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INSTITUT FÜR THEORETISCHE FESTKÖRPERPHYSIK
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Tutorial:

☐ **Group 1,**

☐ **Group 2,**

☐ **Group 3.**

Name:________________________________

Problem set 5 for the course "Theoretical Optics"
Sample Solutions

10 Fresnel Diffraction by a Slit

We consider a monochromatic plane wave impinging normally on an opaque screen with a slit (of width w and height h) in it (Fig. 1).

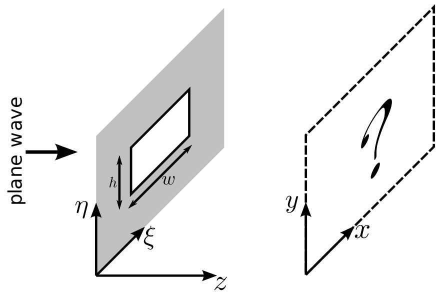

Figure 1: A rectangular shaped aperture.

a) Starting from the Fresnel diffraction integral expressed as

$$U(\mathbf{r}) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{\infty} d\xi d\eta \, U(\xi, \eta) \, e^{i\frac{k}{2z}[(x-\xi)^2+(y-\eta)^2]} \tag{1}$$

with $k = \frac{2\pi}{\lambda}$, put the origin of the coordinate system into the center of the rectangular opening and enter the slit function $U(\xi, \eta)$. Then, normalize all distances to make the expression independent of the wavelength λ. Write the integral as a product of two independent integrals. Please state your substitutions explicitly. [**3** Point(s)]

b) Reformulate the integral from a) in terms of the so-called Fresnel integrals given by

$$C(t) = \int_0^t \cos\left(\frac{\pi}{2}x^2\right) dx, \tag{2}$$

$$S(t) = \int_0^t \sin\left(\frac{\pi}{2}x^2\right) dx. \tag{3}$$

Finally, calculate the intensity $|U(x, y)|^2$ at distance z in terms of C and S. Again, please provide the individual steps leading to your final result. [**5** Point(s)]

c) Use a suitable computer program which is able to evaluate Fresnel integrals (e.g. Mathematica or Maple) to plot the intensity distribution for different aperture sizes and distances. Consider the following cases: $w = h = 1\lambda$, $w = h = 2\lambda$, $w = h = 10\lambda$ and $w = 1\lambda, h = 2\lambda$. For the distances, use $z = 1\lambda,\ 10\lambda,\ 100\lambda,\ 1000\lambda,\ 10^6\lambda$ and $10^9\lambda$. [**2 BONUS** Point(s)]

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a) We place the origin of our coordinate system in the middle of the rectangular slit, where $U(\xi, \eta)$ is equal to 1 in the slit area and 0 outside. Thus the integral boundaries are adjusted to be

$$U(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kz}}{\mathrm{i}\lambda z} \int_{-w/2}^{w/2} \mathrm{d}\xi \int_{-h/2}^{h/2} \mathrm{d}\eta\, \mathrm{e}^{\mathrm{i}\frac{k}{2z}[(x-\xi)^2+(y-\eta)^2]} \tag{4}$$

Now we can make the equation independent of the wavelength, by introducing

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{y} = \frac{y}{\lambda}, \quad \tilde{z} = \frac{z}{\lambda}, \quad \tilde{\xi} = \frac{\xi}{\lambda}, \quad \tilde{\eta} = \frac{\eta}{\lambda}, \quad \tilde{w} = \frac{w}{\lambda}, \quad \tilde{h} = \frac{h}{\lambda}. \tag{5}$$

Then, $\mathrm{d}\tilde{\eta} = \mathrm{d}\eta/\lambda$ and $\mathrm{d}\tilde{\xi} = \mathrm{d}\xi/\lambda$. With these substitutions and $k = 2\pi/\lambda$ at hand, the equation takes the form

$$U(\tilde{\mathbf{r}}) = \frac{\mathrm{e}^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}\tilde{z}\lambda^2}\lambda^2 \int_{-\tilde{w}/2}^{\tilde{w}/2} \mathrm{d}\tilde{\xi} \int_{-\tilde{h}/2}^{\tilde{h}/2} \mathrm{d}\tilde{\eta}\, \mathrm{e}^{\mathrm{i}\pi\frac{1}{\lambda^2\tilde{z}}\lambda^2[(\tilde{x}-\tilde{\xi})^2+(\tilde{y}-\tilde{\eta})^2]} \tag{6}$$

$$= \frac{\mathrm{e}^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}\tilde{z}} \int_{-\tilde{w}/2}^{\tilde{w}/2} \mathrm{d}\tilde{\xi} \int_{-\tilde{h}/2}^{\tilde{h}/2} \mathrm{d}\tilde{\eta}\, \mathrm{e}^{\mathrm{i}\frac{\pi}{\tilde{z}}[(\tilde{x}-\tilde{\xi})^2+(\tilde{y}-\tilde{\eta})^2]}. \tag{7}$$

The integrands depend either on $\tilde{\eta}$ or $\tilde{\xi}$, that's why the integral factorizes to

$$U(\tilde{\mathbf{r}}) = \frac{\mathrm{e}^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}\tilde{z}} \int_{-\tilde{w}/2}^{\tilde{w}/2} \mathrm{d}\tilde{\xi}\, \mathrm{e}^{\mathrm{i}\frac{\pi}{\tilde{z}}(\tilde{x}-\tilde{\xi})^2} \cdot \int_{-\tilde{h}/2}^{\tilde{h}/2} \mathrm{d}\tilde{\eta}\, \mathrm{e}^{\mathrm{i}\frac{\pi}{\tilde{z}}(\tilde{y}-\tilde{\eta})^2}. \tag{8}$$

This is the desired result.

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b) Starting from the previous solution (8), we can define a function F as

$$U(\tilde{\mathbf{r}}) = \frac{\mathrm{e}^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}} \underbrace{\left[\frac{1}{\sqrt{\tilde{z}}} \int_{-\frac{\tilde{w}}{2}}^{\frac{\tilde{w}}{2}} d\tilde{\xi} \mathrm{e}^{\mathrm{i}\frac{\pi}{\tilde{z}}(\tilde{x}-\tilde{\xi})^2}\right]}_{=:F_{\tilde{w}}(\tilde{x},\tilde{z})} \underbrace{\left[\frac{1}{\sqrt{\tilde{z}}} \int_{-\frac{\tilde{h}}{2}}^{\frac{\tilde{h}}{2}} d\tilde{\eta} \mathrm{e}^{\mathrm{i}\frac{\pi}{\tilde{z}}(\tilde{y}-\tilde{\eta})^2}\right]}_{=:F_{\tilde{h}}(\tilde{y},\tilde{z})} \tag{9}$$

First, we consider the function $F_{\tilde{w}}(\tilde{x}, \tilde{z})$, where the argument suggests a substitution of the form

$$\xi' = \sqrt{\frac{2}{\tilde{z}}}\left(\tilde{x} - \tilde{\xi}\right). \tag{10}$$

Here, the factor $\sqrt{2}$ was introduced to make the argument look as needed for Fresnels integrals. Using this substitution, we find the limits of the integration as

$$a = -\frac{\tilde{w}}{2} \Rightarrow \tilde{x}_+ = \sqrt{\frac{2}{\tilde{z}}}\left(\tilde{x} + \frac{\tilde{w}}{2}\right), \quad b = \frac{\tilde{w}}{2} \Rightarrow \tilde{x}_- = \sqrt{\frac{2}{\tilde{z}}}\left(\tilde{x} - \frac{\tilde{w}}{2}\right), \tag{11}$$

For the substitution, we also need the differential as

$$d\xi' = -\sqrt{\frac{2}{\tilde{z}}}d\tilde{\xi} \Rightarrow d\tilde{\xi} = -\sqrt{\frac{\tilde{z}}{2}}d\xi'. \tag{12}$$

