq. (3) can be seen as a unitary change of basis, and unitary changes of basis do not alter the information content. This particular change of basis takes us from a description of electromagnetism based on the electric and magnetic fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$, to a description of electromagnetism based on the $\mathbf{G}_\pm(\mathbf{r},t)$ fields. While we have not "broken" anything with such change of basis, there is a notable difference between the equations in (1) and the equations in (4): The new set of equations is decoupled. The $\mathbf{G}_+(\mathbf{r},t)$ field does not depend at all on or affect the $\mathbf{G}_-(\mathbf{r},t)$ field in any way, and vice versa. In sharp contrast, the time evolution equations in (1) show how the electric and magnetic fields are coupled to each other: The time evolution of $\mathbf{E}(\mathbf{r},t)$ depends on $\mathbf{H}(\mathbf{r},t)$, and vice versa. In this sense, while there is no change on the physical content, the form of Maxwell equations is simpler in (4) than in (1).

The $\mathbf{G}_\pm(\mathbf{r},t)$ fields are called the Riemann-Silberstein vectors [3]. Actually, in these lectures, we will not use the typical version of Riemann-Silberstein vectors, where the electric and magnetic fields used to build them are *real-valued* fields. We will use *complex-valued* electric and magnetic fields instead. The reason for this choice will be made clear later.

3 The $\frac{\nabla\times}{\omega}$ operator and $\mathbf{G}_\pm$ as *the* polarization description

3.1 From the $(\mathbf{r}, t)$ to the $(\mathbf{r}, \omega)$ domain

One obtains a complex-valued field $\mathbf{X}(\mathbf{r}, t)$ with the following definition of its inverse Fourier transform

$$\mathbf{X}(\mathbf{r}, t) = \int_{>0}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, \mathbf{X}(\mathbf{r}, \omega) \exp(-\mathrm{i}\omega t), \tag{5}$$

where only positive frequencies are considered. Such one-sided definition is appropriate in electromagnetism because the same information is contained on both sides of the spectrum. Note that $\omega = 0$ is explicitly taken out of the integral in Eq. (5). This is because we are dealing with electrodynamics and need to exclude electro- and magneto-statics.

We will now use the harmonic decomposition on Eq. (5) to work on the equations in (4). Let us in particular consider the evolution equation for $\mathbf{G}_+(\mathbf{r}, t)$, and substitute $\mathbf{G}_+(\mathbf{r}, t)$ by its decomposition of the kind written in Eq. (5):

$$\partial_t \left[\int_{>0}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, \mathbf{G}_+(\mathbf{r}, \omega) \exp(-\mathrm{i}\omega t)\right] = -\mathrm{i}\nabla\times \left[\int_{>0}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, \mathbf{G}_+(\mathbf{r}, \omega) \exp(-\mathrm{i}\omega t)\right]. \tag{6}$$

After applying the ∂_t on the left-hand side, shifting the curl to the inside of the integral of the right-hand side, and moving the right-hand side onto the left-hand side, we obtain

$$\int_{>0}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, \left[-\mathrm{i}\omega\mathbf{G}_+(\mathbf{r}, \omega) + \mathrm{i}\nabla\times\mathbf{G}_+(\mathbf{r}, \omega)\right] \exp(-\mathrm{i}\omega t) = 0,\ \forall(\mathbf{r}, t). \tag{7}$$

The only way to meet Eq. (7) is that the term within the square-brackets is zero for all $(\mathbf{r}, \omega)$. One can reach this conclusion after thinking about how a sum of weigthed harmonic functions $\alpha(\mathbf{r}, \omega) \exp(-\mathrm{i}\omega t)$ with different frequencies can add up to zero simultaneously for all positions $\mathbf{r}$ and time instances t. We therefore have that the evolution equation for $\mathbf{G}_+(\mathbf{r}, t)$ results in the following equation for $\mathbf{G}_+(\mathbf{r}, \omega)$: $\mathbf{G}_+(\mathbf{r}, \omega) = \frac{\nabla\times}{\omega}\mathbf{G}_+(\mathbf{r}, \omega)$. Very similar steps can be used to show that the set of equations in (4) result in:

$$\begin{aligned} \nabla\cdot\mathbf{G}_+(\mathbf{r}, \omega) = 0,\ \mathbf{G}_+(\mathbf{r}, \omega) &= \frac{\nabla\times}{\omega}\mathbf{G}_+(\mathbf{r}, \omega), \\ \nabla\cdot\mathbf{G}_-(\mathbf{r}, \omega) = 0,\ \mathbf{G}_-(\mathbf{r}, \omega) &= -\frac{\nabla\times}{\omega}\mathbf{G}_-(\mathbf{r}, \omega). \end{aligned} \tag{8}$$

These equations are another form of Maxwell equations. Let us consider just the curl equations.:

$$\boxed{\frac{\nabla\times}{\omega}}\mathbf{G}_\pm(\mathbf{r}, \omega) = \pm\mathbf{G}_\pm(\mathbf{r}, \omega). \tag{9}$$

When we consider the object inside the box as an operator, Eq. (9) is telling us that:

Box 1

- There are two different kinds of solutions of Maxwell equations.
- Both kinds are eigenstates of an operator, which, in the $(\mathbf{r}, \omega)$ representation reads $\frac{\nabla\times}{\omega}$.
- The difference between the two kinds of solutions is their $\frac{\nabla\times}{\omega}$ eigenvalue, which is either +1 or -1.

