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Dr. Boris Narozhny, Institut für Theorie der Kondensierten Materie (TKM), narozhny@tkm.uni-karlsruhe.de
Dr. Panagiotis Kotetes, Institut für Theoretische Festkörperphysik (TFP), panagiotis.kotetes@kit.edu

THEORETICAL OPTICS: EXERCISE SHEET 1

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1. Review of Fourier theory

a. Every function $f(t)$ defined on $[-T/2,+T/2]$ can be expanded in Fourier series:

$$f(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{2\pi nt}{T}\right)+B_n\sin\left(\frac{2\pi nt}{T}\right)\right], \tag{1}$$

with the coefficients A_n and B_n defined as:

$$A_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt, \quad n=0,1,2,\dots, \tag{2}$$

$$B_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt, \quad n=1,2,3,\dots. \tag{3}$$

Expand in Fourier series the following functions (5 points):

i. $f(t)=t$, ii. $f(t)=(t-a)^2$, iii. $f(t)=\sin(\lambda t)$, iv. $f(t)=2-\Theta(-t)$, v. $f(t)=e^{-at}\Theta(t)$,

with $a>0$ and $\Theta(t)$ denotes the Heaviside step function satisfying $\Theta(t<0)=0$ and $\Theta(t\geq 0)=1$.

b. When the interval $[-T/2,+T/2]$ becomes $[-\infty,+\infty]$ a function $f(t)$ can be expanded in a Fourier integral:

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(\omega)e^{-i\omega t}d\omega \quad \text{with} \quad F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{+i\omega t}dt. \tag{4}$$

The function $F(\omega)\equiv\mathcal{F}[f(t)]$ defines the Fourier transform of $f(t)$ and corresponds to the continuum version of the complex coefficients A_n+iB_n encountered in the Fourier series.

Prove the following properties of the Fourier transform (6 points):

i. $\mathcal{F}[f(t-a)]=e^{i\omega a}F(\omega)$, ii. $\mathcal{F}[f(at)]=\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$, iii. $\mathcal{F}\left[\frac{d^n}{dt^n}f(t)\right]=(-i\omega)^nF(\omega)$,

iv. $\mathcal{F}[t^nf(t)]=\left(-i\frac{d}{d\omega}\right)^nF(\omega)$, v. $\mathcal{F}[f(t)g(t)]=\frac{(F*G)(\omega)}{\sqrt{2\pi}}$, vi. $\mathcal{F}[(f*g)(t)]=\sqrt{2\pi}F(\omega)G(\omega)$,

where $(f*g)(t)$ denotes the convolution

$$(f*g)(t)=\int_{-\infty}^{+\infty}f(t-\tau)g(\tau)d\tau. \tag{5}$$

Calculate the Fourier transforms of the following functions (4 points):

$$\text{vii. } f(t) = te^{-at^2}\,, \quad \text{viii. } f(t) = \frac{d\Theta(t)}{dt} \equiv \delta(t)\,, \quad \text{ix. } f(t) = e^{-t}\Theta(t-a)\sin(t-a)\,, \quad \text{x. } f(t) = (\Theta * \sin)(at)\,,$$

where $a > 0$ and $\delta(t)$ corresponds to the Dirac δ-function, which satisfies

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1 \quad \text{and} \quad \delta(t \neq 0) = 0\,. \tag{6}$$

As a matter of fact, $\delta(t)$ becomes infinite at $t = 0$. $\Theta(t)$ and $\delta(t)$ constitute "generalized functions" (in a mathematical language: *distributions*) operating on the usual functions.

c. For a function $f(n)$ depending on a discrete variable $n = 0, 1, \ldots, N-1$ where $N \in \mathbb{N}$ one can define a discrete Fourier transform $F(k)$ through the relations:

$$f(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} F(k) e^{-i\frac{2\pi k}{N}n} \quad \text{and} \quad F(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{+i\frac{2\pi k}{N}n}\,. \tag{7}$$

Retrieve the discrete Fourier transforms for the following functions (3 points):

$$\text{i. } f(n) = \delta(n-r)\,, \quad \text{ii. } f(n) = \cos(\lambda n)\,, \quad \text{iii. } f(n) = \Theta(n-r)e^{-a(n-r)} \ (a > 0) \quad \text{where} \quad r \in [0, N-1]\,. \tag{8}$$

The discrete Fourier transform can be rewritten in an equivalent matrix form:

$$\begin{pmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(N-1) \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_N & \omega_N^2 & \ldots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \ldots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \ldots & \omega_N^{(N-1)^2} \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(N-1) \end{pmatrix} \quad \text{where} \quad \omega_N = e^{i2\pi/N}\,. \tag{9}$$

Calculate the discrete Fourier transform of the function $f(n) = n^2 \sin(n\pi/2)$ for $N = 4$ (2 points).

2. Maxwell's Equations

Maxwell's equations of electromagnetism in vacuum, have the following differential form

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} = \frac{\rho}{\varepsilon_0}\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} = 0\,, \quad \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = -\partial_t \boldsymbol{\mathcal{B}} \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \partial_t \boldsymbol{\mathcal{E}}\,, \tag{10}$$

with ρ the charge density, $\boldsymbol{J}$ the current density and $\partial_t \equiv \partial/\partial t$. Through the defining relations

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\phi - \partial_t \boldsymbol{A} \quad \text{and} \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \boldsymbol{A}\,, \tag{11}$$

the scalar (ϕ) and vector ($\boldsymbol{A}$) electromagnetic potentials are introduced.

i. Using Eq. (11), obtain Maxwell's equations in terms of ϕ and $\boldsymbol{A}$ (7 points).

ii. Study the behaviour of the equations retrieved in i., under gauge transformations (3 points):

$$\phi \to \phi - \partial_t \chi \quad \text{and} \quad \boldsymbol{A}' \to \boldsymbol{A} + \boldsymbol{\nabla}\chi\,.$$