Insertion of these substiutions into $F_{\tilde{w}}(\tilde{x},\tilde{z})$ of (9) yields:

$$\begin{aligned} F_{\tilde{w}}(\tilde{x},\tilde{z}) &= -\frac{1}{\sqrt{\tilde{z}}}\sqrt{\frac{\tilde{z}}{2}}\int_{\tilde{x}_+}^{\tilde{x}_-} d\xi' \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\xi'^2} \qquad (13)\\ &= -\frac{1}{\sqrt{2}}\int_{\tilde{x}_+}^{\tilde{x}_-} d\xi' \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\xi'^2}. \qquad (14) \end{aligned}$$

Since we need the lower limit to be zero to apply Fresnel's integrals, we split the integrations by the scheme

$$\int_{\tilde{x}_+}^{\tilde{x}_-} = \int_{\tilde{x}_+}^{0} + \int_{0}^{\tilde{x}_-} = -\int_{0}^{\tilde{x}_+} + \int_{0}^{\tilde{x}_-}. \tag{15}$$

Thus, with this splitting and (14) we get

$$F_{\tilde{w}}(\tilde{x},\tilde{z}) = \frac{1}{\sqrt{2}}\left(\int_0^{\tilde{x}_+} d\xi' \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\xi'^2} - \int_0^{\tilde{x}_-} d\xi' \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\xi'^2}\right). \tag{16}$$

Now we use Euler's formula $\mathrm{e}^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$ and subsitute Fresnel's integrals to obtain

$$\begin{aligned} F_{\tilde{w}}(\tilde{x},\tilde{z}) &= \frac{1}{\sqrt{2}}\Big[C(\tilde{x}_+) + \mathrm{i}S(\tilde{x}_+) - C(\tilde{x}_-) - \mathrm{i}S(\tilde{x}_-)] \qquad (17)\\ &= \frac{1}{\sqrt{2}}\Big[C(\tilde{x}_+) - C(\tilde{x}_-) + \mathrm{i}\big(S(\tilde{x}_+) - S(\tilde{x}_-)\big)\Big]. \qquad (18) \end{aligned}$$

Now we put it all toghether. Analogous to (19), we define

$$\tilde{y}_+ = \sqrt{\frac{2}{\tilde{z}}}\left(\tilde{y} + \frac{\tilde{h}}{2}\right), \quad \tilde{y}_- = \sqrt{\frac{2}{\tilde{z}}}\left(\tilde{y} - \frac{\tilde{h}}{2}\right), \tag{19}$$

The desired intensity is then given as

$$\begin{aligned} |U(\tilde{\mathbf{r}})|^2 &= |F_{\tilde{w}}(\tilde{x},\tilde{z})|^2 |F_{\tilde{h}}(\tilde{y},\tilde{z})|^2 \qquad \text{(by (9))} \qquad (20)\\ &= \frac{1}{4}\big[|C(\tilde{x}_+) - C(\tilde{x}_-)|^2 + |S(\tilde{x}_+) - S(\tilde{x}_-)|^2\big]\\ &\quad \times \big[|C(\tilde{y}_+) - C(\tilde{y}_-)|^2 + |S(\tilde{y}_+) - S(\tilde{y}_-)|^2\big]. \qquad (21) \end{aligned}$$

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c) The Matlab plots are shown in the following.

Fresnel intensity for short distance z (*different* from Fraunhofer)

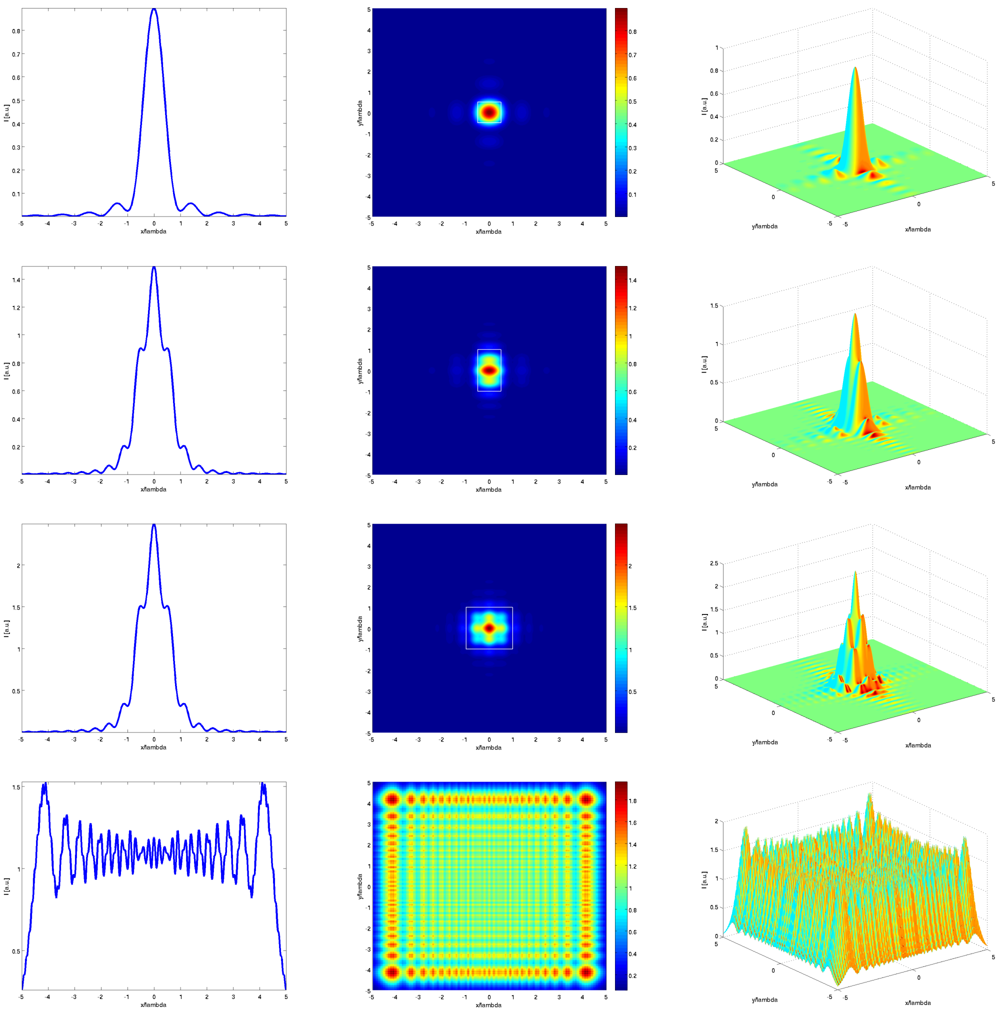

Figure 2: Fresnel intensity in arbitrary units for $z/\lambda = 10^0$. From top to bottom $(w/\lambda, h/\lambda)$: (1,1), (1,2), (2,2), (10,10). Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

Fresnel intensity for large distance z (similar to Fraunhofer)