3.2 The $\frac{\nabla\times}{\omega}$ operator

Apparently, the operator $\frac{\nabla\times}{\omega}$ has a special role in Maxwell equations. Let us investigate some of its properties, starting by its action on electric and magnetic fields. Such action can be easily elucidated by bringing the evolution equations in (1) onto the $(\mathbf{r}, \omega)$ representation:

$$\frac{\nabla\times}{\omega}\mathbf{E}(\mathbf{r},\omega) = \mathrm{i}\mathbf{H}(\mathbf{r},\omega), \ \frac{\nabla\times}{\omega}\mathbf{H}(\mathbf{r},\omega) = -\mathrm{i}\mathbf{E}(\mathbf{r},\omega), \tag{10}$$

which shows that $\frac{\nabla\times}{\omega}$ exchanges electric and magnetic fields. In particular, this means that $\mathbf{E}(\mathbf{r},\omega)$ and $\mathbf{H}(\mathbf{r},\omega)$ *are not* eigenstates of $\frac{\nabla\times}{\omega}$.

If we apply $\frac{\nabla\times}{\omega}$ one more time to Eq. (10)

$$\begin{aligned} \frac{\nabla\times}{\omega}\frac{\nabla\times}{\omega}\mathbf{E}(\mathbf{r},\omega) &= \mathrm{i}\frac{\nabla\times}{\omega}\mathbf{H}(\mathbf{r},\omega) \overset{\text{Eq. (10)}}{=} \mathbf{E}(\mathbf{r},\omega), \\ \frac{\nabla\times}{\omega}\frac{\nabla\times}{\omega}\mathbf{H}(\mathbf{r},\omega) &= -\mathrm{i}\frac{\nabla\times}{\omega}\mathbf{E}(\mathbf{r},\omega) \overset{\text{Eq. (10)}}{=} \mathbf{H}(\mathbf{r},\omega), \end{aligned} \tag{11}$$

we find out that[1] $\frac{\nabla\times}{\omega}\frac{\nabla\times}{\omega} = \left(\frac{\nabla\times}{\omega}\right)^2$ is really the identity operator for solutions of Maxwell equations. This can also be seen after applying $\frac{\nabla\times}{\omega}$ to both sides of Eq. (9). An operator that squares to the identity has only two eigenvalues: +1 and -1. This means that the $\mathbf{G}_{\pm}(\mathbf{r},\omega)$ are all the eigenstates that $\frac{\nabla\times}{\omega}$ has for Maxwellian fields.

3.3 The meaning of $\mathbf{G}_{\pm}$

The $\mathbf{G}_{\pm}(\mathbf{r},\omega)(\mathbf{G}_{\pm}(\mathbf{r},t))$ fields are also apparently special in the context of Maxwell equations. Let us investigate their properties by examining them for particular electromagnetic fields.

We consider a linearly-polarized plane-wave with wavevector $\mathbf{k} = [0, 0, k]$

$$\mathbf{E}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \hat{\mathbf{x}}\exp(\mathrm{i}kz - \mathrm{i}\omega t), \tag{12}$$

where $k = +\sqrt{\mathbf{k}\cdot\mathbf{k}} = \omega$ since we have set $c_0 = 1$. We use Eq. (10) to obtain the corresponding magnetic field applying the determinant mnemonic to compute

[1] The fact that $\nabla\cdot\mathbf{E}(\mathbf{r},\omega) = \nabla\cdot\mathbf{H}(\mathbf{r},\omega) = 0$ is implicitly used.

the curl:

$$\mathrm{i}\mathbf{H}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \frac{\nabla\times}{\omega}\mathbf{E}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \frac{1}{\omega}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ \exp(\mathrm{i}kz) & 0 & 0 \end{vmatrix}\exp(-\mathrm{i}\omega t)$$
$$= \frac{1}{\omega}\begin{bmatrix} 0 \\ \mathrm{i}k\exp(\mathrm{i}kz) \\ 0 \end{bmatrix}\exp(-\mathrm{i}\omega t) = \mathrm{i}\hat{\mathbf{y}}\exp(\mathrm{i}kz - \mathrm{i}\omega t). \tag{13}$$

Note that $\mathbf{H}(\mathbf{r},\omega)$ is also a linearly-polarized plane-wave. We now build $\mathbf{G}_\pm(\mathbf{r},\omega) = [\mathbf{E}(\mathbf{r},\omega) \pm \mathrm{i}\mathbf{H}(\mathbf{r},\omega)]/\sqrt{2}$ using Eq. (12) and Eq. (13):

$$\begin{aligned} \mathbf{G}_+(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) &= \frac{\hat{\mathbf{x}} + \mathrm{i}\hat{\mathbf{y}}}{\sqrt{2}}\exp(\mathrm{i}kz - \mathrm{i}\omega t), \\ \mathbf{G}_-(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) &= \frac{\hat{\mathbf{x}} - \mathrm{i}\hat{\mathbf{y}}}{\sqrt{2}}\exp(\mathrm{i}kz - \mathrm{i}\omega t), \end{aligned} \tag{14}$$

which shows that $\mathbf{G}_\pm(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ are circularly-polarized plane-waves, specifically, $\mathbf{G}_+(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ is left-handed and $\mathbf{G}_-(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ is right-handed.

Let us now examine what happens when the initial $\mathbf{E}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ is circularly-polarized:

$$\mathbf{E}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \frac{\hat{\mathbf{x}} + \mathrm{i}\hat{\mathbf{y}}}{\sqrt{2}}\exp(\mathrm{i}kz - \mathrm{i}\omega t). \tag{15}$$

We could repeat the procedure in Eq. (13) to compute $i\mathbf{H}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \frac{\nabla\times}{\omega}\mathbf{E}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$, but we can also reach the result through a shortcut. The expression for $\mathbf{E}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ in Eq. (15) is actually identical to the one for $\mathbf{G}_+(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ in Eq. (14). This means that it is an eigenstate of $\frac{\nabla\times}{\omega}$ with eigenvalue 1, and hence $i\mathbf{H}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \frac{\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}}}{\sqrt{2}}\exp(\mathrm{i}kz - \mathrm{i}\omega t)$. The corresponding $\mathbf{G}_\pm(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ are then:

$$\mathbf{G}_+(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = (\hat{\mathbf{x}} + \mathrm{i}\hat{\mathbf{y}})\exp(\mathrm{i}kz - \mathrm{i}\omega t),\ \mathbf{G}_-(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \mathbf{0}. \tag{16}$$

It is straightforward to see that when the initial polarization in Eq. (15) is the of opposite polarization handedness ($\frac{\hat{\mathbf{x}}-\mathrm{i}\hat{\mathbf{y}}}{\sqrt{2}}$), then $\mathbf{G}_+(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = \mathbf{0}$ and $\mathbf{G}_+(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = (\hat{\mathbf{x}} - \mathrm{i}\hat{\mathbf{y}})\exp(\mathrm{i}kz - \mathrm{i}\omega t)$.

In summary:

- When the electric field is a single linearly-polarized plane-wave, the magnetic field is also a single linearly-polarized plane-wave, and both $\mathbf{G}_\pm(\mathbf{r},\omega)$ are different than zero and circularly-polarized. $\mathbf{G}_+(\mathbf{r},\omega)$ is left-handed and $\mathbf{G}_-(\mathbf{r},\omega)$ is right-handed.
- When the electric field is a single circularly-polarized plane-wave, the magnetic fields is also circularly-polarized with the same handedness, and then, the $\mathbf{G}_-(\mathbf{r},\omega)(\mathbf{G}_+(\mathbf{r},\omega))$ vanishes for left(right)-handed polarization.

Which indicates that:

Box 2

- $\mathbf{G}_+(\mathbf{r},\omega)$ is always left-hand polarized and $\mathbf{G}_-(\mathbf{r},\omega)$ is always right-hand polarized.
- The $\mathbf{G}_\pm(\mathbf{r},\omega)$, and consequently also the $\mathbf{G}_\pm(\mathbf{r},t)$, split the two handed components of the electromagnetic field.
- The operator $\frac{\nabla\times}{\omega}$ is some kind of handedness operator, whose +1(-1) eigenvalue corresponds to fields with left(right)-handedness.

We are writing the statements in Box 2 by "extrapolating" from the analysis of a single plane-wave of particular frequency ω and momentum $\omega\hat{\mathbf{z}} = [0,0,\omega]$. While it is obvious that the steps in Eqs. (12-16) hold for any ω, it is not so apparently obvious that they hold for any wavevector $\mathbf{k}$. But such is indeed the case, and then, by linearity, they also hold for arbitrary linear combinations of plane-waves, that is, arbitrary fields. Before showing that such is indeed the case, let us reflect on what that means with the help of Fig. 1.

Figure 1 has three panels. Each panel represents three different electromagnetic fields by the sum of the depicted plane-waves. Each plane-wave is characterized by its wavevector and its polarization handedness. The direction and length of the wavevector is represented by the long straight arrows in the figure. The polarization handedness are represented by the curved lines, and encoded by the use of blue color for left-handed and red color for right-handed. On the leftmost panel of Fig. 1, there are five plane-waves with different wavevectors, and all the plane-waves are left-handed. On the central panel, all of the four plane-waves are right-handed. On the rightmost panel there is a mix between left- and right-handed plane-waves.

According to Box 1, and as indicated in the figure, the $\mathbf{G}_-(\mathbf{r},\omega)(\mathbf{G}_-(\mathbf{r},t))$ corresponding to the field in leftmost panel will be zero, the $\mathbf{G}_+(\mathbf{r},\omega)(\mathbf{G}_+(\mathbf{r},t))$ corresponding to the field in the central panel will be zero, and neither of the $\mathbf{G}_\pm(\mathbf{r},\omega)(\mathbf{G}_\pm(\mathbf{r},t))$ corresponding to the field in the rightmost panel will be zero. The $\mathbf{G}_\pm(\mathbf{r},\omega)(\mathbf{G}_\pm(\mathbf{r},t))$ have the ability to split the two handedness components independently of how many different plane-waves compose the field. This is a very remarkable property.

Let us now go back to showing that the $\mathbf{G}_\pm(\mathbf{r},\omega)(\mathbf{G}_\pm(\mathbf{r},t))$ split the two polarization handedness of the electromagnetic field for any linear combination of plane-waves. We will do this by showing that $\mathbf{G}_-(\mathbf{r},\omega) = \mathbf{G}_-(\mathbf{r},t) = \mathbf{0}$ for the field corresponding to the leftmost panel of Fig. 1.