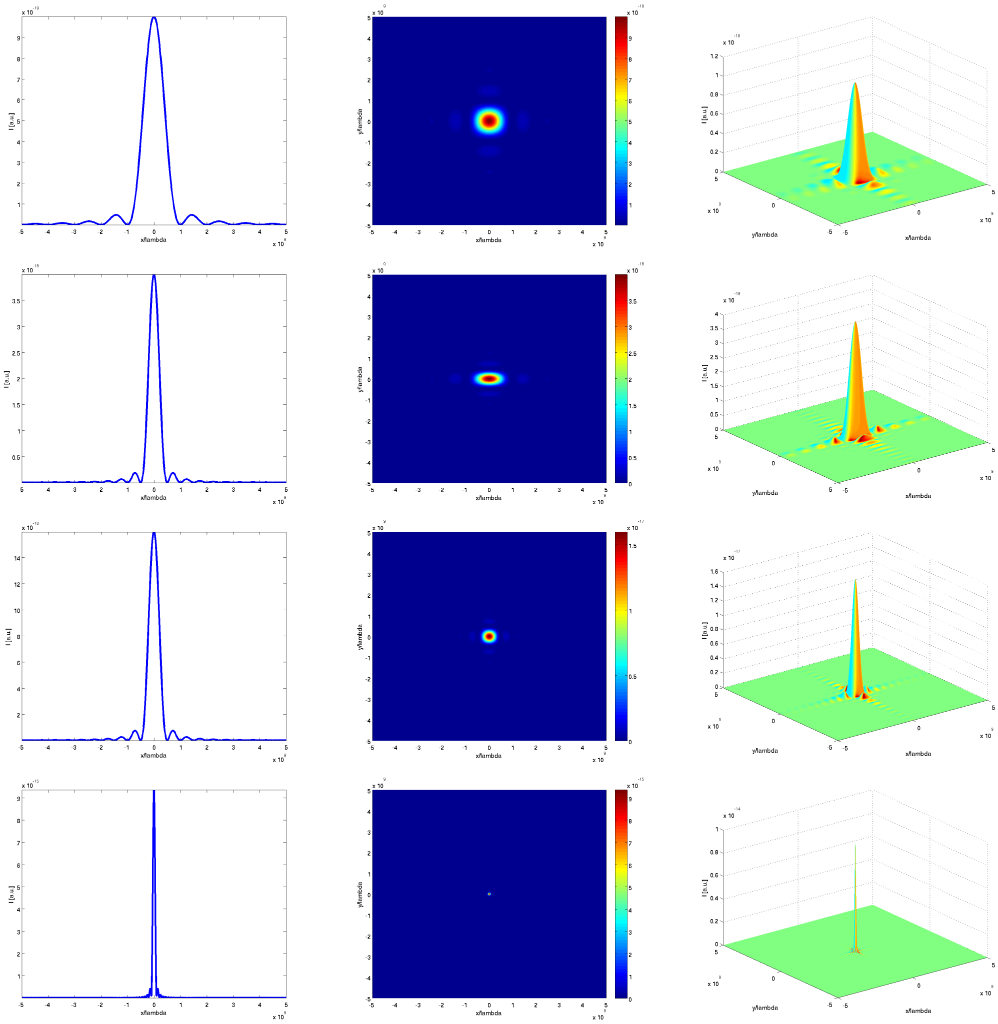

Figure 3: Fresnel intensity in arbitrary units for $z/\lambda = 10^9$ (Compare with Fraunhofer patterns for that distance!). From top to bottom $(w/\lambda, h/\lambda)$: (1,1), (1,2), (2,2), (10,10). Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

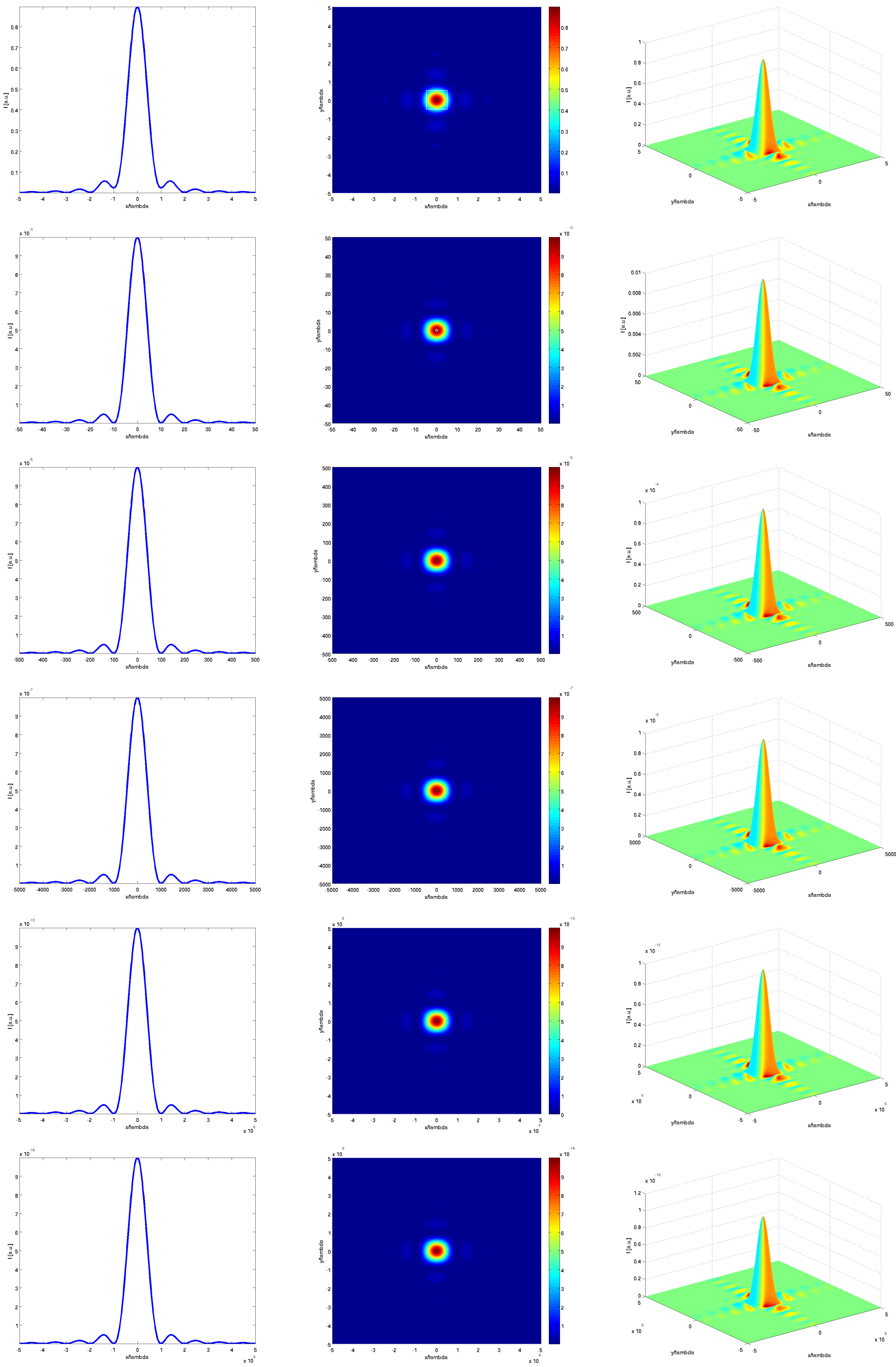

Figure 4: Fresnel Intensity in arbitrary units for $w/\lambda = 1$, $h/\lambda = 1$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

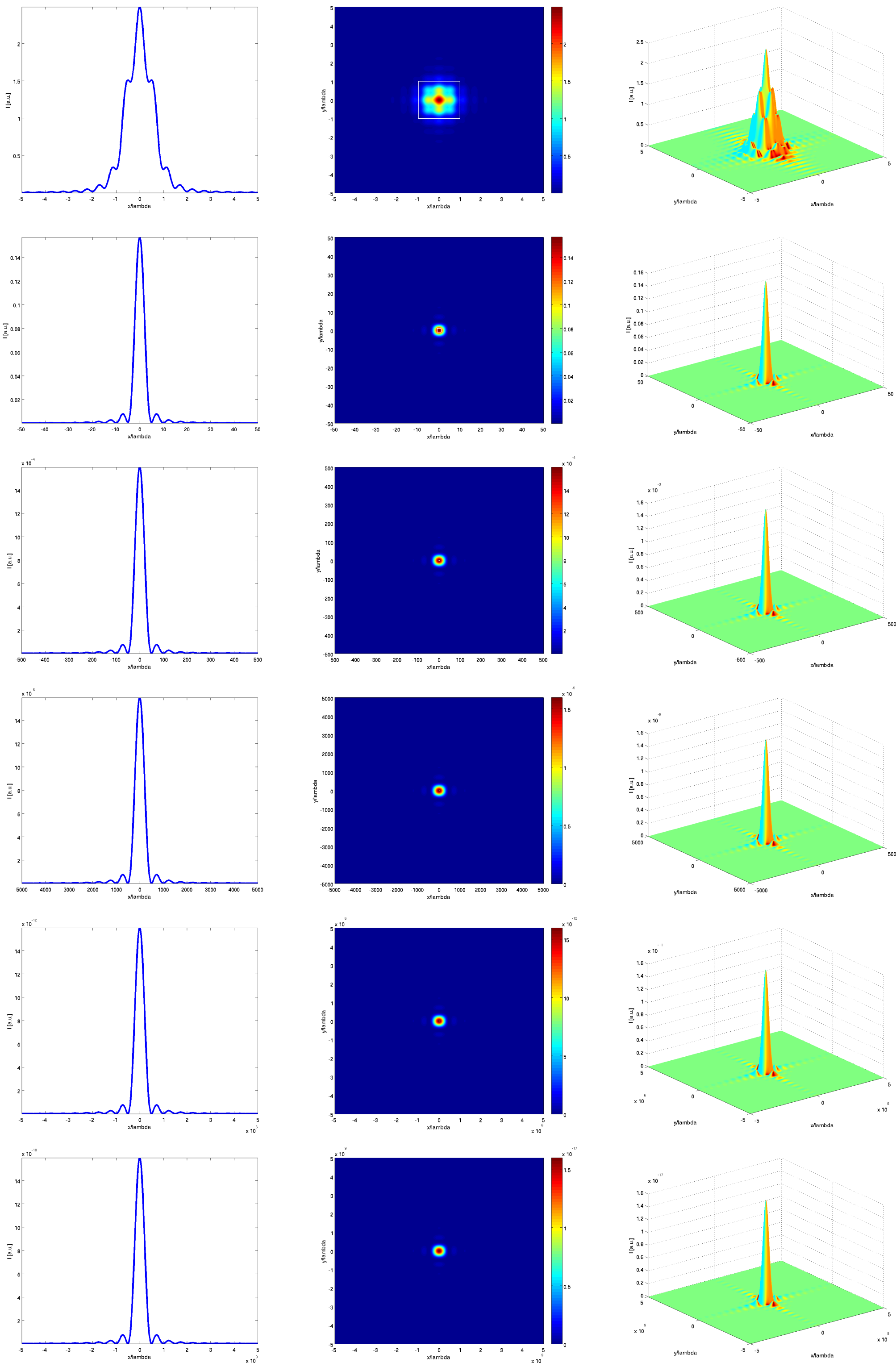

Figure 5: Fresnel Intensity in arbitrary units for $w/\lambda = 2$, $h/\lambda = 2$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

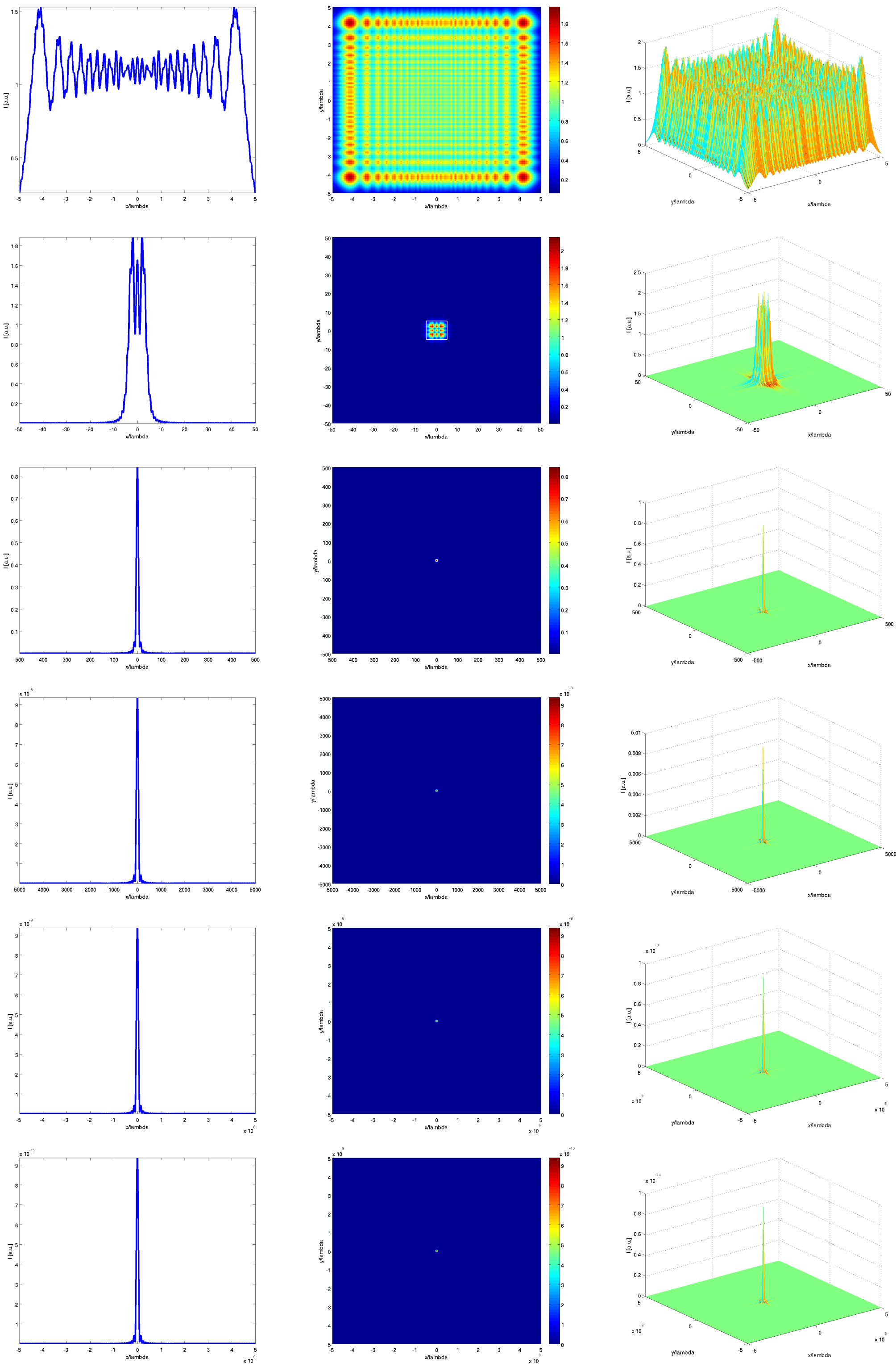

Figure 6: Fresnel Intensity in arbitrary units for $w/\lambda = 10$, $h/\lambda = 10$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