To such end, it is convenient to introduce the following notation for a plane-wave with wavevector $\mathbf{k}$ and handedness $\lambda = \pm 1$:

$$|\mathbf{k}\ \lambda = \pm 1\rangle \equiv \hat{\mathbf{e}}_\pm(\hat{\mathbf{k}}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t) = \begin{bmatrix} \mathrm{i}\sin\phi - \lambda\cos\phi\cos\theta \\ -\mathrm{i}\cos\phi - \lambda\sin\phi\cos\theta \\ \lambda\sin\theta \end{bmatrix} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}kt), \tag{17}$$

where $k = \omega = +\sqrt{\mathbf{k}\cdot\mathbf{k}}$, $\phi = \arctan(k_y, k_x)$, $\theta = \arccos(k_z/k)$, $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$, and the $\hat{\mathbf{e}}_\pm(\hat{\mathbf{k}})$ are polarization vectors corresponding to the two handedness. A plane-wave with wavevector $\mathbf{k}$ and handedness λ can

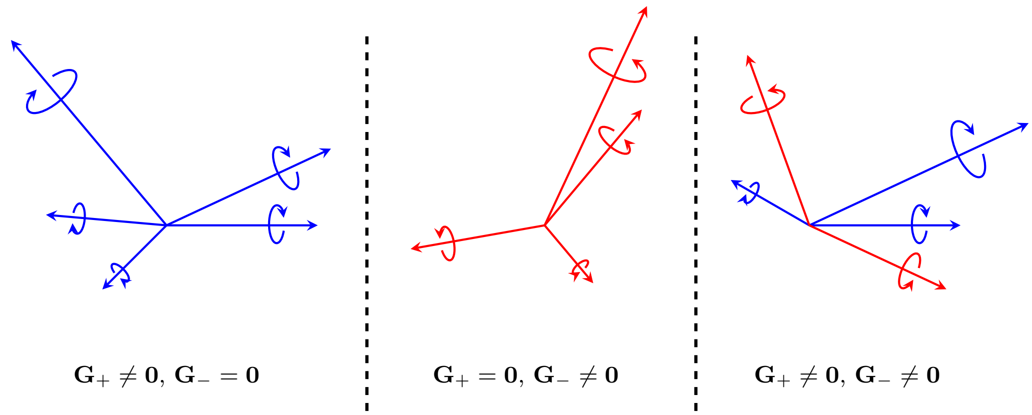

Fig. 1. Three different electromagnetic fields represented as the sum of the depicted plane-waves. The left-handed plane-waves are blue and the right-handed plane-waves are red. The $\mathbf{G}_\pm(\mathbf{r}, \omega)(\mathbf{G}_\pm(\mathbf{r}, t))$ have the very remarkable ability to split the two handed components of a given electromagnetic field, independently of how many different plane-waves compose it.

be obtained by the rotation of a reference plane-wave of the same handedness and frequency [4, Eq. (8.7-1)]:

$$|\mathbf{k}\ \lambda = \pm 1\rangle = R(\hat{\mathbf{k}})|\sqrt{\mathbf{k}\cdot\mathbf{k}}\hat{\mathbf{z}}\ \lambda\rangle, \tag{18}$$

where $R(\hat{\mathbf{k}})$ is a rotation that brings the reference wavevector $[0, 0, \sqrt{\mathbf{k}\cdot\mathbf{k}}\hat{\mathbf{z}}]$ onto the target vector $\mathbf{k}$. The standard way of building such rotation is by a rotation along the $\hat{\mathbf{y}}$ axis by the angle θ followed by a rotation around the $\hat{\mathbf{z}}$ axis by the angle ϕ: $R(\hat{\mathbf{k}}) = R_z(\phi)\, R_y(\theta)$.

The standard reference plane-waves are

$$|k\hat{\mathbf{z}}\ \lambda\rangle \equiv \frac{-\lambda\hat{\mathbf{x}} - \mathrm{i}\hat{\mathbf{y}}}{\sqrt{2}} \exp(\mathrm{i}kz - \mathrm{i}\omega t). \tag{19}$$

In order for us to go forward, and since we have not introduced certain powerful algebraic tools, we will need to accept the following statement without a formal proof:

When a plane-wave of a given handedness λ is rotated, its handedness does not change.

This is intuitively easy to grap. For example, rotating your right arm does not change your right hand into a left hand. Similarly, rotating a screw does not change the handedness of its threads. The statement that we are assuming can be formally written, using the notation $\Lambda \equiv \frac{\nabla\times}{\omega}$, as:

$$\text{If } \Lambda|\mathbf{k}\ \lambda\rangle = \lambda|\mathbf{k}\ \lambda\rangle, \text{ then } \Lambda R|\mathbf{k}\ \lambda\rangle = \lambda R|\mathbf{k}\ \lambda\rangle \text{ for any rotation } R. \tag{20}$$

Once we accept this, the proof is relatively easy. The electric field $\mathbf{E}$ corresponding to the leftmost panel in Fig. 1 can be written as the sum of five different left-hand polarized plane-waves built as in Eq. (18):

$$\sum_{s=1}^{5} \alpha_s R_s(\hat{\mathbf{k}}_s)|k_s\hat{\mathbf{z}}\ +\rangle, \tag{21}$$

where the α_s are complex numbers. Let us choose one of the five plane-waves by fixing $s = n$ so that $\mathbf{E}_n$ is $\alpha_n R_n(\hat{\mathbf{k}}_n)|k_n\hat{\mathbf{z}}\ +\rangle$. According to Eq. (10), its corresponding magnetic field $\mathrm{i}\mathbf{H}_n$ can be computed as $\Lambda\mathbf{E}_n$, namely

$$\Lambda\alpha_n R_n|k_n\hat{\mathbf{z}}\ +\rangle = \alpha_n \Lambda R_n|k_n\hat{\mathbf{z}}\ +\rangle \overset{\text{Eq. (20)}}{=} \alpha_n R_n|k_n\hat{\mathbf{z}}\ +\rangle. \tag{22}$$

That is $\mathbf{E}_n = i\mathbf{H}_n$, implying that $\mathbf{G}_+^n \neq \mathbf{0}$ and $\mathbf{G}_-^n = [\mathbf{E}_n - i\mathbf{H}_n]/\sqrt{2} = \mathbf{0}$, which is true for both $(\mathbf{r},\omega)$- and $(\mathbf{r},t)$-dependent fields. After seeing that $\mathbf{G}_-^n = \mathbf{0}$, it is obvious that the whole $\mathbf{G}_- = \sum_{s=1}^{5} \mathbf{G}_-^s$ will be equal to zero because the analysis with $s = n$ works for any other s.

3.4 The importance of complex fields

This is a good point to motivate the choice of complex-valued electric and magnetic fields, with harmonic decompositions of the kind in Eq. (5), to build the $\mathbf{G}_\pm(\mathbf{r},t)$ in Eq. (3). Let us assume that we use real-valued fields instead, $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ and $\boldsymbol{\mathcal{H}}(\mathbf{r},t)$

$$\boldsymbol{\mathcal{G}}_\pm(\mathbf{r},t) = \frac{\boldsymbol{\mathcal{E}}(\mathbf{r},t) \pm \mathrm{i}\boldsymbol{\mathcal{H}}(\mathbf{r},t)}{\sqrt{2}}. \tag{23}$$

The real-valued fields can be defined as the real part of their complex-valued counterparts: $\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \mathrm{Real}\{\mathbf{E}(\mathbf{r},t)\}$, and $\boldsymbol{\mathcal{H}}(\mathbf{r},t) = \mathrm{Real}\{\mathbf{H}(\mathbf{r},t)\}$.

The $\boldsymbol{\mathcal{G}}_\pm(\mathbf{r},t)$ in Eq. (23) are still complex, but now, since $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ and $\boldsymbol{\mathcal{H}}(\mathbf{r},t)$ are real we have that $\boldsymbol{\mathcal{G}}_\pm(\mathbf{r},t)$ determine each other through complex conjugation:

$$[\boldsymbol{\mathcal{G}}_+(\mathbf{r},t)]^* = [\boldsymbol{\mathcal{G}}_-(\mathbf{r},t)]\,. \tag{24}$$

This is crucially different from the case of $\mathbf{G}_\pm(\mathbf{r},t)$, where such connection does not exist. There is no physical law that, in general and *a priori*, ties one handedness to the other as Eq. (24) implies. Therefore, the $\boldsymbol{\mathcal{G}}_\pm(\mathbf{r},t)$ are not obviously suitable to represent the two handed components of the electromagnetic field. In particular, it can be easily shown that their point-wise squared-norms are equal

$$|\boldsymbol{\mathcal{G}}_+(\mathbf{r},t)|^2 = |\boldsymbol{\mathcal{G}}_-(\mathbf{r},t)|^2, \text{ for all } (\mathbf{r},t). \tag{25}$$

The clean split illustrated in Fig. 1 is hence not possible using the $\boldsymbol{\mathcal{G}}_\pm(\mathbf{r},t)$ since if one of them is zero, so must be the other, according to Eq. (25).

3.5 The $G_\pm$ split as *the* polarization description: Generality and invariance

Let us go back to the $\mathbf{G}_\pm$ and investigate further their ability to characterize the polarization of the electromagnetic field. To such end, we start with the question of *What is polarization?*.

The polarization of a field may be defined as its non-scalar degrees of freedom. That is, those degrees of freedom that scalar fields cannot have. For example, the Higgs boson has energy, but does not have different polarization states. The electromagnetic field has scalar degrees of freedom like energy, momentum, angular momentum, and, *additionally*, can have different polarizations. Non-scalar degrees of freedom are sometimes called internal degrees of freedom. Table 1 lists some polarized and unpolarized fields.

Polarized fields	Unpolarized fields
Electromagnetic field	Temperature
Electrons	Pressure
Neutrinos	The Higgs boson
Gravitational waves	Electric charge density

Table 1. Some fields have a polarization degree of freedom, i.e. a non-scalar degree of freedom, some other fields do not have it.

Let us focus on the electromagnetic field to tackle the question of how to characterize its polarization.