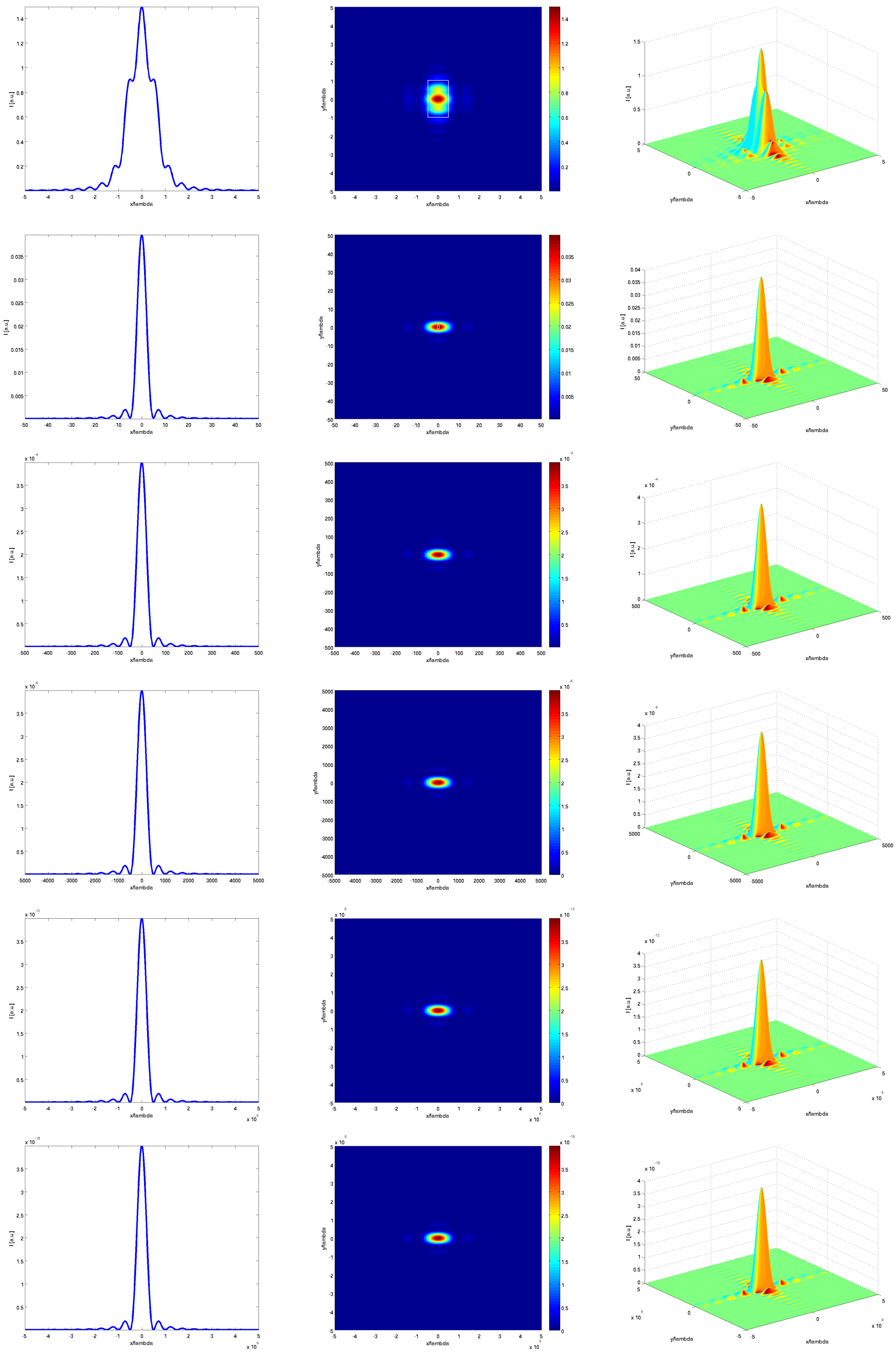

Figure 7: Fresnel Intensity in arbitrary units for $w/\lambda = 1$, $h/\lambda = 2$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

$\boxed{11}$ Fraunhofer Diffraction by a Slit

We consider the same system as in the previous problem, but now using the Fraunhofer approximation.

a) Write down the scalar field on the screen in the Fraunhofer approximation. As before, choose the center of the rectangular slit as the origin, substitute the functions U and make the expression independent of the wavelength λ. Then, perform the integration to obtain the field and the intensity distributions at distance z. Express Your solutions in terms of the unnormalized sinc function $\mathrm{sinc}(x) := \sin(x)/x$.[**4** Point(s)]

b) Again, use a suitable computer program to plot the intensity distributions for the same parameters as given in 12c).[**2 BONUS** Point(s)]

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a) We start with the Fraunhofer approximation given by

$$U(\mathbf{r}) = \frac{e^{\mathrm{i}kz}}{\mathrm{i}\lambda z} e^{\mathrm{i}\frac{k}{2z}\left(x^2+y^2\right)} \iint_{-\infty}^{\infty} U(\xi,\eta) e^{-\mathrm{i}\frac{k}{z}(x\xi+y\eta)} d\xi d\eta. \tag{22}$$

$$= \frac{e^{\mathrm{i}kz}}{\mathrm{i}\lambda z} e^{\mathrm{i}\frac{k}{2z}\left(x^2+y^2\right)} \int_{-w/2}^{w/2} \mathrm{d}\xi \int_{-h/2}^{h/2} \mathrm{d}\eta\, e^{-\mathrm{i}\frac{k}{z}(x\xi+y\eta)}, \qquad \text{(inserted } U \text{ for slit)} \tag{23}$$

Here we entered the same aperture function U as before, which amounts to adjusting the integration boundaries.

Now we rescale all lengths in the system again to get rid of the wavelength by introducing the substitutions

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{y} = \frac{y}{\lambda}, \quad \tilde{z} = \frac{z}{\lambda}, \quad \tilde{\xi} = \frac{\xi}{\lambda}, \quad \tilde{\eta} = \frac{\eta}{\lambda}, \quad \tilde{w} = \frac{w}{\lambda}, \quad \tilde{h} = \frac{h}{\lambda}. \tag{24}$$

Then, $\mathrm{d}\tilde{\eta} = \mathrm{d}\eta/\lambda$ and $\mathrm{d}\tilde{\xi} = \mathrm{d}\xi/\lambda$. With these substitutions and $k = 2\pi/\lambda$ at hand, the equation takes the form

$$U(\mathbf{r}) = \frac{e^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}\tilde{z}\lambda^2} e^{\mathrm{i}\frac{2\pi}{2\tilde{z}\lambda^2}\lambda^2\left(\tilde{x}^2+\tilde{y}^2\right)} \lambda^2 \int_{-\tilde{w}/2}^{\tilde{w}/2} \mathrm{d}\tilde{\xi} \int_{-\tilde{h}/2}^{\tilde{h}/2} \mathrm{d}\tilde{\eta}\, e^{-\mathrm{i}\frac{2\pi}{\tilde{z}\lambda^2}\lambda^2\left(\tilde{x}\tilde{\xi}+\tilde{y}\tilde{\eta}\right)} \tag{25}$$

$$= \frac{e^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}\tilde{z}} e^{\mathrm{i}\frac{2\pi}{2\tilde{z}}\left(\tilde{x}^2+\tilde{y}^2\right)} \int_{-\tilde{w}/2}^{\tilde{w}/2} \mathrm{d}\tilde{\xi} \int_{-\tilde{h}/2}^{\tilde{h}/2} \mathrm{d}\tilde{\eta}\, e^{-\mathrm{i}\frac{2\pi}{\tilde{z}}\left(\tilde{x}\tilde{\xi}+\tilde{y}\tilde{\eta}\right)}, \tag{26}$$

$$= \frac{e^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}\tilde{z}} e^{\mathrm{i}\frac{2\pi}{2\tilde{z}}\left(\tilde{x}^2+\tilde{y}^2\right)} \underbrace{\int_{-\tilde{w}/2}^{\tilde{w}/2} \mathrm{d}\tilde{\xi}\, e^{-\mathrm{i}\frac{2\pi}{\tilde{z}}\left(\tilde{x}\tilde{\xi}\right)}}_{=:G_{\tilde{w}}(\tilde{x},\tilde{z})} \underbrace{\int_{-\tilde{h}/2}^{\tilde{h}/2} \mathrm{d}\tilde{\eta}\, e^{-\mathrm{i}\frac{2\pi}{\tilde{z}}\left(\tilde{y}\tilde{\eta}\right)}}_{=:G_{\tilde{h}}(\tilde{y},\tilde{z})}, \tag{27}$$