When dealing with a single plane-wave, the concept of polarization is quite intuitive. Polarization is typically understood as the direction of the field oscillations in a plane transverse to the wavevector. Fixing one such plane, the linear, circular, or generally elliptical polarization of the plane-wave is identified by the geometrical figure that the "tip" of the electric field "draws" on the plane with the passing of time. This understanding of polarization, though, is quite restricted to single plane-wave cases, and not applicable in general. For example, consider the following electric field:

$$\begin{aligned}&\sqrt{2}\mathbf{E}(\mathbf{r},t) = \\ &\hat{\mathbf{x}}\exp\left(\mathrm{i}2\omega(z-t)\right) - \mathrm{i}\hat{\mathbf{y}}\left[\exp\left(\mathrm{i}\omega(x-t)\right) + \exp\left(\mathrm{i}2\omega(z-t)\right)\right] - \hat{\mathbf{z}}\exp\left(\mathrm{i}\omega(x-t)\right).\end{aligned} \tag{26}$$

It is apparent the field contains at least two different plane-waves of different frequency and wavevector directions. The previous definition of polarization cannot be applied because it is impossible to find a plane transverse to both wavevectors. We may attempt to define polarization by fixing a point $\mathbf{r}_0$ and observing the trajectory of the "tip" of the electric field $\mathbf{E}(\mathbf{r}_0, t)$ as time elapses. Or we may fix $t = t_0$ and observe the trajectory of $\mathbf{E}(\mathbf{r}, t_0)$ for some $\mathbf{r}$ in a given volume or surface. Some exercises with Eq. (26) quickly show the complexity of these kind of approaches. But, the fact is that the field in Eq. (26) is just

$$|[\omega, 0, 0]\ +\rangle + |[0, 0, 2\omega]\ +\rangle, \tag{27}$$

that is, it is a $\mathbf{G}_+$.

We will now argue that the $\mathbf{G}_+/\mathbf{G}_-$ split is the most general and robust way to characterize the polarization degree of freedom of the electromagnetic field.

The general applicability of $\mathbf{G}_\pm$ is already clear from the previous discussion: It applies to general linear combinations of plane-waves. Let us discuss its robustness. By that, we mean the resilience upon transformations, i.e., invariance. We have already heuristically discussed how the $\mathbf{G}_\pm$ character does not change upon rotations. The linearly-polarized states provides a counter-example. A rotation along the axis defined by the wavevector of a plane-wave transforms the two orthogonal linear polarization vectors onto each other. Linear polarizations are hence not invariant under rotations.

Let us keep using the analogy of the threads of a screw to investigate the transformation properties of the handed $\mathbf{G}_\pm$. In the same way that the threads do not change handedness when the screw is rotated, they also do not change handedness upon:

– Spatial translations: Moving the screw from point A to point B.
– Time translations: Letting time pass or moving time backwards.

So, the handedness defined by $\mathbf{G}_\pm$ is invariant under time and space translations and rotations. We only need to add the Lorentz transformations to complete the Poincaré group, that is, the group of transformations of special relativity. And we can do that. A Lorent boost, or Lorentz transformation, changes an inertial reference frame into another intertial reference frame. The new reference frame moves with a constant velocity $\boldsymbol{\beta}$ with respect to the original one. Let us consider the transformation of the electric and magnetic fields upon a Lorentz boost characterized by the real-valued 3-vector $\boldsymbol{\beta}$. The relationship between the $\mathbf{E}$ and $\mathbf{H}$ fields in the original reference frame and the $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ fields in the boosted reference frame is [5, Eq. (11.149)]:

$$\begin{aligned} ⓐ\!: \bar{\mathbf{E}} &= \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{H}\right) - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right), \\ ⓑ\!: \bar{\mathbf{H}} &= \gamma\left(\mathbf{H} - \boldsymbol{\beta}\times\mathbf{E}\right) - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{H}\right), \end{aligned} \tag{28}$$

where $\gamma = (1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{-1/2}$. We now perform the same manipulation that took the equations in (1) to the equations in (2):

$$\begin{aligned} \frac{ⓐ + \mathrm{i}ⓑ}{\sqrt{2}}\!: \bar{\mathbf{G}}_+ &= \gamma\mathbf{G}_+ - \mathrm{i}\boldsymbol{\beta}\times\mathbf{G}_+ - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{G}_+\right) \\ \frac{ⓐ - \mathrm{i}ⓑ}{\sqrt{2}}\!: \bar{\mathbf{G}}_- &= \gamma\mathbf{G}_- + \mathrm{i}\boldsymbol{\beta}\times\mathbf{G}_- - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{G}_-\right). \end{aligned} \tag{29}$$

Equation (29) shows that a boosts does not mix $\mathbf{G}_+$ and $\mathbf{G}_-$. Incidentally, Eq. (28) shows that the notions of electric and magnetic field are not relativistically invariant: A boost mixes electric and magnetic fields.

The conclusion is that the characterization of polarization by means of $\mathbf{G}_\pm$ is relativistically invariant. Actually, the $\mathbf{G}_\pm$ character is invariant under a larger group: The conformal group, which is actually the largest group of invariance of Maxwell equations [6]. Therefore, the $\mathbf{G}_\pm$ character is as robust as anything can be in electromagnetism.

4 An even simpler form of Maxwell equations and the $\mathrm{i}\hat{\mathbf{k}}\times$ operator

We will now simplify Maxwell equations further by the use of the following decomposition of an $(\mathbf{r}, t)$-dependent field:

$$\begin{aligned} \mathbf{X}(\mathbf{r},t) &= \int_{\mathbb{R}^3-\{\mathbf{0}\}} \frac{d\mathbf{k}}{\sqrt{(2\pi)^3}}\, \mathbf{X}(\mathbf{k}) \exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}kt\right) \\ &= \int_{>0}^{\infty} \frac{dk}{\sqrt{2\pi}} k^2 \left[\int_{\mathbb{S}^2} \frac{d\hat{\mathbf{k}}}{2\pi}\, \mathbf{X}(k,\hat{\mathbf{k}}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\right] \exp(-\mathrm{i}kt), \end{aligned} \tag{30}$$

where, in the second line, we have split the initial $d\mathbf{k}$ integral into its radial and angular parts