The integral for the function G can be evaluated as

$$
\begin{aligned}
G_{\tilde{w}}(\tilde{x},\tilde{z}) &= \int_{-\frac{\tilde{w}}{2}}^{\frac{\tilde{w}}{2}} e^{-\mathrm{i}\frac{2\pi}{\tilde{z}}\tilde{x}\tilde{\xi}} d\tilde{\xi} && (28)\\
&= \left[-\frac{\tilde{z}}{\mathrm{i}2\pi\tilde{x}} e^{-\mathrm{i}\frac{2\pi}{\tilde{z}}\tilde{x}\tilde{\xi}}\right]_{\tilde{\xi}=-\frac{\tilde{w}}{2}}^{\tilde{\xi}=\frac{\tilde{w}}{2}} \qquad \text{(elementary integration)} && (29)\\
&= -\frac{\tilde{z}}{\mathrm{i}2\pi\tilde{x}} \underbrace{\left[e^{-\mathrm{i}2\pi\frac{\tilde{x}}{\tilde{z}}\frac{\tilde{w}}{2}} - e^{\mathrm{i}2\pi\frac{\tilde{x}}{\tilde{z}}\frac{\tilde{w}}{2}}\right]}_{=-2\mathrm{i}\sin\left(2\pi\frac{\tilde{x}}{\tilde{z}}\frac{\tilde{w}}{2}\right)} \qquad \text{(evaluate at boundaries)} && (30)\\
&= \tilde{w}\frac{\sin\left(2\pi\frac{\tilde{x}}{\tilde{z}}\frac{\tilde{w}}{2}\right)}{2\pi\frac{\tilde{x}}{\tilde{z}}\frac{\tilde{w}}{2}} && (31)\\
&= \tilde{w}\,\mathrm{sinc}(2\pi\frac{\tilde{x}}{\tilde{z}}\frac{\tilde{w}}{2}), && (32)
\end{aligned}
$$

where the (unnormalized) sinc function (Latin, *sinus cardinalis*) is defined as $\mathrm{sinc}(x) := \sin(x)/x$. **There is also the normalized version for this function with $x \mapsto \pi x$ in the definition. The normalized version would be even nicer to express the solution here, but the use of the unnormalized sinc function is more common in physics. When using this function in computer programs, one should take care which version is actually used then.**

Now, we can write down the full expression for U as

$$
U(\tilde{\mathbf{r}}) = \frac{e^{\mathrm{i}2\pi\tilde{z}}}{\mathrm{i}\tilde{z}} e^{\mathrm{i}\frac{2\pi}{2\tilde{z}}\left(\tilde{x}^2+\tilde{y}^2\right)} \tilde{w}\tilde{h}\,\mathrm{sinc}(2\pi\frac{\tilde{x}}{\tilde{z}}\frac{\tilde{w}}{2})\,\mathrm{sinc}(2\pi\frac{\tilde{y}}{\tilde{z}}\frac{\tilde{h}}{2}). \tag{33}
$$

From this we also find the intensity as

$$
I(\tilde{\mathbf{r}}) = \frac{(\tilde{w}\tilde{h})^2}{\tilde{z}^2}\,\mathrm{sinc}(\pi\frac{\tilde{x}}{\tilde{z}}\tilde{w})^2\,\mathrm{sinc}(\pi\frac{\tilde{y}}{\tilde{z}}\tilde{h})^2. \tag{34}
$$

This is the desired solution.

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b) The Matlab plots are shown in the following.

Fraunhofer intensity for short and large distance z (similar to Fresnel)

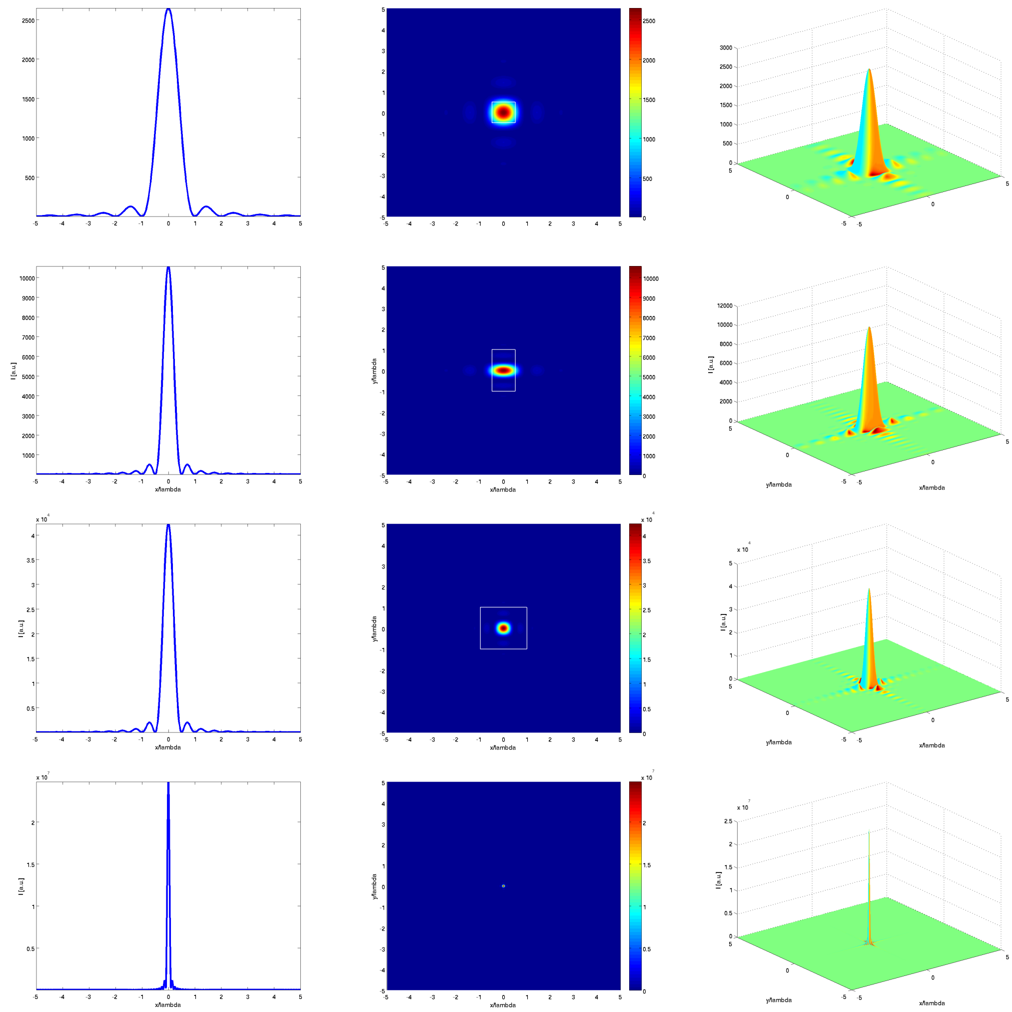

Figure 8: Fraunhofer intensity in arbitrary units for $z/\lambda = 10^0$ (all other distances look basically the same, see following plots). From top to bottom $(w/\lambda, h/\lambda)$: (1,1), (1,2), (2,2), (10,10). Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

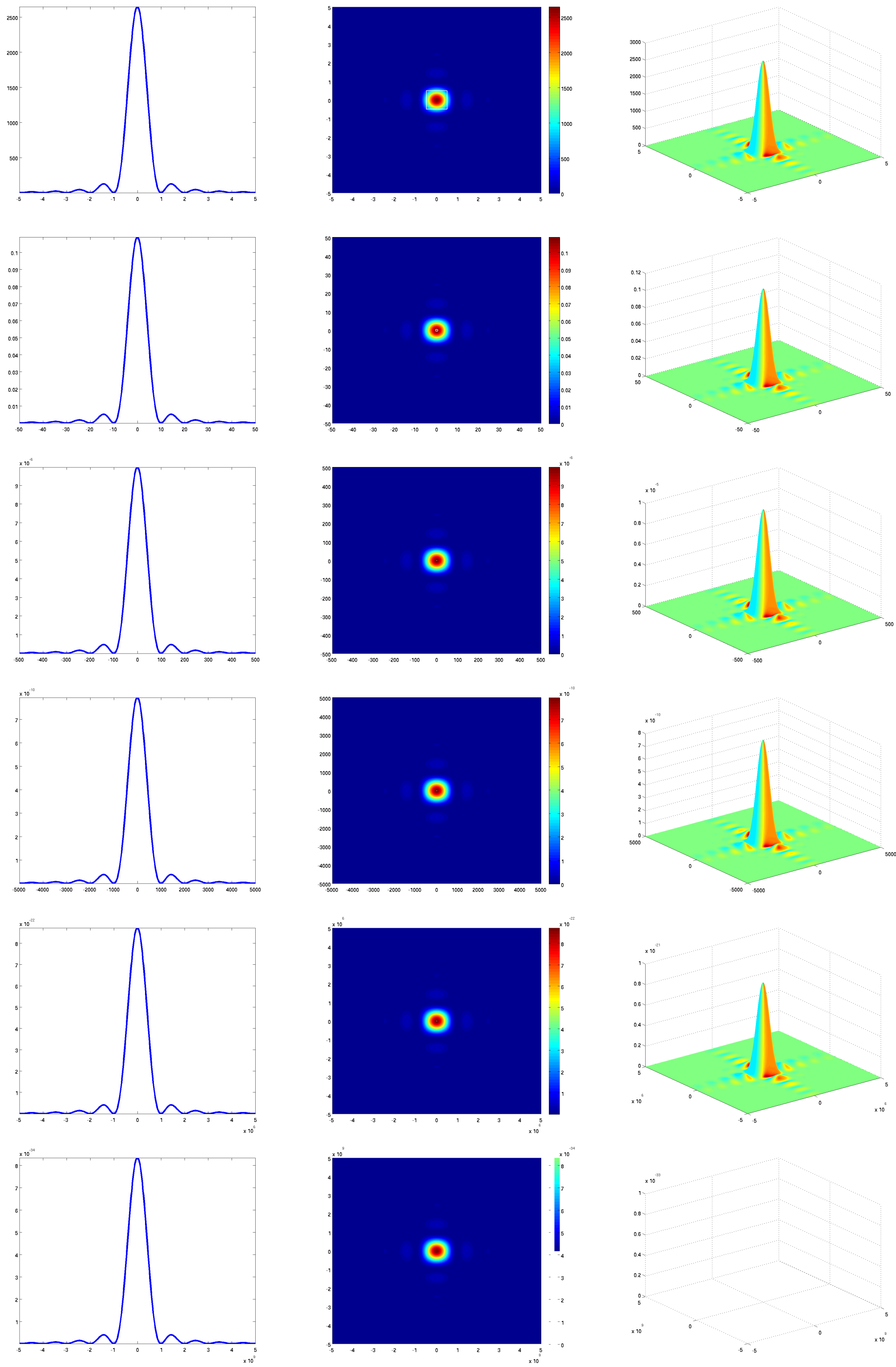

Figure 9: Fraunhofer Intensity in arbitrary units for $w/\lambda = 1$, $h/\lambda = 1$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

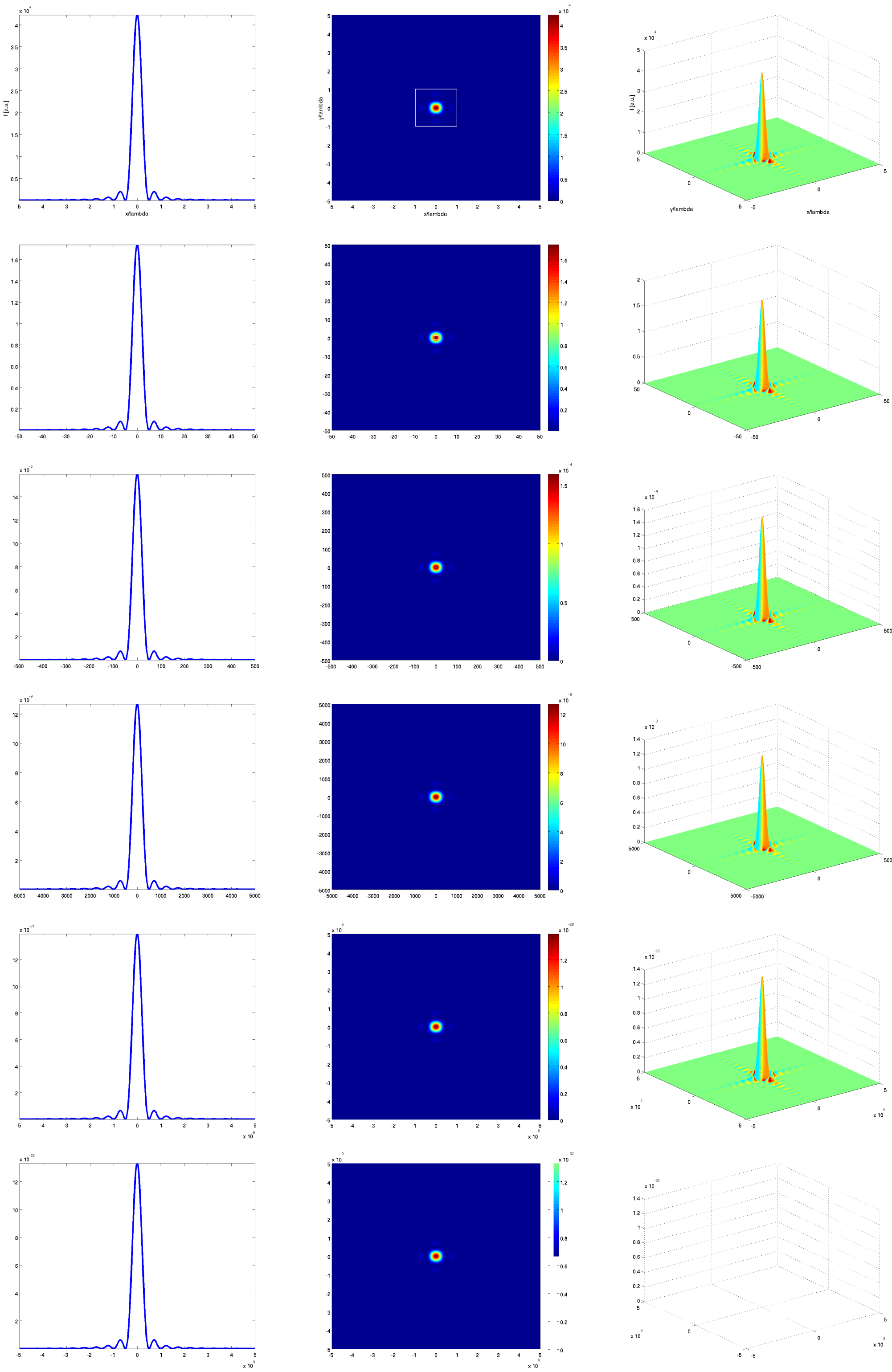

Figure 10: Fraunhofer Intensity in arbitrary units for $w/\lambda = 2$, $h/\lambda = 2$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