$$\int_{\mathbb{R}^3-\{\mathbf{0}\}} \frac{d\mathbf{k}}{\sqrt{(2\pi)^3}} = \int_{>0}^{\infty} \frac{dk}{\sqrt{2\pi}} k^2 \int_{\mathbb{S}^2} \frac{d\hat{\mathbf{k}}}{2\pi} = \int_{>0}^{\infty} \frac{dk}{\sqrt{2\pi}} k^2 \int_0^{\pi} d\theta \sin\theta \int_{-\pi}^{\pi} d\phi, \tag{31}$$

where θ and ϕ are defined below Eq. (17) and $\mathbb{S}^2$ is the 2-sphere in $\mathbf{k}$ space with radius k. The removal of the origin $\mathbf{k} = \mathbf{0}$ responds again to the exclusion of electro- and magneto-statics. Recalling that $k = \omega$, and comparing Eq. (30) with Eq. (5), we establish that:

$$\mathbf{X}(\mathbf{r},\omega) = k^2 \int_{\mathbb{S}^2} \frac{d\hat{\mathbf{k}}}{2\pi}\, \mathbf{X}(k,\hat{\mathbf{k}}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \tag{32}$$

which we will now use to work on the equations in (9). To such end, we need to know what is the effect of $\frac{\nabla\times}{\omega}$ in a field $\mathbf{X}(\mathbf{r},\omega)$ decomposed as in Eq. (32):

$$\begin{aligned}
&\frac{\nabla\times}{\omega}\mathbf{X}(\mathbf{r},\omega) = k^2\int_{\mathbb{S}^2}\frac{d\hat{\mathbf{k}}}{2\pi}\,\frac{\nabla\times}{\omega}\mathbf{X}(k,\hat{\mathbf{k}})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\\
&=\frac{k^2}{\omega}\int_{\mathbb{S}^2}\frac{d\hat{\mathbf{k}}}{2\pi}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial_x & \partial_y & \partial_z\\ X_x(k,\hat{\mathbf{k}})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) & X_y(k,\hat{\mathbf{k}})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) & X_z(k,\hat{\mathbf{k}})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\end{vmatrix}\\
&=\frac{\mathrm{i}k^2}{\omega}\int_{\mathbb{S}^2}\frac{d\hat{\mathbf{k}}}{2\pi}\begin{bmatrix}X_z(k,\hat{\mathbf{k}})k_y - X_y(k,\hat{\mathbf{k}})k_z\\ -X_z(k,\hat{\mathbf{k}})k_x + X_x(k,\hat{\mathbf{k}})k_z\\ X_y(k,\hat{\mathbf{k}})k_x - X_x(k,\hat{\mathbf{k}})k_y\end{bmatrix}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\\
&=k^2\int_{\mathbb{S}^2}\frac{d\hat{\mathbf{k}}}{2\pi}\,\frac{\mathrm{i}\mathbf{k}\times}{\omega}\mathbf{X}(k,\hat{\mathbf{k}})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\\
&=k^2\int_{\mathbb{S}^2}\frac{d\hat{\mathbf{k}}}{2\pi}\,\mathrm{i}\hat{\mathbf{k}}\times\mathbf{X}(k,\hat{\mathbf{k}})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}).
\end{aligned} \tag{33}$$

The one-before-last equality can be easily verified using the expression of the cross-product between two 3-vectors.

Using Eq. (33), we can write Eq. (9) as:

$$k^2\int_{\mathbb{S}^2}\frac{d\hat{\mathbf{k}}}{2\pi}\,\left[\mathrm{i}\hat{\mathbf{k}}\times\mathbf{G}_\pm(\mathbf{k})\mp\mathbf{G}_\pm(\mathbf{k})\right]\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})=\mathbf{0}\text{ for all }\mathbf{r}, \tag{34}$$

which implies

$$\mathrm{i}\hat{\mathbf{k}}\times\mathbf{G}_\pm(\mathbf{k})=\pm\mathbf{G}_\pm(\mathbf{k}). \tag{35}$$

It is clear that $\mathrm{i}\hat{\mathbf{k}}\times$ in Eq. (35) represents, in the $\mathbf{k}$ domain, the same operator represented by $\frac{\nabla\times}{\omega}$ in the $(\mathbf{r},\omega)$ domain.

A very similar procedure allows to show that

$$\nabla\cdot\mathbf{G}_\pm(\mathbf{r},\omega)=0 \implies \mathrm{i}\mathbf{k}\cdot\mathbf{G}_\pm(\mathbf{k})=0, \tag{36}$$

and the four Maxwell equations can finally be written in $\mathbf{k}$ domain as:

$$\boxed{\mathrm{i}\mathbf{k}\cdot\mathbf{G}_\pm(\mathbf{k})=0,\ \mathrm{i}\hat{\mathbf{k}}\times\mathbf{G}_\pm(\mathbf{k})=\pm\mathbf{G}_\pm(\mathbf{k}).} \tag{37}$$

The equations in (37) are a very simple form of Maxwell equations. All the partial derivatives are eliminated and Maxwell equations become algebraic equations in (37). These are easier to interpret. For example, the $\mathrm{i}\mathbf{k}\cdot\mathbf{G}_\pm(\mathbf{k})=0$ conditions say that, given a wavevector $\mathbf{k}$, the polarization of the field must be orthogonal to it. There cannot be any longitudinal components along $\mathbf{k}$. We say that Maxwell solutions are transverse.

5 Summary

The $\mathbf{G}_\pm$ forms of Maxwell equations

$$\mathrm{i}\partial_t \mathbf{G}_\pm(\mathbf{r},t) = \pm\nabla\times\mathbf{G}_\pm(\mathbf{r},t) \ \ , \ \nabla\cdot\mathbf{G}_\pm(\mathbf{r},t)=0, \tag{38}$$

$$\boxed{\frac{\nabla\times}{\omega}}\mathbf{G}_\pm(\mathbf{r},\omega) = \pm\mathbf{G}_\pm(\mathbf{r},\omega) \ \ , \ \nabla\cdot\mathbf{G}_\pm(\mathbf{r},\omega)=0, \tag{39}$$

$$\boxed{\mathrm{i}\hat{\mathbf{k}}\times}\mathbf{G}_\pm(\mathbf{k}) = \pm\mathbf{G}_\pm(\mathbf{k}) \ \ , \ \mathrm{i}\mathbf{k}\cdot\mathbf{G}_\pm(\mathbf{k})=0. \tag{40}$$

The conversion of the Maxwell equations from their typical form featuring electric and magnetic fields to their form featuring the $\mathbf{G}_\pm$ Riemann-Silberstein-like fields reveals a simple underlying structure:

The $\mathbf{G}_\pm$ Maxwell structure

- There are two decoupled and fundamentally different kinds of solutions of Maxwell equations: $\mathbf{G}_+$ and $\mathbf{G}_-$.
- The $\mathbf{G}_\pm$ are eigenstates of the operator inside the boxes in Eqs. (39-40) with corresponding eigenvalues ± 1.
- The $\mathbf{G}_\pm$ split the handedness content of the field into the left- and right-handed components, corresponding to $\mathbf{G}_+$ and $\mathbf{G}_-$, respectively.

The $\frac{\nabla\times}{\omega} \equiv \mathrm{i}\hat{\mathbf{k}}\times$ operator is actually the well-known helicity operator. These two forms of the helicity operator are particular representations of its abstract definition. The helicity operator Λ is defined as the projection of the angular momentum operator vector $\mathbf{J}$ onto the direction of the linear momentum operator vector $\mathbf{P}$:

$$\Lambda = \frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|}. \tag{41}$$

Helicity is not only relevant in electromagnetism, but also in the general field of high energy physics. Essentially, helicity is one of the key operators used in the identification of physical particles and fields with the representations of the Poincaré group [7]. This identification is the basic idea behind our understanding of the standard model and its extensions.

6 Concluding remarks

One can see the $\mathbf{G}_\pm$ forms of Maxwell equations as one of the bases for an algebraic approach to light-matter interactions. The other bases of such approach are the setting of Hilbert spaces, and the scattering operator for modeling the interaction of material objects with the electromagnetic field. With such tools, the powerful ideas of symmetries and conservation laws can be applied to light-matter interactions in a straightforward way [8]. In this regard, the advantage of considering $\mathbf{G}_\pm$ and the helicity operator is that helicity is the generator of a fundamental symmetry in electromagnetism: The electromagnetic duality symmetry. Helicity and duality are in the same kind of relationship as angular momentum and rotations, or linear momentum and translations. The one-to-one connection to a fundamental symmetry does not exist for other ways of characterizing polarization [9, Sec. 2.4.2]. The use of helicity and duality allows the use of symmetries and conservation laws for treating polarization in light-matter interactions [9]. Importantly, the handedness-splitting ability of $\mathbf{G}_{\pm}$ actually extends to evanescent plane-waves, which are ubiquitous in the near-fields around illuminated objects (see e.g. [9, Fig. 2.7]).

When a given formulation simplifies the theoretical description of physical phenomena, it is often the case that such formulation also helps in practical scenarios. The use of the $\mathbf{G}_{\pm}$ fields is not an exception. The practical application of the algebraic approach has lead, for example, to criteria for the optimal sensing of chiral molecules, to the symmetry conditions for zero reflection, and to a quantitative understanding of near-field directional coupling.

References

1. Bialynicki-Birula, I.: Photon wave function, Prog. Optics 36, 245 (1996).
2. Bialynicki-Birula, I., Bialynicka-Birula, Z.: Beams of electromagnetic radiation carrying angular momentum: The riemann-silberstein vector and the classical-quantum correspondence, Opt. Commun. 264(2), 342 (2006).
3. Bialynicki-Birula, I., Bialynicka-Birula, Z.: The role of the Riemann-Silberstein vector in classical and quantum theories of electromagnetism, J. Phys. A: Math. Theor. 46(5), 053001 (2013).
4. Tung, W.-K.: Group Theory in Physics. World Scientific, New Jersey, (1985).
5. Jackson, J. D.: Classical Electrodynamics. Wiley, New York, (1998).
6. Bateman, H.: The transformation of the electrodynamical equations, Proceedings of the London Mathematical Society s2-8(1), 223 (1910).
7. Wigner, E.: On unitary representations of the inhomogeneous lorentz group, Annals of Mathematics 40(1), 149 (1939).
8. Fernandez-Corbaton, I., Rockstuhl, C.: Unified theory to describe and engineer conservation laws in light-matter interactions, Phys. Rev. A 95, 053829 (2017).
9. Fernandez-Corbaton, I., Helicity and duality symmetry in light matter interactions: Theory and applications. Ph.D. thesis, Macquarie University (2014)