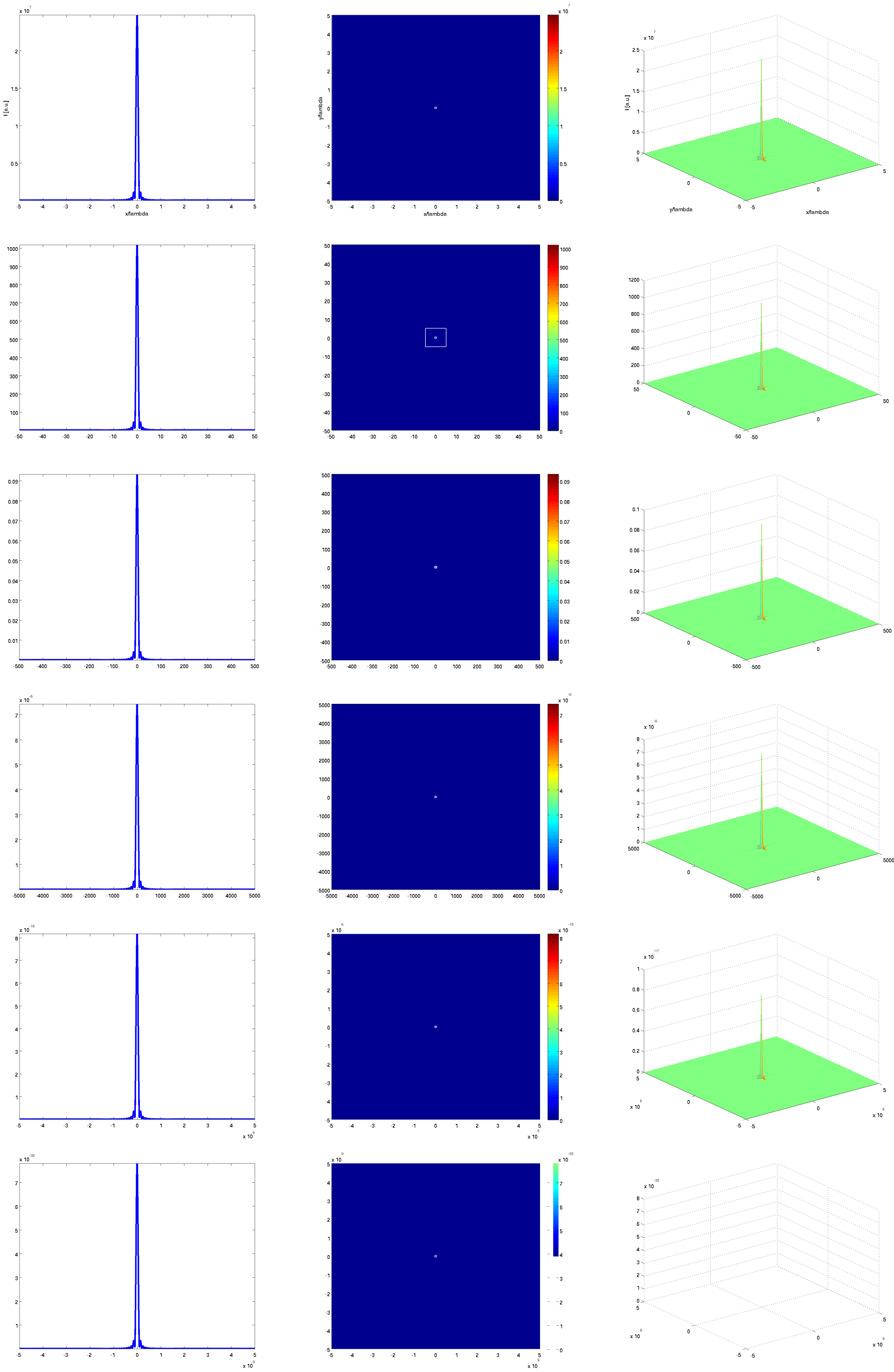

Figure 11: Fraunhofer Intensity in arbitrary units for $w/\lambda = 10$, $h/\lambda = 10$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

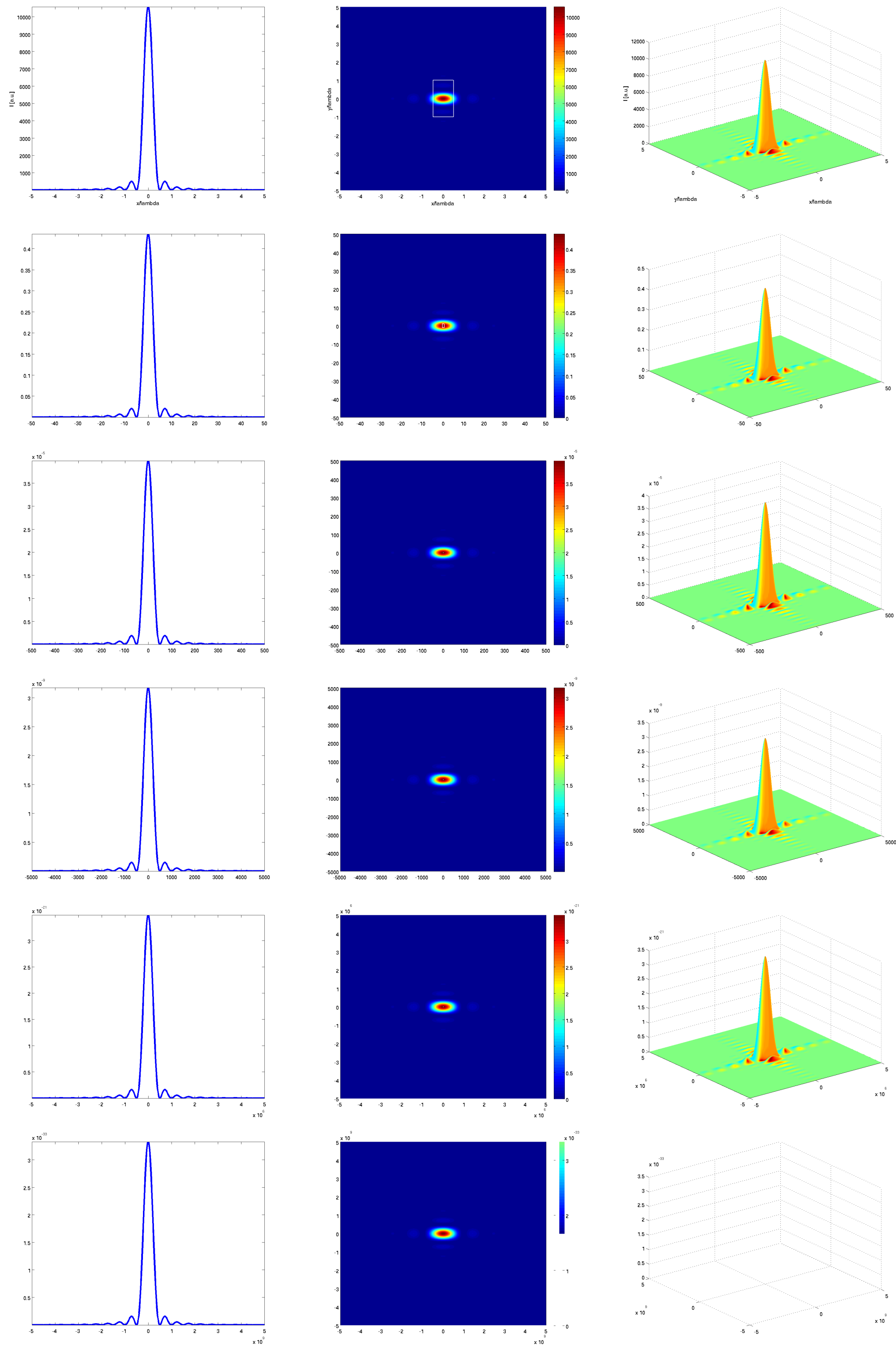

Figure 12: Fraunhofer Intensity in arbitrary units for $w/\lambda = 1$, $h/\lambda = 2$, from top to bottom: $z/\lambda = 10^0, 10^1, 10^2, 10^3, 10^6, 10^9$. Left: $I(x/\lambda)$, Middle: $I(x/\lambda, y/\lambda)$ (with slit), Right: same as in middle.

— Hand in solutions in lecture on 11.06.2011 —